~ Диференцијабилност ~

прво - подсетање - теорија pdf

1. $f'(x) = ?$

a) $f(x) = 2x^3 \cdot e^x \cdot \ln x$

$[(\alpha \cdot x^n)' = \alpha \cdot n \cdot x^{n-1}]$, α ↑ const

$$f'(x) = (2x^3 \cdot (e^x \cdot \ln x))' = (2x^3)' \cdot (e^x \ln x) + 2x^3 \cdot (e^x \ln x)'$$

$$= 2 \cdot 3 \cdot x^2 \cdot e^x \cdot \ln x + 2x^3 \cdot ((e^x)' \cdot \ln x + e^x \cdot (\ln x)')$$

$$= 6x^2 e^x \cdot \ln x + 2x^3 \cdot (e^x \cdot \ln x + e^x \cdot \frac{1}{x})$$

$$= 6x^2 \cdot e^x \cdot \ln x + 2x^3 \cdot e^x \cdot \ln x + 2x^3 \cdot e^x \cdot \frac{1}{x} = \dots$$

$2 \cdot 3x^2$

(!) $(f_1 \cdot f_2)' = f_1' \cdot f_2 + f_1 \cdot f_2'$

$(f_1 \cdot f_2 \cdot f_3)' = f_1' \cdot f_2 \cdot f_3 + f_1 \cdot f_2' \cdot f_3 + f_1 \cdot f_2 \cdot f_3'$

индукција → $(f_1 \cdot f_2 \cdots f_n)' = f_1' \cdot f_2 \cdots f_n + f_1 \cdot f_2' \cdot f_3 \cdots f_n + \dots + f_1 \cdots f_{n-1} \cdot f_n'$

(б) $f(x) = \sqrt{1+\sqrt[3]{1+x^4}}$

$$\left(\sqrt{h}\right)' = \underbrace{\frac{1}{2}\cdot\frac{1}{\sqrt{h}}}_{\text{извод корена}}\cdot h' \qquad (h = h(x))$$

$\sqrt{h} = h^{1/2}$

$(x^{\alpha})' = \alpha \cdot x^{\alpha-1}$

$(\sqrt{x})' = \frac{1}{2}\cdot x^{1-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$$f'(x) = \frac{1}{2}\cdot\frac{1}{\sqrt{1+\sqrt[3]{1+x^4}}}\cdot\left(1+\sqrt[3]{1+x^4}\right)'$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{1+\sqrt[3]{1+x^4}}}\cdot\left(0+\frac{1}{3}\cdot(1+x^4)^{-2/3}\cdot\underbrace{(1+x^4)'}_{4x^3}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{1+\sqrt[3]{1+x^4}}}\cdot\frac{1}{3}\cdot\frac{1}{\sqrt[3]{(1+x^4)^2}}\cdot 4x^3 = \ldots$$

(в) $f(x) = x^x$

☺ $f(x) = u(x)^{t(x)}$

$f'(x) = \left(e^{t(x)\cdot \ln(u(x))}\right)'$

$\leftarrow f(x) = e^{x\cdot\ln x}$

$$f'(x) = \left(e^{x\ln x}\right)' = e^{x\cdot\ln x}\cdot(x\cdot\ln x)'$$

$$= x^x\cdot\left(x\cdot\frac{1}{x} + 1\cdot\ln x\right)$$

$$= x^x\cdot(1+\ln x)$$

(I) $f(x) = x^{x^x} = e^{x^x \cdot \ln x}$

$$f'(x) = \left(e^{x^x \cdot \ln x}\right)' = e^{x^x \cdot \ln x} \cdot \left(x^x \cdot \ln x\right)'$$

$$= e^{x^x \cdot \ln x} \cdot \Big[\underbrace{(x^x)'}_{\substack{(б)\ \| \\ x^x \cdot (1+\ln x)}} \cdot \ln x + x^x \cdot \underbrace{(\ln x)'}_{\frac{1}{x}}\Big]$$

$$= x^{x^x} \cdot \left(x^x \cdot (1+\ln x) \cdot \ln x + x^x \cdot \frac{1}{x}\right)$$

$$= x^{x^x} \cdot x^x \cdot \left(\ln x + (\ln x)^2 + \frac{1}{x}\right)$$

* $f(x) = \left(\dfrac{\arcsin(\sin^2 x)}{\arccos(\cos^2 x)}\right)^{\operatorname{arcctg}^2 x}$ $\quad f'(x) = ?$

* $f(x) = (\operatorname{arcctg} x)^{x^2}$ $\quad f'(x) = ?$

за вежбу

② Тангента? $f(x)=3x^2+e^x-x$ у тачки $x_0=1$

$f(x_0)=f(1)=3\cdot 1^2+e^1-1=e+2$

$A(1, e+2)$

праву? $f'(x)=3\cdot 2x+e^x-1=6x+e^x-1$

$f'(1)=5+e$ $=k$ - коеф правца тан. у тачки A

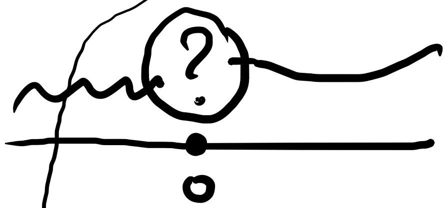
$(x_0,y_0)\in t$
k - коеф. пр. $\Big\}\rightarrow$ $y-y_0=k\cdot(x-x_0)$

→ тангента: $y-(e+2)=(5+e)(x-1)$

$y=(5+e)x-3$

③ Одредити $a, b \in \mathbb{R}$ т.д. f буде диф. на $\mathbb{R}$ (ако $\exists$)

$$f(x) = \begin{cases} x \cdot e^{-bx} + a, & x < 0 \\ x^3 + bx, & x \geq 0 \end{cases}$$



НЕПРЕКИДНОСТ:

за $x \neq 0$: ✓

за $x = 0$: $f(0) = 0 = \lim_{x \to 0+} f(x)$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x \cdot e^{-bx} + a) = a$$

f непр. у $x = 0 \iff \boxed{a = 0}$

Даље:

$$f(x) = \begin{cases} x \cdot e^{-bx}, & x < 0 \\ x^3 + bx, & x \geq 0 \end{cases}$$

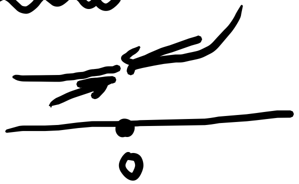
f $\quad f(x) = \begin{cases} x \cdot e^{-bx}, & x<0 \\ x^3+bx, & x \geq 0 \end{cases}$

$x \neq 0$

$x<0: \quad f'(x) = (x \cdot e^{-bx})' = e^{-bx} + x \cdot e^{-bx} \cdot (-b)$ ✓

$x>0: \quad f'(x) = 3x^2+b$ ✓

$x=0$



$$f'_+(0) = \lim_{h\to 0_+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0_+} \frac{h^3+bh-0}{h} = \lim_{h \to 0_+}(h^2+b) = \underline{b}$$

$$f'_-(0) = \lim_{h\to 0_-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0_-} \frac{h \cdot e^{-bh}-0}{h} = \lim_{h \to 0_-} e^{-bh} = \underline{1}$$

f παρ. στο $0 \iff f'_+(0) = f'_-(0)$

$\iff \boxed{b=1}$

$\boxed{a=0,\ b=1}$

④ $f(x)=\sqrt{1-e^{-x^2}}$ диф. на D_f.

$D_f=?$ $1-e^{-x^2}\geq 0 \iff 1\geq e^{-x^2} \iff 0\geq -x^2 \iff$ (Т)

($1 = e^0$)

$\Rightarrow$ $D_f=\mathbb{R}$

НЕПРЕКИДНОСТ: f непр. на $\mathbb{R}$

ДИФЕРЕНЦИЈАБИЛНОСТ:

$$f'(x)=\frac{1}{2}\cdot\frac{1}{\sqrt{1-e^{-x^2}}}\cdot(1-e^{-x^2})'=\frac{1}{2}\cdot\frac{1}{\sqrt{1-e^{-x^2}}}\cdot e^{-x^2}\cdot 2x$$

$$=\frac{x\cdot e^{-x^2}}{\sqrt{1-e^{-x^2}}}$$

за $1-e^{-x^2}\neq 0$

$\iff e^{-x^2}\neq 1$

$\iff x\neq 0$

$x\neq 0 \rightarrow f$ диф. у x

За $x=0$: $f'(0) = \lim_{h\to 0}\frac{f(h)-0}{h}=\lim_{h\to 0}\frac{\sqrt{1-e^{-h^2}}}{h}$ (ако ∃; $f(0)=0$)

$\left[\frac{e^t-1}{t}\to 1,\ t\to 0\right]$

h под корен? $+/-$

$$f'_+(0)=\lim_{h\to 0+}\frac{\sqrt{1-e^{-h^2}}}{h}=\lim_{h\to 0+}\sqrt{\frac{1-e^{-h^2}}{h^2}}$$

($h>0$, $h=\sqrt{h^2}$)

$$=\lim_{h\to 0+}\sqrt{\frac{e^{-h^2}-1}{-h^2}}=1$$

($-h^2\to 0$, $\frac{e^{-h^2}-1}{-h^2}\to 1$)

$$f'_-(0)=\lim_{h\to 0-}\frac{\sqrt{1-e^{-h^2}}}{h}=\lim_{h\to 0-}-\sqrt{\frac{1-e^{-h^2}}{h^2}}$$

$h<0$ $h=-\sqrt{h^2}$

$$=\lim_{h\to 0-}-\sqrt{\frac{e^{-h^2}-1}{-h^2}}=-1$$

($\to 1$)

$f'_+(0)\neq f'_-(0)$ (1, -1) $\Rightarrow$ $\not\exists f'(0)$ — f није диф. у 0

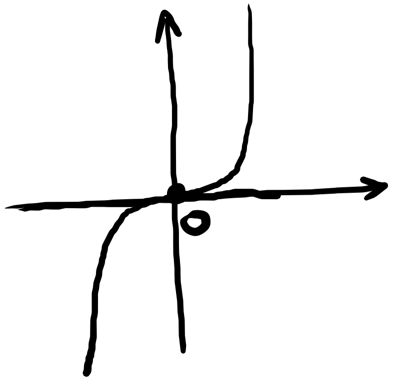
f диф на $(-\infty,0)\cup(0,+\infty)$

Днф $\Rightarrow$ непр $\quad$ /7

7 непр $\Rightarrow$ 7Днф.

5. $f(x) = x \cdot |x|$ диф. ?

$$f(x) = \begin{cases} x^2, x \geq 0 \\ -x^2, x \leq 0 \end{cases}$$



Непр: ✓ $\mathbb{R}$

Днф: $x \neq 0$: $x > 0$ $f'(x) = 2x$

$x < 0$ $f'(x) = -2x$

$$x = 0: \quad f'(0) \underset{\substack{\uparrow \\ \text{ако } \exists \\ \lim}}{=} \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h \cdot |h| - 0}{h} = \lim_{h \to 0} |h| = 0$$

$\boxed{f'(0) = 0}$ $\Rightarrow$ $\boxed{f \text{ диф. на } \mathbb{R}}$

6. $f(x)=|x^2-x-12|\cdot\cos\frac{x\pi}{2}$ диф. ?

$\left[\ |x|'=\begin{cases}1, & x>0\\ -1, & x<0\end{cases}\ \right]$

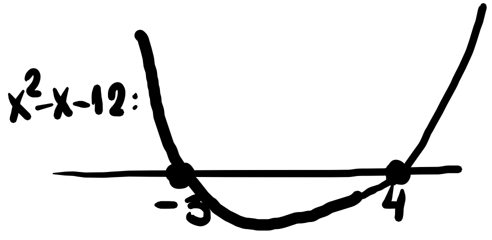
$D_f=\mathbb{R}$ $\quad$ f непр. на $\mathbb{R}$

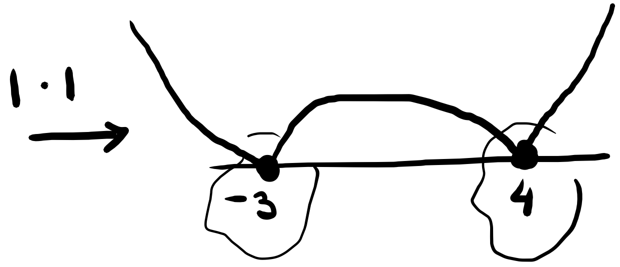
диф:

$x^2-x-12=0$

$(x-4)(x+3)=0$

$x_1=-3,\ x_2=4$





$\boxed{x\neq -3,4 \quad f \text{ диф.}}$ (као производ две диф.)

$x=-3$?

$x=4$?

$\boxed{x=-3}$

$$f'(-3)\overset{\text{деф}}{=}\lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h} \quad (f(-3)=0)$$

$$=\lim_{h\to 0}\frac{|(-3+h)^2-(-3+h)-12|\cdot\cos\frac{(-3+h)\pi}{2}}{h}$$

$\cos\left(\alpha-\frac{3\pi}{2}\right)=\cos\left(\alpha+\frac{\pi}{2}\right)=-\sin\alpha$

$\cos\frac{(-3+h)\pi}{2}=-\sin\frac{h\pi}{2}$

$$=\lim_{h\to 0}\frac{|h^2-7h|\cdot\left(-\sin\frac{h\pi}{2}\right)}{h}=\lim_{h\to 0}\underbrace{|h^2-7h|}_{\to 0}\cdot\left(-\frac{\sin\frac{h\pi}{2}}{\frac{h\pi}{2}}\right)\cdot\frac{\pi}{2}=\boxed{0}$$

($\frac{h\pi}{2}\to 0$, $\frac{\sin\frac{h\pi}{2}}{\frac{h\pi}{2}}\to 1$)

$f'(-3)=0$ $\quad$ f диф у -3 ✓

X=4

$$\lim_{h\to 0}\frac{f(4+h)-\overset{0}{f(4)}}{h}=\lim_{h\to 0}\frac{|(4+h)^2-(4+h)-12|\cdot\cos\frac{(4+h)\pi}{2}}{h}$$

$\leftarrow$ $\cos(\alpha+2\pi)=\cos\alpha$

$$=\lim_{h\to 0}\frac{|h^2+7h|\cdot\cos\frac{h\pi}{2}}{h}\quad \left(\cos\frac{h\pi}{2}\to 1\right)\quad=\lim_{h\to 0}\frac{|h^2+7h|}{h}$$

$\left[\,h=\operatorname{sgn}h\cdot|h|\,\right]$

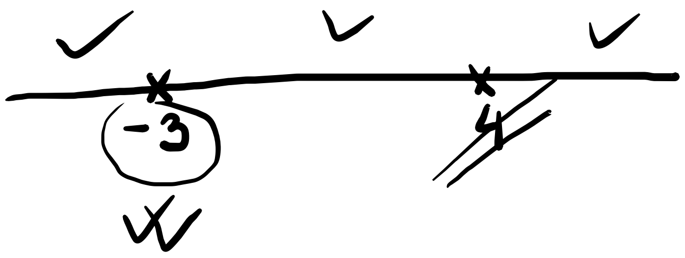
$$=\lim_{h\to 0}\frac{|h^2+7h|}{\operatorname{sgn}h\,|h|}=\lim_{h\to 0}\underbrace{\frac{1}{\operatorname{sgn}h}}_{\pm 1}\cdot\underbrace{|h+7|}_{\to 7}\qquad \nexists$$

$$f'_+(4)=\lim_{h\to 0+}\underbrace{\frac{1}{\operatorname{sgn}h}}_{=1}\cdot\underbrace{|h+7|}_{\to 7}=7$$

$$f'_-(4)=\lim_{h\to 0-}\underbrace{\frac{1}{\operatorname{sgn}h}}_{-1}\cdot|h+7|=-7$$

$\neq \;\Rightarrow\; \nexists\, f'(4)$

f nije dif. u 4



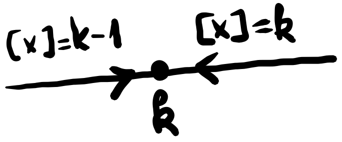
Zaključak:
f dif. na $(-\infty,4)\cup(4,+\infty)$

* за вежбу: диф. $f(x) = \sin x \cdot |\sin x|$ $\quad$ [$\sin x = 0$, $x = k\pi$]

7. $f(x) = [x] \cdot \sin \pi x$ непр.? диф.? на $\mathbb{R}$.

НЕПР: $[x]$ непр. на $\bigcup_{k\in\mathbb{Z}} (k,k+1)$, $\sin \pi x$ непр. на $\mathbb{R}$ $\Big\} \Rightarrow f$ непр. на $\bigcup_{k\in\mathbb{Z}} (k,k+1)$

$x = k \in \mathbb{Z}$: $\quad f(k) = \underbrace{[k]}_{k} \cdot \underbrace{\sin k\pi}_{0} = 0$

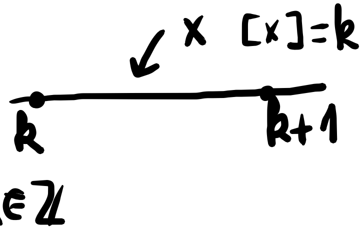


$$\lim_{x\to k-} \underbrace{[x]}_{k-1} \cdot \underbrace{\sin \pi x}_{\sin k\pi = 0} = (k-1)\cdot 0 = 0 = f(k) \checkmark \qquad \lim_{x\to k+} \underbrace{[x]}_{k} \cdot \underbrace{\sin \pi x}_{\sin k\pi = 0} = k \cdot 0 = 0 = f(k) \checkmark$$

f непр. у $k \in \mathbb{Z}$

$\Rightarrow$ **f непр. на $\mathbb{R}$**

ДИФ:



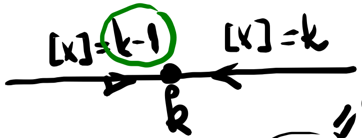
$k \in \mathbb{Z}$: $\quad x \in (k,k+1)$ $\quad f(x) = \underset{\text{const}}{k} \cdot \sin \pi x$

$f'(x) = k \cdot \cos \pi x \cdot \pi$

f диф. у x $\Big\}$ **f диф. на $\bigcup_{k\in\mathbb{Z}} (k,k+1)$**

$\mathbb{Z}$?

$$\boxed{x = k \in \mathbb{Z}}$$



$$\left\{ f(x) = [x]\cdot \sin \pi x \right\}$$

$$\left[\sin(\alpha + \pi k) = (-1)^k \cdot \sin\alpha \right]$$

$$f'_+(k) = \lim_{h\to 0+} \frac{f(k+h) - f(k)}{h} = \lim_{h\to 0+} \frac{[k+h]\cdot \sin\pi(k+h) - 0}{h}$$

($f(k) = 0$, $[k+h] = k$)

$$= \lim_{h\to 0+} \frac{k\cdot \sin(h\pi)\cdot(-1)^k}{h} = \lim_{h\to 0+} k\cdot(-1)^k \cdot \frac{\sin h\pi}{h\pi}\cdot \pi$$

($h\pi \to 0$, $\frac{\sin h\pi}{h\pi} \to 1$)

$$= \boxed{k\cdot(-1)^k\cdot \pi}$$

$$f'_-(k) = \lim_{h\to 0-} \frac{f(k+h) - f(k)}{h} = \lim_{h\to 0-} \frac{[k+h]\cdot \sin(\pi(k+h)) - 0}{h}$$

($f(k) = 0$, $[k+h] = k-1$)

$$= \lim_{h\to 0-} \frac{(k-1)\cdot(-1)^k\cdot \sin h\pi}{h\pi}\cdot \pi = \boxed{(k-1)\cdot(-1)^k\cdot \pi}$$

($h\pi \to 0$, $\frac{\sin h\pi}{h\pi} \to 1$)

$$f'_+(k) \neq f'_-(k) \Rightarrow \boxed{f \text{ није диф. у } k \in \mathbb{Z}}$$

[T] $f: (a,b) \to (\alpha,\beta)$ бијекција, диф у $x_0 \in (a,b)$, $f'(x_0) \neq 0$

$f^{-1}: (\alpha,\beta) \to (a,b)$ диф у $f(x_0) = y_0$

$$\Rightarrow (f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}$$

8. $g(y) = \operatorname{arctg} y$, $y \in \mathbb{R}$ $\quad g'(y) = ?$

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{\operatorname{tg} x} \mathbb{R}$, $\quad \mathbb{R} \xrightarrow{\operatorname{arctg} y} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f(x) = \operatorname{tg} x \to$ инверзна $g(y) = f^{-1}(y) = \operatorname{arctg} y$

$g'(y_0) = ?$ $\quad y_0 = \operatorname{tg} x_0$, $x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\quad f'(x_0) = \frac{1}{\cos^2 x_0}$

$$g'(y_0) \overset{T}{=} \frac{1}{f'(x_0)} = \frac{1}{1/\cos^2 x_0} = \cos^2 x_0 = \frac{1}{1+\operatorname{tg}^2 x_0} = \frac{1}{1+y_0^2}$$

($\cos^2 x_0$: изразити преко y_0)

[$\cos^2 x_0 = \ldots y_0 \ldots$

$y_0 = \operatorname{tg} x_0$

$\cos^2 x_0 = \frac{1}{1+\operatorname{tg}^2 x_0}$, $\quad x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$]

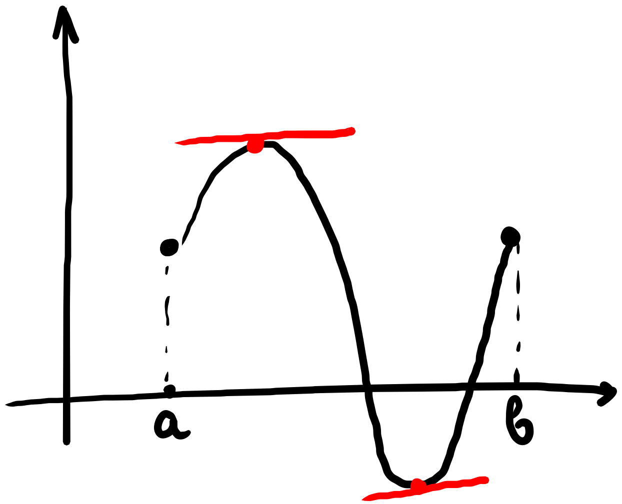
$$\Rightarrow (\operatorname{arctg})'(y) = \frac{1}{1+y^2}$$

за вежбу:

- $g(y) = \arccos y$, $y \in (-1,1)$ $\qquad g'(y) = ?$
- $g(y) = \arcsin y$, $y \in (-1,1)$ $\qquad g'(y) = ?$

~ Основне теореме диференцијалног рачуна ~

РОЛОВА ТЕОРЕМА



$f : [a,b] \to \mathbb{R}$ непрекидна

f диф. на (a,b)

$f(a) = f(b)$

$\Rightarrow \exists\, c \in (a,b) \quad f'(c) = 0$

1. $a_0, a_1, \dots, a_n \in \mathbb{R}$ тд. важи: $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_n}{n+1} = 0$

Док. да полином $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ има нулу у $\mathbb{R}$

? $\exists c \in \mathbb{R}$ $p(c) = 0$

$2 \nmid n \Rightarrow$ бар 1 нула у $\mathbb{R}$

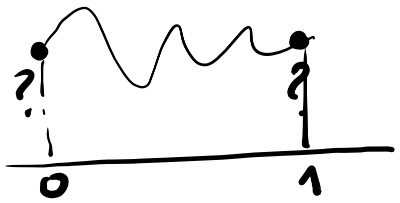
$2 \mid n$? $\cup$ x^2+5

☺ тражимо $f(x)$ тд. $f'(x) = p(x)$

$$a_k \cdot x^k = \left(a_k \cdot \frac{x^{k+1}}{k+1}\right)', \quad \forall k \in \mathbb{N}_0$$

$$\boxed{f(x)} = a_0 x + a_1 \cdot \frac{x^2}{2} + a_2 \cdot \frac{x^3}{3} + \dots + a_n \cdot \frac{x^{n+1}}{n+1}$$

$f'(x) = p(x)$ ✓



$f(0) = 0$

$f(1) = a_0 + \frac{a_1}{2} + \dots + \frac{a_n}{n+1} = 0$

f непр. на $[0,1]$, диф. на $(0,1)$

$f(0) = f(1)$ $\Longrightarrow$ из Роловe т. $\exists c \in (0,1)$ $f'(c) = 0$ тд. $\boxed{p(c) = 0}$ ✓ □