Асимптотске релације о и ~

Деф $f=o(g),\ x\to a \iff \exists$ околина U тачке a и $\exists$ беск. мала ф-ја α, $\mathring{U}=U\setminus\{a\}$

$\lim\limits_{x\to a}\alpha(x)=0$

тд. $f(x)=\alpha(x)\cdot g(x),\ \forall x\in\mathring{U}$

* α - беск мала кад $x\to a$: $\lim\limits_{x\to a}\alpha(x)=0$ $\quad \underbrace{\alpha(x)}_{\downarrow 0}=\alpha(x)\cdot 1 \to \boxed{\alpha=o(1),\ x\to a}$

* ако $g(x)\neq 0$ за $\forall x\in\mathring{U}$: $\boxed{f=o(g),\ x\to a \iff \lim\limits_{x\to a}\frac{f(x)}{g(x)}=0}$

Пр. x^n $\quad x^3, x^{15}$

$x\to 0$

$x^{15}=o(x^3),\ x\to 0 \Leftarrow \underbrace{x^{15}}_{f}=\underbrace{x^{12}}_{\substack{\alpha(x)\\ \downarrow x\to 0\\ 0}}\cdot\underbrace{x^3}_{g}$

II начин: $\lim\limits_{x\to 0}\frac{x^{15}}{x^3}=\lim\limits_{x\to 0}x^{12}=0$

$x\to+\infty$

$x^3=o(x^{15}),\ x\to+\infty$

$\lim\limits_{x\to+\infty}\frac{x^3}{x^{15}}=\lim\limits_{x\to+\infty}\frac{1}{x^{12}}=0$

(!)

$m>n$

$x^m=o(x^n),\ x\to 0$

$m>n$

$x^n=o(x^m),\ x\to+\infty$

$(x\to-\infty)$

* $f=o(g),\ x\to a$: $f \ll g,\ x\to a$ ускоро ☺ $\ln x \ll x^k \ll a^x,\ x\to+\infty$, $k>0$, $a>1$

Својства

1° $o(f)+o(f)=o(f),\ x\to a$

2° $o(o(f))=o(f),\ x\to a$

3° $f\cdot o(g)=o(f\cdot g),\ x\to a$

4° $o(f)\cdot o(g)=o(f\cdot g),\ x\to a$

5° c-const $\neq 0$: $o(cf)=o(f),\ x\to a$

Пр: 1° $f_1=o(f),\ x\to a \rightsquigarrow f_1(x)=\alpha_1(x)\cdot f(x),\ x\in \mathring{U}_1$, $\lim\limits_{x\to a}\alpha_1(x)=0$

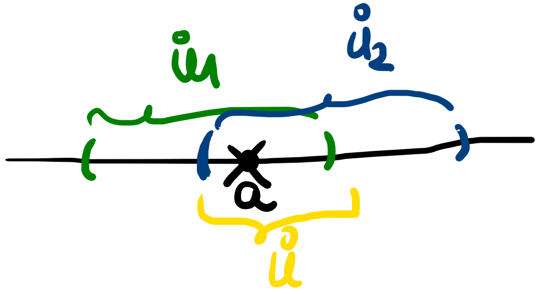
$f_2=o(f),\ x\to a \rightsquigarrow f_2(x)=\alpha_2(x)\cdot f(x),\ x\in \mathring{U}_2$, $\lim\limits_{x\to a}\alpha_2(x)=0$

? $f_1+f_2=o(f),\ x\to a$

$\Downarrow \oplus$

$$f_1(x)+f_2(x)=\underbrace{(\alpha_1(x)+\alpha_2(x))}_{\xrightarrow{x\to a}\,0} f(x),\quad x\in \underbrace{\mathring{U}_1\cap\mathring{U}_2}_{\mathring{U}}$$

$\Rightarrow f_1+f_2=o(f),\ x\to a$ ✓



* $o(x)-o(x) \cancel{=} 0$

* $\dfrac{o(x)}{x} = \dfrac{1}{x}\cdot o(x) = o\left(\dfrac{1}{x}\cdot x\right) = o(1) \to 0 \quad , x\to 0$

$\lim\limits_{x\to 0} o(1) = 0$

$\cancel{(1+o(1))^{\frac{1}{x}} = 1^{\frac{1}{x}} = 1}$, $x \to 0$

$\lim\limits_{x\to 0}(1+\underbrace{x}_{=o(1)})^{1/x} = e$

* $o(c\cdot x + o(x)) = o(x) \quad , x\to a$ (c-const) $\quad \left[\, o(c\cdot f + o(f)) = o(f) \,\right]$

$f_1(x) = o(c\cdot x + o(x)) \overset{?}{\Rightarrow} f_1(x) = o(x) \quad , x\to a$

$$\lim_{x\to a}\frac{f_1(x)}{x} = \lim_{x\to a}\frac{o(c\cdot x + o(x))}{x} = \lim_{x\to a}\underbrace{\frac{o(c\cdot x+o(x))}{c\cdot x + o(x)}}_{=o(1),\ x\to a\ \to 0}\ \underbrace{\frac{c\cdot x + o(x)}{x}}_{=c+\frac{o(x)}{x} = c + o(1) \to c} = 0\cdot c = 0$$

$\Rightarrow f_1 = o(x), x\to a$

Деф. $f \sim g, x \to a \iff \exists$ околност U на точка a и ф-я γ в тази околност

т.ч. $\lim\limits_{x \to a} \gamma(x) = 1$ и важи $\underline{\underline{f(x) = \gamma(x) \cdot g(x)}}$, $\forall x \in \mathring{U} = U \setminus \{a\}$

* ако $g(x) \neq 0, \forall x \in \mathring{U}$:

$$\boxed{f \sim g,\ x \to a \iff \lim_{x \to a} \frac{f(x)}{g(x)} = 1}$$

* $$\boxed{f \sim g,\ x \to a \iff f = g + o(g),\ x \to a}$$

$$f - g = o(g),\ x \to a$$

* $f \sim g, x \to a \Rightarrow o(f) = o(g), x \to a$

$f \sim g,\ x\to a \iff f = g + o(g),\ x \to a$

$\frac{\sin x}{x} \xrightarrow[x\to 0]{} 1$: $\sin x \sim x,\ x\to 0 \longrightarrow \sin x = x + o(x),\ x\to 0$

$\frac{\ln(1+x)}{x} \xrightarrow[x\to 0]{} 1$: $\ln(1+x) \sim x,\ x\to 0 \longrightarrow \ln(1+x) = x + o(x),\ x\to 0$

$\frac{(1+x)^\alpha - 1}{\alpha x} \xrightarrow[x\to 0]{} 1$: $(1+x)^\alpha - 1 \sim \alpha x,\ x\to 0 \longrightarrow (1+x)^\alpha - 1 = \alpha x + \underbrace{o(\alpha x)}_{o(x)},\ x\to 0$

$(1+x)^\alpha = 1 + \alpha \cdot x + o(x),\ x\to 0$

$\frac{e^x - 1}{x} \to 1,\ x\to 0$: $e^x - 1 \sim x,\ x\to 0 \longrightarrow e^x = 1 + x + o(x),\ x\to 0$

$\frac{1-\cos x}{x^2} \to \frac{1}{2},\ x\to 0$: $1-\cos x \sim \frac{1}{2}x^2,\ x\to 0$

$\cos x - 1 \sim -\frac{1}{2}x^2,\ x\to 0 \longrightarrow \cos x = 1 - \frac{1}{2}x^2 + o\left(-\frac{1}{2}x^2\right),\ x\to 0$

$\cos x = 1 - \frac{1}{2}x^2 + o(x^2),\ x\to 0$

1. $a, b, c > 0$
$a, b, c \neq 0$

Док. $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}} = \sqrt[3]{abc}$

$\left[M_x = \left(\dfrac{a^x+b^x+c^x}{3}\right)^{1/x} \right.$

M_2 - квадр. ср. M_1 - аритм.

$\left. \lim\limits_{x\to 0} M_x = \sqrt[3]{abc} =: M_0 \right]$

$\lim\limits_{x\to 0}\dfrac{a^x-1}{x} = \ln a \qquad a^x-1 \sim \ln a\cdot x \qquad a^x-1 = \ln a\cdot x + o(\underbrace{\ln a}_{\text{const}}\cdot x),\ x\to 0$

$$\boxed{a^x = 1+\ln a\cdot x + o(x),\ x\to 0}$$

$$\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}} = e^{\overbrace{\frac{1}{x}\cdot\ln\left(\frac{a^x+b^x+c^x}{3}\right)}^{f(x)}}$$

$$\boxed{f(x)} = \frac{1}{x}\cdot\ln\left(\frac{a^x+b^x+c^x}{3}\right) = \frac{1}{x}\cdot\ln\left(\frac{1}{3}\left(1+\ln a\cdot x+o(x)+1+\ln b\cdot x+o(x)+1+\ln c\cdot x+o(x)\right)\right) =$$

$$= \frac{1}{x}\cdot\ln\left(\frac{1}{3}\left(3+x\cdot(\underbrace{\ln a+\ln b+\ln c}_{\ln(abc)})+o(x)\right)\right) = \frac{1}{x}\ln\left(1+\frac{\ln(abc)}{3}\cdot x+\underbrace{\frac{o(x)}{3}}_{=o(x)}\right)$$

$\left[o(x)+o(x)+o(x) = o(x) \right]$, $x\to 0$

$$= \frac{1}{x}\cdot\ln\left(1+\underbrace{\ln\sqrt[3]{abc}\cdot x+o(x)}_{t\to 0,\ x\to 0}\right) = \frac{1}{x}\cdot\left(\overbrace{\ln\sqrt[3]{abc}\cdot x+o(x)}^{t}+\underbrace{\overbrace{o\left(\underbrace{\ln\sqrt[3]{abc}}_{\text{const}}\cdot x+o(x)\right)}^{o(t)}}_{\text{тобрз.}\ =o(x)}\right)$$

$\left[\ln(1+t) = t+o(t),\ t\to 0 \right]$

$$= \frac{1}{x}\cdot\left(\ln\sqrt[3]{abc}\cdot x+\underbrace{o(x)+o(x)}_{o(x)}\right) = \ln\sqrt[3]{abc}+\frac{o(x)}{x} = \boxed{\ln\sqrt[3]{abc}+o(1),\ x\to 0}$$

$$\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x} = \lim_{x\to 0} e^{f(x)} = \lim_{x\to 0} e^{\ln\sqrt[3]{abc}+\cancel{o(1)}\to 0} = e^{\ln\sqrt[3]{abc}} = \sqrt[3]{abc}$$ ✓ ◻

② $L = \lim\limits_{x\to 0} \dfrac{1-(\cos x)^{\sin x}}{x^3} = \ ?$

$$(\cos x)^{\sin x} = e^{\sin x \cdot \ln(\cos x)}$$

$$\underbrace{\sin x \cdot \ln(\cos x)}_{x\to 0} = (x+o(x))\cdot \ln\Big(1\underbrace{-\tfrac12 x^2+o(x^2)}_{t\to 0,\ x\to 0}\Big) \overset{\ln(1+t)}{=} (x+o(x))\cdot\Big(\overbrace{-\tfrac12 x^2+o(x^2)}^{t} + \overbrace{\underbrace{o\big(-\tfrac12 x^2+o(x^2)\big)}_{=o(x^2)}}^{o(t)}\Big)$$

$$\oplus = o(x^2)$$

$$= (x+o(x))\cdot\left(-\tfrac12 x^2+o(x^2)\right)$$

$$= -\tfrac12 x^3 + \underbrace{x\cdot o(x^2)}_{o(x^3)} + \underbrace{o(x)\cdot\left(-\tfrac12 x^2\right)}_{o(-\frac12 x^3) = o(x^3)} + \underbrace{o(x)\cdot o(x^2)}_{o(x^3)}$$

$$= -\tfrac12 x^3 + o(x^3)$$

$f\cdot o(g) = o(f\cdot g)$
$o(f)\cdot o(g) = o(f\cdot g)$

$$\underbrace{\boxed{(\cos x)^{\sin x}}}_{x\to 0} = e^{-\frac12 x^3+o(x^3)} \overset{e^t,\ t\to 0}{=} 1 + \overbrace{-\tfrac12 x^3+o(x^3)}^{t} + \underbrace{\overbrace{o\big(-\tfrac12 x^3+o(x^3)\big)}^{o(t)}}_{=o(x^3)} = 1-\tfrac12 x^3+o(x^3),\ x\to 0$$

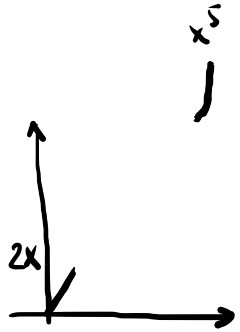
$$L = \lim_{x\to 0}\frac{\cancel{1}-\left(\cancel{1}-\frac12 x^3+o(x^3)\right)}{x^3} = \lim_{x\to 0}\left(\frac12+\frac{o(x^3)}{x^3}\right) = \lim_{x\to 0}\Big(\frac12+\underbrace{o(1)}_{\to 0}\Big) = \boxed{\frac12}$$

∎

③ Одредити константу c и $n\in\mathbb{N}$ тд. $f(x)\sim c\cdot x^n,\ x\to 0$

а) $f(x)=2x-3x^2+x^5$

б) $f(x)=\sqrt{1+x}-\sqrt{1-x}$



а) $f(x)=2x-3x^2+x^5$
$$=2x\cdot\underbrace{\left(1-\frac{3}{2}x+\frac{x^4}{2}\right)}_{=\gamma(x)\to 1,\ x\to 0}$$

$$\boxed{f(x)\sim 2x,\ x\to 0}$$

у $+\infty$ другачије! :)

$$f(x)=2x-3x^2+x^5=x^5\cdot\Big(\underbrace{\underbrace{\frac{2}{x^4}-\frac{3}{x^3}}_{\to 0,\ x\to+\infty}}_{\theta(x)\to 0,\ x\to+\infty}+1\Big)$$

$$\boxed{f(x)\sim x^5,\ x\to+\infty}$$

б) $$f(x)=\sqrt{1+x}-\sqrt{1-x}=(1+x)^{1/2}-(1-x)^{1/2}=\cancel{1}+\frac{1}{2}x+o(x)-\Big(\cancel{1}+\frac{1}{2}(-x)+\underbrace{o(-x)}_{=o(x)}\Big)$$

$[(1+x)^\alpha=1+\alpha x+o(x),\ x\to 0]$

$$=x+o(x)-o(x)=\boxed{x+o(x)}=x\cdot\Big(\underbrace{1+\frac{o(x)}{x}}_{\to 1,\ x\to 0}\Big)$$

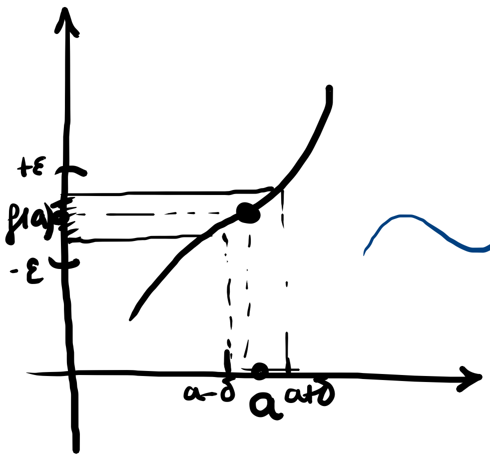
$$\boxed{f(x)\sim x,\ x\to 0}$$

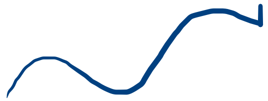
Непрекидност

$f: A \to \mathbb{R} \quad a \in A$

f је непр. у $a \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)\ |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

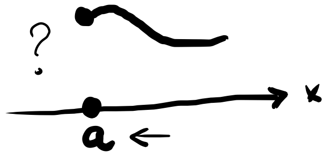
$\iff \lim\limits_{x \to a} f(x) = f(a)$



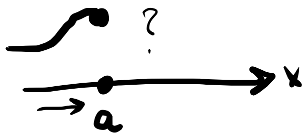


f непрекидна на скупу $A \iff (\forall a \in A)$ f непр. у тачки a

f непр. здесна у $a \iff \lim\limits_{x \to a+} f(x) = f(a)$

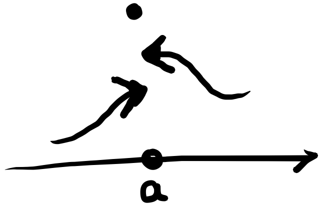


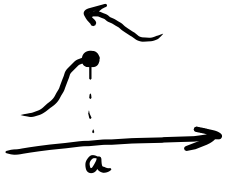
f непр. слева у $a \iff \lim\limits_{x \to a-} f(x) = f(a)$



f непр. у $a \iff \lim\limits_{x \to a-} f(x) = f(a) = \lim\limits_{x \to a+} f(x)$

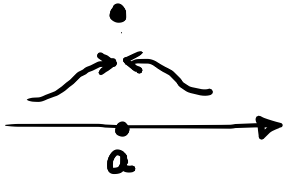
АКО f није непр. у a → f има ПРЕКИД у a





1) ПРЕКИД I ВРСТЕ : $\exists \lim\limits_{x\to a-} f(x)$ и $\exists \lim\limits_{x\to a+} f(x)$

специјално, ако $\lim\limits_{x\to a-} f(x) = \lim\limits_{x\to a+} f(x) \quad (\neq f(a))$



ОТКЛОЊИВ ПРЕКИД

2) иначе → ПРЕКИД II ВРСТЕ

=

збир, разлика, производ, количник $\frac{f}{g}$, $g\neq 0$ непр. ф-ја → непр. функција

елементарне ф-је → непрекидне на својим доменима

$\sin x, e^x, \cos x, a^x, x^k, \dots$

① Испитати непрекидност:

а) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

б) $g(x) = \begin{cases} \dfrac{\sin x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

а) $x \neq 0$: $\sin x$, x – непрекидне

$\Rightarrow \dfrac{\sin x}{x}$ је непрекидна $\forall x$ ✓ (1)

$x = 0$: $f(0) = 1$

$$\lim_{x\to 0} f(x) \overset{?}{=} f(0)$$

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{?}{=} 1 \quad ✓$$

$\Rightarrow f$ је непр. у тачки 0 (2)

(1),(2) $\Rightarrow$ $\boxed{f \text{ је непр. на скупу } \mathbb{R}}$

б) $x \neq 0$: непр. као количник непрекидних ✓

$x = 0$: $g(0) = 1$

$$\cancel{\lim_{x\to 0} \frac{\sin x}{|x|}} \overset{?}{=} 1$$

НЕ ПОСТОЈИ!

$$\lim_{x\to 0+} \frac{\sin x}{\underbrace{|x|}_{\overset{>0}{=}x}} = \lim_{x\to 0+} \frac{\sin x}{x} = 1 = g(0)$$

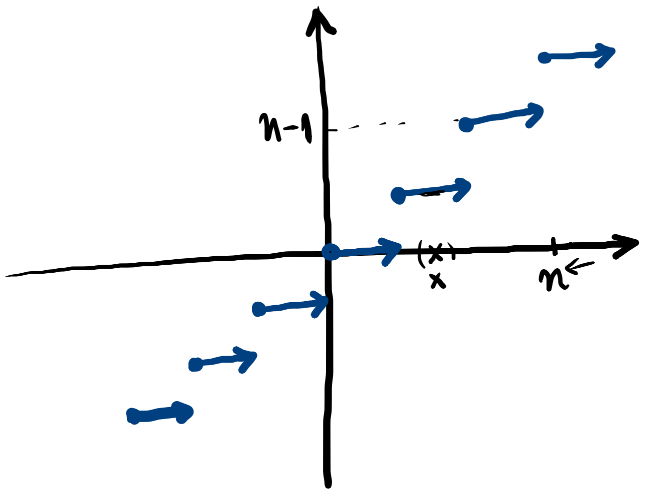
непр. здесна

$$\lim_{x\to 0-} \frac{\sin x}{\underbrace{|x|}_{\overset{<0}{=}-x}} = \lim_{x\to 0-} \frac{\sin x}{-x} = -1 \neq g(0)$$

није непр. слева!

$\Rightarrow$ $\underline{g \text{ није непр. у } 0}$

$\boxed{f(x)=[x]}$



$x\notin\mathbb{Z}$ $\quad x\in(n,n+1)$

$[x]=n$ ↗ локално константна

$\boxed{f \text{ је непр. у } x\notin\mathbb{Z}}$

$x\in\mathbb{Z}$ $\quad x=n\in\mathbb{Z}$

$\lim\limits_{x\to n+}\underset{n}{[x]} = n = [n]$ f непр. здесна у n

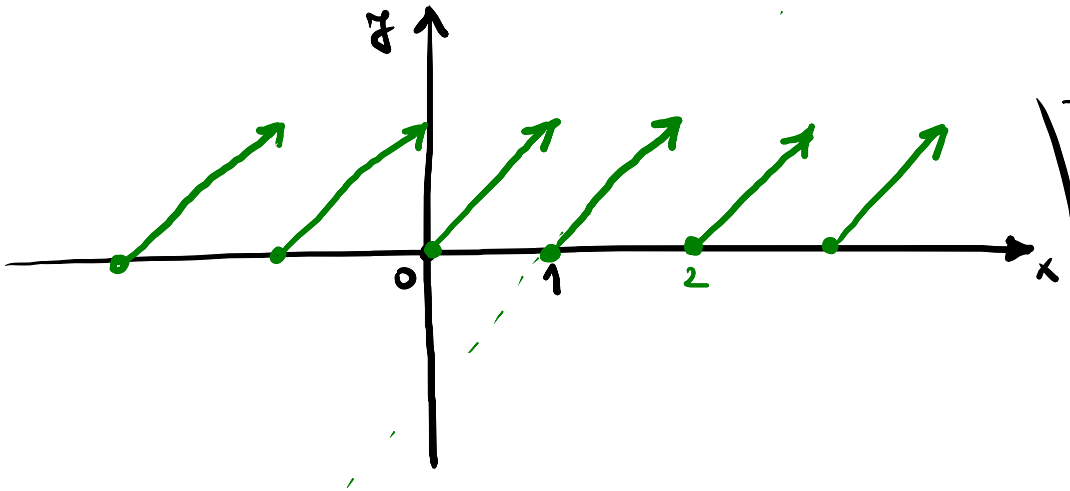
$\lim\limits_{x\to n-}\underbrace{[x]}_{=n-1} = n-1 \neq [n]$ f није непр. слева у n

$\rightsquigarrow$ $\boxed{f \text{ није непр. у } n\in\mathbb{Z}.}$

$\boxed{g(x)=x-[x]=\{x\}}$ – разломљени део x

$0\leq x-[x]<1$

$$g(x)=\begin{cases} \vdots \\ x+1, & x\in[-1,0) \\ x, & x\in[0,1) \\ x-1, & x\in[1,2) \\ x-2, & x\in[2,3) \\ \vdots \end{cases}$$



$\boxed{g \text{ непр у } x \Leftrightarrow x\notin\mathbb{Z}}$

② Испитати непрекидност: $f(x) = [x]\cdot\left([x] - (-1)^{[x]}\cdot \cos(\pi\cdot x)\right)$

$x \notin \mathbb{Z}$: $x \in (n, n+1)$, $n \in \mathbb{Z}$ → за $x \in (n, n+1)$ $\quad [x] = n$

$\underbrace{n \quad [x]=n \quad n+1}$

$f(x) = n\cdot\left(n - (-1)^n \cdot \cos(\pi\cdot x)\right) \Rightarrow$ f непр у x

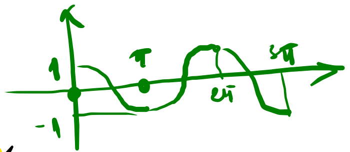
константе — само ово зависи од x, непр. ф-ја

$x = n \in \mathbb{Z}$: $\quad \lim\limits_{x\to n-} f(x) \overset{?}{=} f(n) \overset{?}{=} \lim\limits_{x\to n+} f(x)$

$\boxed{f(n)} = \underbrace{[n]}_{n}\cdot\left([n] - (-1)^{[n]}\cos(\pi\cdot n)\right) = n\left(n - \underbrace{(-1)^n\cdot(-1)^n}_{=1}\right) = \boxed{n\cdot(n-1)}$

☺ $n \in \mathbb{Z}$

$\cos n\pi = (-1)^n$



$\boxed{\lim\limits_{x\to n+} f(x)} = \lim\limits_{x\to n+} \underbrace{[x]}_{n}\cdot\left(\underbrace{[x]}_{n} - (-1)^{\overset{n}{[x]}}\underbrace{\cos(x\cdot\pi)}_{\to \cos n\pi = (-1)^n}\right) = n\cdot(n-(-1)^n(-1)^n) = n(n-1) = \boxed{f(n)}$

$\boxed{\lim\limits_{x\to n-} f(x)} = \lim\limits_{x\to n-} \underbrace{[x]}_{n-1}\cdot\left(\underbrace{[x]}_{n-1} - (-1)^{\overset{n-1}{[x]}}\underbrace{\cos(x\pi)}_{\to \cos n\pi = (-1)^n}\right) = (n-1)\cdot\underbrace{\left((n-1) - \underbrace{(-1)^{n-1}\cdot(-1)^n}_{=-1}\right)}_{n} = (n-1)\cdot n = \boxed{f(n)}$

$\Rightarrow$ f непр у $n \in \mathbb{Z}$ и

Закључак: f непр на целом скупу $\mathbb{R}$ ✓ :)