Асимптотске ознаке o и $\sim$

Деф $\boxed{o}$ $f=o(g),\ x\to a \iff \exists$ околина U тачке a и беск. мала ф-ја α $\left(\lim\limits_{x\to a}\alpha(x)=0\right)$ у $\mathring{U}=U\setminus\{a\}$

т.д. $f(x)=\underline{\underline{\alpha(x)}}\cdot g(x),\ \forall x\in\mathring{U}$

Став ако $g(x)\neq 0,\ \forall x\in\mathring{U}$: $\boxed{f=o(g),\ x\to a \iff \lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=0}$

- x^n $\quad x^{15}=o(x^3),\ x\to 0$ јер $\lim\limits_{x\to 0}\dfrac{x^{15}}{x^3}=\lim\limits_{x\to 0}x^{12}=0$

  према деф.: $x^{15}=x^{12}\cdot x^3$, $\quad x^{12}=\alpha(x)\to 0,\ x\to 0$

  $m>n$: $\quad x^m=o(x^n),\ x\to 0$

  за $+\infty$: $x^3=o(x^{15}),\ x\to+\infty$: $\lim\limits_{x\to+\infty}\dfrac{x^3}{x^{15}}=\lim\limits_{x\to+\infty}\dfrac{1}{x^{12}}=0$

  $m>n$: $\quad x^n=o(x^m),\ x\to+\infty$ $\quad(x\to-\infty)$

- $\ln x=o(x^k)$, $k>0$, $\quad x^k=o(a^x)$, $a>1$, $\quad x\to+\infty$

Ознака: $f=o(g),\ x\to a \ \to\ f\ll g,\ x\to a$

свойства:

1) $o(f)+o(f)=o(f),\ x\to a$

2) $o(o(f))=o(f),\ x\to a$

3) $f\cdot o(g)=o(f\cdot g),\ x\to a$

4) $o(f)\cdot o(g)=o(f\cdot g),\ x\to a$

5) $c=const\neq 0:\ o(c\cdot f)=o(f),\ x\to a$

1) $f_1=o(f),\ x\to a \rightsquigarrow f_1(x)=\alpha_1(x)\cdot f(x),\ x\in \mathring{U}_1,\ \lim\limits_{x\to a}\alpha_1(x)=0$

$f_2=o(f),\ x\to a \rightsquigarrow f_2(x)=\alpha_2(x)\cdot f(x),\ x\in \mathring{U}_2,\ \lim\limits_{x\to a}\alpha_2(x)=0$

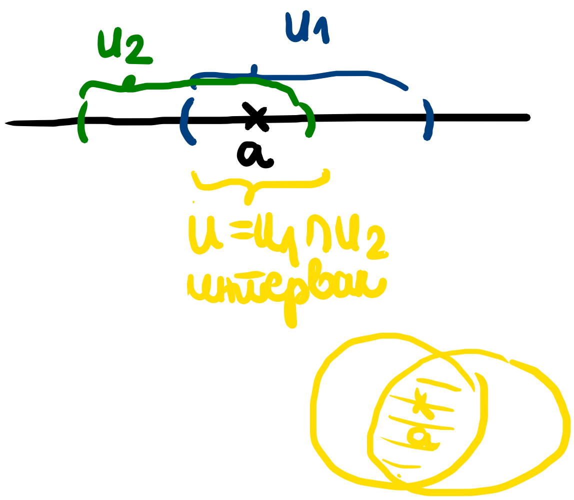
$\oplus$

? $f_1+f_2=o(f),\ x\to a$

$$\downarrow$$

$$f_1(x)+f_2(x)=\underbrace{(\alpha_1(x)+\alpha_2(x))}_{\xrightarrow{x\to a}\,0}\cdot f(x),\quad x\in\underbrace{\mathring{U}_1\cap\mathring{U}_2}_{\mathring{U}}$$

$\Rightarrow f_1+f_2=o(f),\ x\to a$



① $o(c\cdot x+o(x)) = o(x)$, $x\to 0$ (c-конст. $c\neq 0$)

$$\lim_{x\to 0}\frac{o(c\cdot x+o(x))}{x} \overset{?}{=} 0 \qquad \lim_{x\to 0}\frac{o(c\cdot x+o(x))}{x} = \lim_{x\to 0}\overbrace{\frac{o(c\cdot x+o(x))}{c\cdot x+o(x)}}^{\to 0}\cdot\underbrace{\frac{c\cdot x+o(x)}{x}}_{\substack{\| \\ c+\frac{o(x)}{x}\to c}} = 0\cdot c = \boxed{0}$$

✓

$$* \quad \frac{o(f)}{f} = o\left(\frac{1}{f}\cdot f\right) = o(1)\to 0$$

$f_1 = o(f)$ $\quad f_1(x) = \alpha(x)\cdot f(x)$ $\quad \dfrac{o(f)}{f} \leftarrow \dfrac{f_1}{f} = \dfrac{\alpha\cdot f}{f} = \alpha \to 0$ $\Bigg\}$ у доказателствата полезна

Деф. $f\sim g,\ x\to a \iff \exists$ окол. $\mathring{U}$ на a и ф-я γ на $\mathring{U}$ т.ч. $\lim\limits_{x\to a}\gamma(x)=1$ и $f(x)=\gamma(x)\cdot g(x),\ \forall x\in\mathring{U}$

ако $g(x)\neq 0,\ x\in\mathring{U}$: $\quad \boxed{f\sim g,\ x\to a \iff \lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = 1}$

* $f \sim g,\ x \to a \iff f = g + o(g),\ x \to a$ $\quad (f - g = o(g),\ x \to a)$

* $f \sim g,\ x \to a \Rightarrow o(f) = o(g),\ x \to a$
  ($g \neq 0$ на $\mathring{U}$)

$\dfrac{\sin x}{x} \to 1,\ x \to 0$ $\qquad \sin x \sim x,\ x \to 0$ $\longrightarrow$ $\boxed{\sin x = x + o(x),\ x \to 0}$

$\dfrac{\ln(1+x)}{x} \to 1,\ x \to 0$ $\qquad \ln(1+x) \sim x,\ x \to 0$ $\longrightarrow$ $\boxed{\ln(1+x) = x + o(x),\ x \to 0}$

$\dfrac{(1+x)^\alpha - 1}{\alpha x} \to 1,\ x \to 0$ $\qquad (1+x)^\alpha - 1 \sim \alpha x,\ x \to 0$

$(1+x)^\alpha - 1 = \alpha x + \underbrace{o(\alpha x)}_{o(x)},\ x \to 0$ $\longrightarrow$ $\boxed{(1+x)^\alpha = 1 + \alpha x + o(x),\ x \to 0}$

$\dfrac{e^x - 1}{x} \to 1,\ x \to 0$ $\qquad e^x - 1 \sim x,\ x \to 0$ $\longrightarrow$ $\boxed{e^x = 1 + x + o(x),\ x \to 0}$

$\dfrac{1 - \cos x}{\frac{1}{2}x^2} \to 1,\ x \to 0$ $\qquad 1 - \cos x \sim \frac{1}{2}x^2,\ x \to 0$ $\longrightarrow$ $\boxed{\cos x = 1 - \frac{1}{2}x^2 + o(x^2),\ x \to 0}$

$\cos x - 1 \sim -\frac{1}{2}x^2,\ x \to 0$

.....

① $a,b,c>0$, $a,b,c\neq 1$. Док: $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{1/x}=\sqrt[3]{abc}$

$$\frac{a^x-1}{x}\to \ln a,\ x\to 0 \qquad \frac{a^x-1}{\ln a\cdot x}\to 1,\ x\to 0 \qquad a^x-1=\ln a\cdot x+\underbrace{o(\ln a\cdot x)}_{o(x)},\ x\to 0 \qquad \boxed{a^x=1+\ln a\cdot x+o(x),\ x\to 0}$$

$$\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}=e^{\underbrace{\frac{1}{x}\cdot\ln\left(\frac{a^x+b^x+c^x}{3}\right)}_{f(x)}}$$

$$\boxed{f(x)}=\frac{1}{x}\cdot\ln\left(\frac{a^x+b^x+c^x}{3}\right)=\frac{1}{x}\cdot\ln\left(\frac{1}{3}\left(1+\ln a\cdot x+o(x)+1+\ln b\cdot x+o(x)+1+\ln c\cdot x+o(x)\right)\right)$$

$$=\frac{1}{x}\cdot\ln\left(\frac{1}{3}\left(3+\underbrace{(\ln a+\ln b+\ln c)}_{\ln abc}\cdot x+o(x)\right)\right)$$

$\ulcorner o(x)+o(x)+o(x)=o(x)$, $x\to 0\lrcorner$

$$=\frac{1}{x}\cdot\ln\left(1+\frac{1}{3}\ln(abc)\cdot x+o(x)\right)=\frac{1}{x}\cdot\ln\left(1+\underbrace{\ln\sqrt[3]{abc}\cdot x+o(x)}_{t\to 0,\ x\to 0}\right)$$

$\ulcorner \frac{1}{n}\cdot o(x)=o(x)\lrcorner$

$\ulcorner \ln(1+t)=t+o(t)$, $t\to 0\lrcorner$

$$=\frac{1}{x}\cdot\left(\overbrace{\ln\sqrt[3]{abc}\cdot x+o(x)}^{t}+\underbrace{\overbrace{o\left(\ln\sqrt[3]{abc}\cdot x+o(x)\right)}^{o(t)}}_{=o(x)}\right)$$

$$=\frac{1}{x}\cdot\left(\ln\sqrt[3]{abc}\cdot x+\underbrace{o(x)+o(x)}_{o(x)}\right)=\ln\sqrt[3]{abc}+\frac{o(x)}{x}=\boxed{\ln\sqrt[3]{abc}+o(1)} \qquad \boxed{x\to 0}$$

$$\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}=\lim_{x\to 0} e^{\underbrace{\frac{1}{x}\cdot\ln(\dots)}_{f(x)}}=\lim_{x\to 0} e^{\ln\sqrt[3]{abc}+\cancel{o(1)}\to 0}=e^{\ln\sqrt[3]{abc}}=\boxed{\sqrt[3]{abc}}\quad\blacksquare$$

$$2)\quad L=\lim_{x\to 0}\frac{1-(\cos x)^{\sin x}}{x^3}=?$$

$$(\cos x)^{\sin x}=e^{\underline{\sin x\cdot\ln(\cos x)}}$$

$$\boxed{\sin x\cdot\ln(\cos x)}_{x\to 0}=(x+o(x))\cdot\ln\Big(1\overbrace{-\tfrac{1}{2}x^2+o(x^2)}^{t\to 0}\Big)\overset{\ln(1+t)}{=\!=}(x+o(x))\cdot\Big(\overbrace{-\tfrac{1}{2}x^2+o(x^2)}^{t}+\overbrace{\underbrace{o\big(-\tfrac{1}{2}x^2+o(x^2)\big)}_{=o(x^2)}}^{o(t)}\Big)$$

$$+\ o(x^2)$$

$$=(x+o(x))\cdot\left(-\tfrac{1}{2}x^2+o(x^2)\right)$$

$$=-\tfrac{1}{2}x^3+\underbrace{x\cdot o(x^2)}_{\substack{o(x^3)\\ f\cdot o(g)=o(f\cdot g)}}+\underbrace{o(x)\cdot -\tfrac{1}{2}x^2}_{\substack{=o(-\frac{1}{2}x^3)\\ =o(x^3)}}+\underbrace{o(x)\cdot o(x^2)}_{\substack{o(x^3)\\ o(f)\cdot o(g)=o(f\cdot g)}}$$

$$=-\tfrac{1}{2}x^3+o(x^3)+o(x^3)+o(x^3)=\boxed{-\tfrac{1}{2}x^3+o(x^3)}$$

$$\boxed{e^{\sin x\ln(\cos x)}}_{x\to 0}=e^{\underbrace{-\frac{1}{2}x^3+o(x^3)}_{t\to 0}}\overset{e^t=1+t+o(t)}{=\!=}1+-\tfrac{1}{2}x^3+o(x^3)+\underbrace{o\left(\tfrac{1}{2}x^3+o(x^3)\right)}_{o(x^3)}=\boxed{1-\tfrac{1}{2}x^3+o(x^3)}$$

$$L = \lim_{x\to 0} \frac{1-\left(1-\frac{1}{2}x^3+o(x^3)\right)}{x^3} = \lim_{x\to 0} \frac{1}{2} - \underbrace{\frac{o(x^3)}{x^3}}_{\substack{=o(1)\\ \to 0}} = \boxed{\frac{1}{2}}$$

$$\underbrace{o(x)}_{f_1(x)} - \underbrace{o(x)}_{f_2(x)} \cancel{=} 0$$

$$o(x) - o(x) = o(x), \quad x \to 0$$

* ~ $o(1)$ ~~ , $x \to a$

овде стоји нека ~~непозната~~ ф-ја $f(x) = o(1)$ $\quad f(x) = \overline{f(x)} \cdot 1$

о којој једино што знамо јесте да $\lim_{x\to a} f(x) = 0$

* $\lim_{x\to 0} (1+o(1))^{\frac{1}{x}} \cancel{=} \lim_{x\to 0} 1^{\frac{1}{x}} = 1$

$$\lim_{x\to 0} (1+\underset{\substack{\shortparallel\\ o(1),\, x\to 0}}{x})^{\frac{1}{x}} = e$$

$$\lim_{x\to 0} \left(\alpha + \overset{0}{o(1)} \cdot \overset{0}{o(1)} + \overset{0}{o(1)}\right) = \alpha \ ✓$$

③ $c = const = ?$ т.ч. $f(x) \sim c \cdot x^n,\ x \to 0$

$n \in \mathbb{N}$?

a) $f(x) = 2x - 3x^2 + x^5$

б) $f(x) = \sqrt{1+x} - \sqrt{1-x}$

a) $f(x) = 2x - 3x^2 + x^5 \overset{?}{\sim} c \cdot x^n,\ x \to 0$

$$f(x) = 2x \cdot \underbrace{\left(1 - \frac{3}{2}x + \frac{x^4}{2}\right)}_{\gamma(x) \to 1,\ x \to 0}$$

$$f(x) = 2x \cdot \underbrace{\gamma(x)}_{\to 1,\ x \to 0}$$

$$\boxed{f(x) \sim 2x,\ x \to 0}$$

$c = 2,\ n = 1$

у $+\infty$:

$$2x - 3x^2 + x^5 = x^5 \cdot \underbrace{\left(1 - \frac{3}{x^3} + \frac{2}{x^4}\right)}_{\to 1,\ x \to +\infty},\ x \to +\infty$$

$$\boxed{f(x) \sim x^5,\ x \to +\infty}$$

δ) $f(x) = \sqrt{1+x} - \sqrt{1-x}$

$$(1+x)^{\alpha} = 1+\alpha x + o(x), \quad x \to 0$$

$$= (1+x)^{\frac{1}{2}} - (\underbrace{1-x}_{\to 0})^{1/2}$$

$$= \cancel{1} + \underbrace{\frac{1}{2}\cdot x}_{} + o(x) - \left(\cancel{1} + \underbrace{\frac{1}{2}\cdot(-x)}_{} + \underbrace{o(-x)}_{= o(x)}\right)$$

$x \to 0$

$$= x + \underbrace{o(x) - o(x)}_{o(x)}$$

$$= x + o(x) \qquad \left(= \underline{x} \cdot \underbrace{\left(1 + \frac{o(x)}{x}^{\to 0}\right)}_{\to 1}\right)$$

$$\Rightarrow \boxed{f(x) \sim x,\ x \to 0}$$

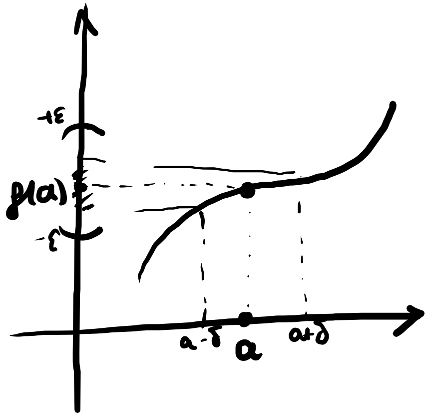
$f \sim g,\ x \to a \iff f = g + o(g),\ x \to a$

Непрекидност

$f: A \to \mathbb{R}$ $a \in A$

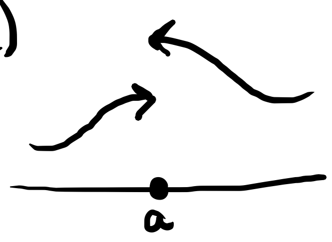
f непр. у $a \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)\ |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

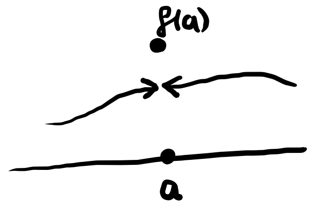
$\iff \lim\limits_{x \to a} f(x) = f(a)$



f није непр. у a → f има прекид у a (a тачка прекида)

1) прекид I врсте : ако $\exists \lim\limits_{x \to a-} f(x)$ и $\exists \lim\limits_{x \to a+} f(x)$



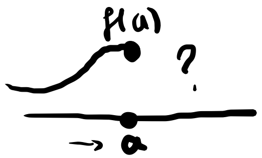


ако додатно: $\lim\limits_{x \to a-} f(x) = \lim\limits_{x \to a+} f(x)$ $(\neq f(a))$

отклоњив прекид

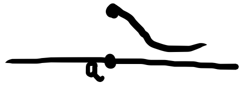
2) иначе, прекид II врсте

f непр. слева у $a \iff \lim\limits_{x \to a-} f(x) = f(a)$



f непр. здесна у a

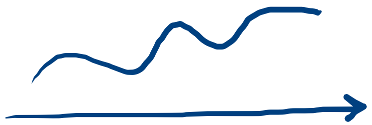
$\iff \lim\limits_{x \to a+} f(x) = f(a)$



f непр. у $a \iff$

$$\boxed{\lim_{x \to a-} f(x) = f(a) = \lim_{x \to a+} f(x)}$$

f непр. на скупу $A \iff f$ непр. у a за $\forall a \in A$



① Испитати непрекидност:

а) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$x \neq 0$: $f(x) = \dfrac{\sin x}{x}$ — $\sin x$ непр. ф-ја, x непр. ф-ја

f непр. у x као количник две непр. ф-је (1)

$x = 0$: $f(0) = 1$

$\lim\limits_{x \to 0} f(x) \overset{?}{=} f(0)$

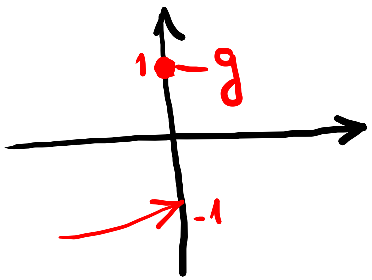
$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ ✓

$\Rightarrow f$ непр. у 0 (2)

(1),(2) $\Rightarrow$ $\boxed{f \text{ је непр. на } \mathbb{R}}$

б) $g(x) = \begin{cases} \dfrac{\sin x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$x \neq 0$: $g(x) = \dfrac{\sin x}{|x|}$ — непр. ф. $\Rightarrow g$ непр.



$x = 0$: $g(0) = 1$

$\lim\limits_{x \to 0} g(x) \overset{?}{=} g(0)$

$\cancel{\lim\limits_{x \to 0} \dfrac{\sin x}{|x|}}$ не постоји

$\lim\limits_{x \to 0+} \dfrac{\sin x}{\underbrace{|x|}_{x}} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1 = g(0)$

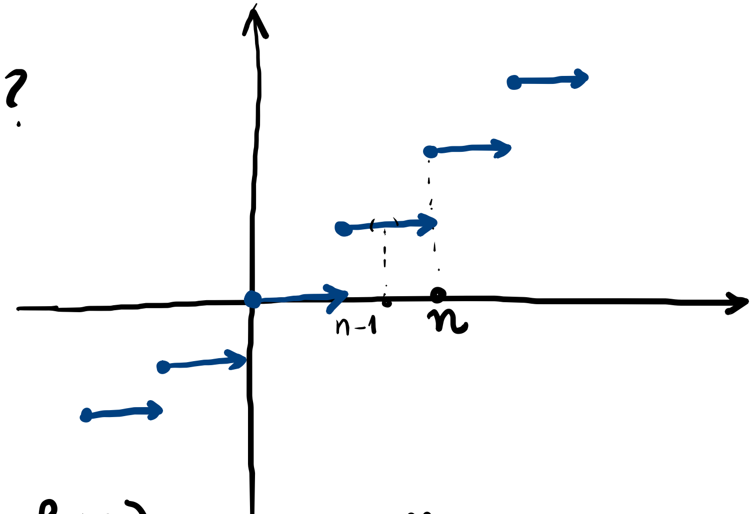
непр. здесна у 0

$1 \neq -1$

$\lim\limits_{x \to 0-} \dfrac{\sin x}{\underbrace{|x|}_{-x}} = \lim\limits_{x \to 0-} -\underbrace{\boxed{\dfrac{\sin x}{x}}}_{\to 1} = -1 \neq g(0)$

није непр. слева $\Rightarrow$ $\boxed{g \text{ није непр. у } 0}$

$\boxed{f(x)=[x]}$ непрекидност?



$x\notin\mathbb{Z} \to f$ непр. у x

$x\in\mathbb{Z}$: $x=n\in\mathbb{Z}$

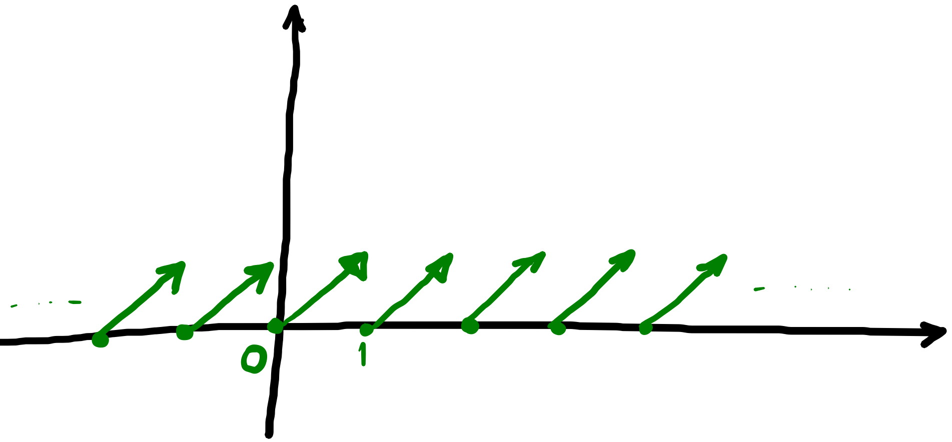
$$\left.\begin{aligned} &\lim_{x\to n-}[\underset{\hat{n}}{x}] = n-1 \\ &\lim_{x\to n+}[\underset{\check{n}}{x}] = n = [n] \end{aligned}\right\}$$ f непр. здесна у n, није непр. слева у n

$\boxed{g(x)=x-[x]=\{x\}}$

разломљени део од x

$$x-[x]=\begin{cases} \vdots \\ x+1, & x\in[-1,0) \\ x, & x\in[0,1) \\ x-1, & x\in[1,2) \\ x-2, & x\in[2,3) \\ \vdots \\ x-n, & x\in[n,n+1) \\ \vdots \end{cases}$$

$\{x\}$ периодична

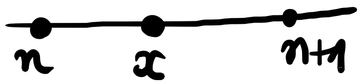


$\{x\}$ непр. за $x\notin\mathbb{Z}$

а за $x\in\mathbb{Z}$ непр. здесна, није слева $\leftarrow$ није непрекидна!

② $f(x) = [x]\cdot\left([x] - (-1)^{[x]}\cdot\cos\pi x\right)$ непрекидност ?

$x \notin \mathbb{Z}$



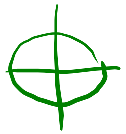
$x \in (n, n+1),\ n\in\mathbb{Z}$

за такве x: $[x]=n$ $\longrightarrow$ $f(x) = n\cdot\left(n - (-1)^n\cdot\underbrace{\cos\pi x}_{\text{непр.}}\right)$ $\leftarrow$ непр. ✓

$\boxed{f \text{ непр. на } (n,n+1) \text{ за } \forall n\in\mathbb{Z}}$

$x\in\mathbb{Z}$: $x = n \in \mathbb{Z}$

$$f(n) = \underbrace{[n]}_{n}\cdot\left([n] - (-1)^{[n]}\cdot\cos\pi n\right) = n\left(n - \underbrace{(-1)^n\cdot(-1)^n}_{=1}\right) = \boxed{n(n-1)}$$



☺ $\cos\pi n = (-1)^n$, $n\in\mathbb{Z}$

$$\lim_{x\to n+} f(x) = \lim_{x\to n+} \underbrace{[x]}_{\to n}\cdot\left([x]-(-1)^{[x]}\cdot\cos\pi x\right) = n\cdot\left(n-(-1)^n\cdot\underbrace{\cos\pi n}_{=(-1)^n}\right)$$

$$= n\cdot(n-1) = f(n) \ ✓$$

$$\lim_{x\to n-} f(x) = \lim_{x\to n-} \underbrace{[x]}_{n-1}\left(\underbrace{[x]}_{n-1}-\underbrace{(-1)^{[x]}}_{(-1)^{n-1}}\cdot\underbrace{\cos\pi x}_{\to\cos\pi n = (-1)^n}\right) = (n-1)\cdot\underbrace{\left(n-1-\underbrace{(-1)^{n-1}\cdot(-1)^n}_{=-1}\right)}_{n} = (n-1)\cdot n = f(n) \ ✓$$

$\Rightarrow \boxed{f \text{ непр. у } n,\ \forall n\in\mathbb{Z}}$