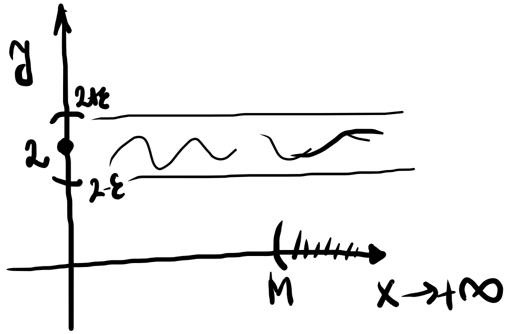
~ Граничне вредности функција - наставак ~

① По дефиницији: $\lim\limits_{x\to+\infty} \underbrace{\dfrac{2x+7}{x-3}}_{f(x)} = 2.$



$\lim\limits_{x\to+\infty} f(x) = 2 \iff (\forall \varepsilon>0)(\exists M>0)\ \forall x>M \quad |f(x)-2|<\varepsilon$

ε-фиксирано, $\varepsilon>0 \to \boxed{?M}$

$|f(x)-2|<\varepsilon \iff \left|\dfrac{2x+7}{x-3} - 2\right| < \varepsilon$

$\iff \left|\dfrac{13}{x-3}\right| < \varepsilon$

$\iff \dfrac{13}{\varepsilon} < |x-3|$ ← ово важи сигурно за $x > 3+\dfrac{13}{\varepsilon}$

узмимо $M = 3+\dfrac{13}{\varepsilon}$

$x>M \Rightarrow |f(x)-2|<\varepsilon$

$\forall \varepsilon>0$

$\lim\limits_{x\to+\infty} f(x) = 2$ ✓

Важни лимеси:

* $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, $\lim\limits_{x\to 0} \dfrac{\operatorname{tg} x}{x} = 1$, $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

* $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = \lim\limits_{x\to 0} \Big(\underbrace{\dfrac{1-\cos x}{x^2}}_{\to 1/2} \cdot \underset{\to 0}{x} \Big) = \dfrac{1}{2} \cdot 0 = 0$

* $\lim\limits_{x\to +\infty} \left(1+\dfrac{1}{x}\right)^x = e$, $\lim\limits_{x\to 0} (1+x)^{1/x} = e$

$(x\to -\infty)$

* $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$, $\lim\limits_{x\to 0} \dfrac{\log_a(1+x)}{x} = \log_a e \quad \left(= 1/\ln a\right)$

* $\lim\limits_{x\to 0} \dfrac{e^x-1}{x} = 1$, $\lim\limits_{x\to 0} \dfrac{a^x-1}{x} = \ln a \quad (a>0, a\neq 1)$

* $\lim\limits_{x\to 0} \dfrac{(1+x)^\alpha - 1}{x} = \alpha$

$\alpha \in \mathbb{R}$

* коначне тачке можемо транслирати у нулу

$\lim\limits_{x\to 5} f(x) \underset{t\to 0}{\overset{x=t+5}{=\!=}} \lim\limits_{t\to 0} f(t+5)$

$= \dots$

② Доказати: $\lim\limits_{x\to 0}\frac{(1+x)^{\alpha}-1}{x}=\alpha,\ \alpha\in\mathbb{R}$

$\left[\ \frac{e^{t}-1}{t}\to 0,\ t\to 0\ \right]$

$$L=\lim_{x\to 0}\frac{e^{\overbrace{\alpha\cdot\ln(1+x)}^{\to 0}}-1}{x}=\lim_{x\to 0}\underbrace{\frac{e^{\alpha\cdot\ln(1+x)}-1}{\alpha\cdot\ln(1+x)}}_{\to 1}\cdot\alpha\underbrace{\frac{\ln(1+x)}{x}}_{\to 1}$$

$$=1\cdot\alpha\cdot 1=\boxed{\alpha}$$

$x\to 0$

$1+x\to 1$

$\ln(1+x)\to\ln 1=0$

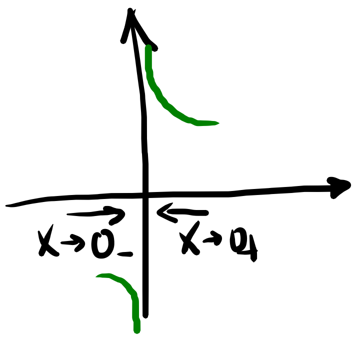
Непрекидност функције ln !

③ Шта је са $\lim\limits_{x\to 0}\frac{\cos x}{x}$?

$x\to 0$ $\quad\frac{\overbrace{\cos x}^{\to 1}}{x\to 0_{+/-}}$???

$x\to 0_{+}$ $\quad\lim\limits_{x\to 0_{+}}\frac{\overbrace{\cos x}^{\to 1}}{x\to 0_{+}}=+\infty$

$x\to 0_{-}$ $\quad\lim\limits_{x\to 0_{-}}\frac{\overbrace{\cos x}^{\to 1}}{x\to 0_{-}}=-\infty$



$$\Rightarrow\ \neg\exists\lim_{x\to 0}\frac{\cos x}{x}$$

$$* \lim_{x\to \sqcup} \frac{f(x)}{g(x)}$$

? Ког облика је $\frac{f(x)}{g(x)}$?

$x \to a$ $\quad f(x)\to 5$, $g(x)\to 2$ $\quad \frac{f(x)}{g(x)} \to \frac{5}{2}$

НЕОДРЕЂЕНИ ИЗРАЗИ: „$\frac{0}{0}$", „$\frac{\infty}{\infty}$"

„$\infty-\infty$", „$0\cdot\infty$", „1^{∞}"

4. $\lim\limits_{x\to 0} \frac{\sin 15x - \sin 3x}{\sin x}$ „$\frac{0}{0}$"

$$\overset{:x}{=} \lim_{x\to 0} \frac{\frac{\sin 15x}{x} - \frac{\sin 3x}{x}}{\frac{\sin x}{x}}$$

$$= \lim_{x\to 0} \frac{\frac{\sin 15x}{15x}\cdot 15 - \frac{\sin 3x}{3x}\cdot 3}{\frac{\sin x}{x}}$$

$\frac{\sin 15x}{15x} \to 1$, $\frac{\sin 3x}{3x} \to 1$, $\frac{\sin x}{x} \to 1$

$$= \frac{15-3}{1} = \boxed{12}$$

$\left[\frac{\sin x}{x} \to 1\right.$ кад $\left. x\to 0\right]$

$x\to 0$

$15x \to 0$

$\frac{\sin 15x}{15x} \to 1$

(лимес сложене ф-је

⑤ $a \in \mathbb{R}$ $\quad \lim\limits_{x \to a} \dfrac{\sin x - \sin a}{x - a}$ $\quad$ "$\frac{0}{0}$" $\quad$ непр. $\sin x$: $\sin x \xrightarrow[x \to a]{} \sin a$

$$= \lim_{x \to a} \frac{2 \cdot \sin \frac{x-a}{2} \cdot \cos \frac{x+a}{2}}{x - a}$$

$$= \lim_{x \to a} \underbrace{\frac{\sin \frac{x-a}{2}}{\frac{x-a}{2}}}_{\to 1} \cdot \underbrace{\cos \frac{x+a}{2}}_{\to \cos a}$$

$\frac{x-a}{2} \to 0$, $\frac{x+a}{2} \to a$; $\to \cos a$ (непр. cos)

$$= 1 \cdot \cos a$$

$$= \boxed{\cos a}$$

⑥ $\lim\limits_{x \to 0} \dfrac{e^{ax} - e^{bx}}{x}$ $\quad a, b \in \mathbb{R},\ a, b \neq 0$ $\quad \frac{0}{0}$

$$= \lim_{x \to 0} \underbrace{\frac{e^{ax} - 1}{ax}}_{\to 1} \cdot a + \cancel{\frac{1}{x}} - \underbrace{\frac{e^{bx} - 1}{bx}}_{\to 1} \cdot b - \cancel{\frac{1}{x}}$$

$ax \to 0$

$$= 1 \cdot a - 1 \cdot b = \boxed{a - b}$$

$\left[\lim\limits_{x \to 0} \dfrac{e^{ax} - e^{bx}}{x} \neq \lim\limits_{x \to 0} \dfrac{e^{ax}}{x} - \lim\limits_{x \to 0} \dfrac{e^{bx}}{x} \right]$

$\nexists$

⑦ $\lim\limits_{x \to 5} \dfrac{\operatorname{tg} \pi x}{x - \sqrt{8}}$

$x \to 5$

$\operatorname{tg} \pi x \to \operatorname{tg} 5\pi = 0$

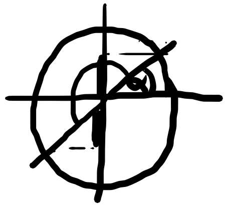
$x - \sqrt{8} \to 5 - \sqrt{8} \neq 0$

$$= \frac{0}{5 - \sqrt{8}} = \boxed{0}$$

! није неодређен израз !

⑧ $\lim\limits_{x\to 1} \dfrac{1-x^2}{\sin \pi x} \overset{\text{„}0/0\text{"}}{=} \left(\begin{matrix} x=1+t \\ t\to 0 \end{matrix} \right)$

$= \lim\limits_{t\to 0} \dfrac{1-(1+t)^2}{\sin \pi(1+t)}$ $\quad \left[\sin(\alpha+\pi) = -\sin\alpha \right]$



$= \lim\limits_{t\to 0} \dfrac{1-(1+t)^2}{-\sin \pi t}$

$= \lim\limits_{t\to 0} + \underbrace{\dfrac{(1+t)^2-1}{t}}_{\to 2} \cdot \underbrace{\dfrac{\overset{\to 0}{t\cdot\pi}}{\sin \pi t}}_{\to 1} \cdot \dfrac{1}{\pi}$

$= 2\cdot 1\cdot \dfrac{1}{\pi} = \boxed{\dfrac{2}{\pi}}$

$x\to a \quad f(x) \to \alpha$
$x\to a \quad g(x) \to \beta$

$\dfrac{f(x)}{g(x)} \to \dfrac{\alpha}{\beta}$

$f(x) = 1 = \alpha$ const

$\dfrac{1}{g(x)} \to \dfrac{1}{\beta}, \; x\to a$

⑨ $\lim\limits_{x\to +\infty} x\cdot\left(\ln(x+1) - \ln x\right)$ „$\infty\cdot(\infty-\infty)$"

$= \lim\limits_{x\to+\infty} x\cdot \ln\left(1+\dfrac{1}{x}\right)$ „$\infty\cdot 0$"

$= \lim\limits_{x\to+\infty} \ln\Big(\underbrace{\left(1+\dfrac{1}{x}\right)^x}_{\to e}\Big)$

$\overset{\text{непр. ln}}{=} \ln e = \boxed{1}$

10. $\lim\limits_{x\to0} \frac{1}{x}\cdot \ln\sqrt{\frac{1+x}{1-x}}$ $\qquad \frac{1+x}{1-x} \xrightarrow[x\to0]{} 1$

"$\frac{0}{0}$" $= \lim\limits_{x\to0} \frac{1}{x}\cdot\frac{1}{2}\cdot \ln\Big(\underbrace{\frac{1+x}{1-x}}_{1+\frac{2x}{1-x}}\Big)$ $\qquad \lim\limits_{t\to0}\frac{\ln(1+t)}{t} \to 1$

$= \lim\limits_{x\to0} \frac{1}{\cancel{2x}}\cdot \underbrace{\frac{\ln\left(1+\frac{2x}{1-x}\right)}{\frac{2x}{1-x}}}_{\to 1}\cdot \underbrace{\frac{\cancel{2x}}{1-x}}_{\to 1}$ $\qquad \frac{2x}{1-x}\to 0$

$= 1\cdot 1 = \boxed{1}$

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11. $\lim\limits_{x\to0} (\cos x)^{\frac{1}{3x^2}}$ $\qquad 1^{\infty}$ $\qquad \lim\left(1+\frac{1}{x}\right)^{x} \xrightarrow[x\to+\infty]{} e$

$= \lim\limits_{x\to0} e^{\frac{1}{3x^2}\cdot \ln(\underbrace{\cos x}_{\to 1})}$

$= \lim\limits_{x\to0} e^{\frac{1}{3x^2}\cdot \frac{\ln(1+(\cos x-1))}{\cos x - 1}(\cos x - 1)}$, $\quad \cos x - 1 \to 0$ $\qquad \frac{1-\cos x}{x^2} \xrightarrow[x\to0]{} \frac{1}{2}$

$= \lim\limits_{x\to0} e^{\overbrace{\frac{\ln(1+(\cos x-1))}{\cos x-1}}^{\to 1}\cdot\left(-\overbrace{\frac{1-\cos x}{x^2}}^{\to \frac{1}{2}}\right)\cdot\frac{1}{3}}$

$= e^{1\cdot -\frac{1}{2}\cdot\frac{1}{3}} = e^{-1/6} = \boxed{\frac{1}{\sqrt[6]{e}}}$

непр. ф-је $e^{\square}$

12. $m, n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$

$$L = \lim_{x\to 0} \frac{\sqrt[m]{\cos\alpha x} - \sqrt[n]{\cos\beta x}}{x^2}$$

"$0/0$"

$\ulcorner \frac{(1+t)^{\alpha}-1}{t} \to \alpha,\ t\to 0 \lrcorner$

$$= \lim_{x\to 0} \frac{\big(1+\underbrace{(\cos\alpha x-1)}_{\to 0}\big)^{1/m} - \big(1+\underbrace{(\cos\beta x-1)}_{\to 0}\big)^{1/n}}{x^2}$$

$$= \lim_{x\to 0} \underbrace{\frac{\big(1+(\cos\alpha x-1)\big)^{1/m} - 1}{\cos\alpha x - 1}}_{\to 1/m} \cdot \underbrace{\frac{(\cos\alpha x-1)}{x^2}}_{\to -\frac{1}{2}\alpha^2} + \cancel{\frac{1}{x^2}} - \underbrace{\frac{\big(1+(\cos\beta x-1)\big)^{\frac{1}{n}} - 1}{\cos\beta x - 1}}_{\to 1/n} \cdot \underbrace{\frac{\cos\beta x - 1}{x^2}}_{\to -\frac{1}{2}\beta^2} - \cancel{\frac{1}{x^2}}$$

$\cos\beta x - 1 \to 0$

$$= \frac{1}{m}\cdot\frac{-1}{2}\alpha^2 - \frac{1}{n}\cdot\frac{-1}{2}\beta^2$$

$$= \boxed{\frac{1}{2}\cdot\left(\frac{\beta^2}{n} - \frac{\alpha^2}{m}\right)} \quad \square$$

$\ulcorner \frac{\cos\alpha x - 1}{x^2} = -\underbrace{\frac{1-\cos\overset{\to 0}{\alpha x}}{(\alpha x)^2}}_{\to \frac{1}{2}}\alpha^2 \rightsquigarrow -\frac{1}{2}\alpha^2 \lrcorner$

$\dfrac{\text{полином}}{\text{полином}}$

$x \to +\infty$

$$\lim_{x\to+\infty} \frac{a_m x^m + a_{m-1}x^{m-1} + \dots + a_1 x + a_0}{b_k \cdot x^k + \dots + b_1 x + b_0} = \begin{cases} +\infty, & m>k \text{ и } a_m, b_k \text{ ист. знака} \\ -\infty, & m>k \text{ и } a_m, b_k \text{ разл. знака} \\ 0, & m<k \\ \dfrac{a_m}{b_k}, & m=k \end{cases}$$

$a_m, b_k \neq 0$

⑬ $m, n \in \mathbb{N}$

$$\lim_{x\to 1} \frac{x^m - 1}{x^n - 1} \underset{t\to 0}{\overset{x=t+1}{=}} \lim_{t\to 0} \frac{(t+1)^m - 1}{(t+1)^n - 1} =$$

$$= \lim_{t\to 0} \underbrace{\frac{(t+1)^m - 1}{t}}_{\to m} \cdot \underbrace{\frac{t}{(t+1)^n - 1}}_{\to 1/n}$$

$$= m \cdot \frac{1}{n}$$

$$= \boxed{\frac{m}{n}}$$

II начин:

$$x^m - 1 = (x-1)(x^{m-1} + x^{m-2} + \dots + x^2 + x + 1)$$

$$\lim_{x\to 1} \frac{x^m - 1}{x^n - 1} = \lim_{x\to 1} \frac{\cancel{(x-1)}\overbrace{(x^{m-1} + \dots + x + 1)}^{\to m}}{\cancel{(x-1)}\underbrace{(x^{n-1} + \dots + x + 1)}_{\to n}}$$

$$= \boxed{\frac{m}{n}}$$

14. $a \neq 0$: $\lim\limits_{x\to+\infty}\left(\dfrac{x+a}{x-a}\right)^{x}$ $\quad (1^{\infty})$

$$\frac{x+a}{x-a} \to \frac{1}{1} = 1, \quad x\to+\infty$$

$$\boxed{\left(1+\frac{1}{t}\right)^{t} \to e, \quad t\to+\infty, \ t\to-\infty}$$

$$= \lim_{x\to+\infty}\left(1+\frac{2a}{x-a}\right)^{x}$$

$$= \lim_{x\to+\infty}\underbrace{\left(1+\frac{1}{\underbrace{\frac{x-a}{2a}}_{t\,\to+\infty}}\right)^{\overset{t}{\frac{x-a}{2a}}}}_{\to e}{}^{\frac{2a}{x-a}\cdot x \;\to\; 2a} = \boxed{e^{2a}}$$

$$\frac{2a}{x-a}\cdot x = \frac{2ax}{x-a} \to 2a$$

15. $$L = \lim_{x\to-\infty}\left(\frac{3x^2-x+1}{3x^2+x+1}\right)^{\frac{x^2+1}{x-1}} = \lim_{x\to-\infty}\left(1+\frac{-2x}{3x^2+x+1}\right)^{\frac{x^2+1}{x-1}} =$$

$1^{-\infty}$! $\lim\limits_{x\to-\infty}\dfrac{\overset{\to+\infty}{x^2+1}}{\underset{\to-\infty}{x-1}} = -\infty$

$$= \lim_{x\to-\infty}\underbrace{\left(1+\frac{1}{\underbrace{\frac{3x^2+x+1}{-2x}}_{\to+\infty}}\right)^{\frac{3x^2+x+1}{-2x}}}_{e}\cdot{}^{\frac{-2x}{3x^2+x+1}\cdot\frac{x^2+1}{x-1}}$$

$$\frac{-2x}{3x^2+x+1}\cdot\frac{x^2+1}{x-1} = \frac{-2x^3-2x}{3x^3+\text{mai mici}} \to -\frac{2}{3}$$

$$\Rightarrow \boxed{L = e^{-2/3}}$$

(16.) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+b^2}+x} \underset{t\to+\infty}{\overset{t=-x}{=}} \lim\limits_{t\to+\infty} \dfrac{\sqrt{t^2+a^2}-t}{\sqrt{t^2+b^2}-t}$ $:t$

$\left[\ \dfrac{\sqrt{t^2}}{t} \overset{(t>0)}{\underset{!}{=}} \dfrac{\sqrt{t^2}}{\sqrt{t^2}} = 1\ \right]$

$\dfrac{+\infty-\infty}{+\infty-\infty}$

$$= \lim_{t\to+\infty} \frac{\sqrt{1+\frac{a^2}{t^2}}-1}{\sqrt{1+\frac{b^2}{t^2}}-1} = \lim_{t\to+\infty} \underbrace{\frac{\left(1+\frac{a^2}{t^2}\right)^{1/2}-1}{\frac{a^2}{t^2}\to 0}}_{\to 1/2} \cdot \frac{a^2}{\cancel{t^2}} \cdot \underbrace{\frac{b^2/t^2}{\left(1+\frac{b^2}{t^2}\right)-1}}_{\to \frac{1}{1/2}=2} \cdot \frac{\cancel{t^2}}{b^2}$$

$$= \frac{1}{2}\, a^2 \cdot 2 \cdot \frac{1}{b^2} = \boxed{\frac{a^2}{b^2}}$$

(17.) $\lim\limits_{x\to\pi} \dfrac{\sqrt{1-\operatorname{tg}x}-\sqrt{1+\operatorname{tg}x}}{\sin 2x}$ $\quad \sqrt{\ }+\sqrt{\ }$

"$\dfrac{0}{0}$"

$$= \lim_{x\to\pi} \frac{(1-\operatorname{tg}x)-(1+\operatorname{tg}x)}{2\sin x\cdot\cos x\cdot\left(\sqrt{1-\operatorname{tg}x}+\sqrt{1+\operatorname{tg}x}\right)}$$

$= -2\operatorname{tg}x = -2\dfrac{\sin x}{\cos x}$

$$= \lim_{x\to\pi} \frac{-\cancel{2}\cdot\frac{\cancel{\sin x}}{\cos x}}{\cancel{2\sin x}\cdot\cos x\cdot\left(\sqrt{\ }+\sqrt{\ }\right)}$$

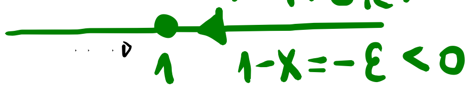
$$= \lim_{x\to\pi} \frac{-1}{\underbrace{(\cos x)^2}_{\to 1}\underbrace{\left(\sqrt{1-\operatorname{tg}x}+\sqrt{1+\operatorname{tg}x}\right)}_{\to 2}} = \boxed{\frac{-1}{2}}$$

18. Одредити леви и десни лимес ф-је f у датој тачки:

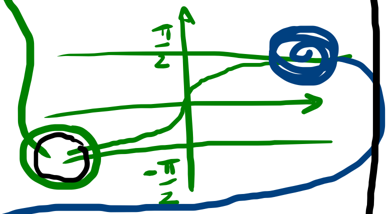
а) $f(x) = \operatorname{arcctg} \frac{1}{1-x}$, $x \to 1_+$, $x \to 1_-$

$$\lim_{x \to 1_+} \operatorname{arcctg} \underbrace{\frac{1}{\underbrace{1-x}_{\to 0_-}}}_{\to -\infty} = \boxed{-\frac{\pi}{2}}$$

$x = 1+\varepsilon,\ \varepsilon > 0$



$1 - x = -\varepsilon < 0$

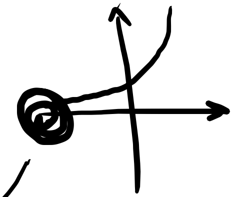


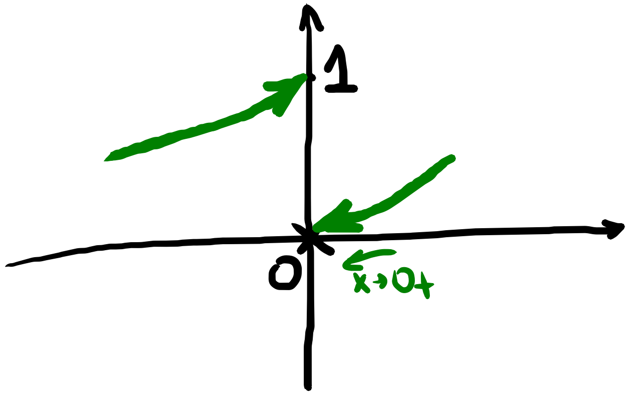
$$\lim_{x \to 1_-} \operatorname{arcctg} \underbrace{\frac{1}{\underbrace{1-x}_{\to 0_+}}}_{\to +\infty} = \boxed{\frac{\pi}{2}}$$

б) $f(x) = \frac{1}{1+e^{1/x}}$, $x \to 0_+$, $x \to 0_-$

$$\lim_{x \to 0_+} \frac{1}{1 + \underbrace{e^{\overbrace{1/x}^{\to +\infty}}}_{\to +\infty}} \overset{1/\infty}{=} \boxed{0}$$

$$\lim_{x \to 0_-} \frac{1}{1 + \underbrace{e^{\overbrace{1/x}^{\to -\infty}}}_{\to 0}} = \frac{1}{1+0} = \boxed{1}$$





За вибору

* $\lim\limits_{x\to 0} \dfrac{1-\sqrt[3]{1+x}}{\ln(1-x)}$

* $\lim\limits_{x\to 0} \dfrac{\log_{10}(1+\sin x)}{\operatorname{tg} 2x}$

* $\lim\limits_{x\to 0} \dfrac{\ln(\cos ax)}{\ln(\cos bx)}$ $\quad a, b \neq 0$

* $\lim\limits_{x\to 1} \dfrac{(1+x)^x - 2}{x-1}$

* $\lim\limits_{x\to 1} (1-x)\cdot \operatorname{tg}\dfrac{\pi x}{2}$

* $\lim\limits_{x\to 0} \dfrac{\sqrt{\cos x} - \sqrt[3]{\cos x}}{\sin^2 x}$

* $\lim\limits_{x\to 4} \dfrac{\sqrt{1+2x} - 3}{\sqrt{x} - 2}$

* $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}$

* $\lim\limits_{x\to +\infty} \left(\dfrac{7x^2 - x}{7x^2 + x}\right)^{\frac{x^3+1}{x^2-2}}$