~ Граничне вредности функција - наставак ~

Важни лимеси

- $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, $\lim\limits_{x\to 0} \frac{\operatorname{tg} x}{x} = 1$, $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

$$* \lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \underbrace{\boxed{\frac{1-\cos x}{x^2}}}_{\to 1/2} \cdot \underbrace{x}_{\to 0} \overset{\text{Т о производу лимеса}}{=} \frac{1}{2}\cdot 0 = 0$$

$\ulcorner x\to 0$

$\cos x \to 1$

$1-\cos x \to 1-1 = 0$

$\frac{1-\cos x}{x^2 \to 0}$ "$\frac{0}{0}$"

- $\lim\limits_{\substack{x\to +\infty \\ (x\to -\infty)}} \left(1+\frac{1}{x}\right)^x = e$, $\lim\limits_{t\to 0} (1+t)^{1/t} = e$
- $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$, $\lim\limits_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$
- $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$, $\lim\limits_{x\to 0} \frac{a^x-1}{x} = \ln a$
- $\alpha \in \mathbb{R}$: $\lim\limits_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$

☺ коначне тачке можемо транслирати у 0

① Доказати: $\underbrace{\lim_{x\to 0} \frac{(1+x)^{\alpha}-1}{x}}_{} = \alpha$

$\left[\frac{e^{x}-1}{x} \to 1,\ x\to 0\right]$

$$L = \lim_{x\to 0} \underbrace{\frac{e^{\overbrace{\alpha\cdot\ln(1+x)}^{t}}-1}{\underbrace{\alpha\cdot\ln(1+x)}_{t}}}_{\to 1} \cdot \underbrace{\frac{\alpha\cdot\ln(1+x)}{x}}_{\to 1,\ x\to 0} = 1\cdot\alpha\cdot 1 = \alpha$$

✓

$\frac{e^{t}-1}{t}$ $\quad t=\alpha\ln(\underbrace{1+x}_{\to 1}) \to ? = 0$

$x\to 0 \qquad \ln(1+x)\to \ln 1 = 0$

$t\to 0 \qquad \frac{e^{t}-1}{t} \to 1$

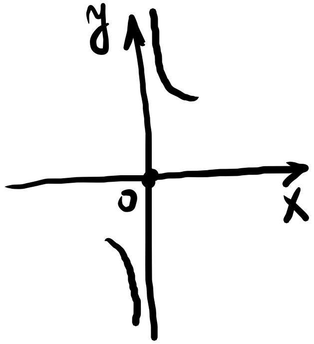
② Шта је са лимесом:

$\lim_{x\to 0} \frac{\cos x}{x}$?

$\cancel{\frac{0}{0}} \qquad \frac{1}{0}$?

$x\to 0+ \qquad \lim_{x\to 0+} \frac{\overbrace{\cos x}^{\to 1}}{\underbrace{x}_{\to 0+}} = +\infty$

$x\to 0- \qquad \lim_{x\to 0-} \frac{\overbrace{\cos x}^{\to 1}}{\underbrace{x}_{\to 0-}} = -\infty$



$\neg\exists \lim_{x\to 0} \frac{\cos x}{x}$

③ $\lim\limits_{x\to 0} \frac{\sin 5x - \sin 3x}{\sin x} = ?$

! Ког облика је израз $\frac{f \to 7}{g \to -10} \to -\frac{7}{10}$

$„\frac{0-0}{0}" \quad „\frac{0}{0}"$

Неодређен:

$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, \infty-\infty, 0\cdot\infty, 0^0, 1^\infty$

$$L = \lim_{x\to 0} \frac{\frac{\sin 5x}{x} - \frac{\sin 3x}{x}}{\frac{\sin x}{x}} = \lim_{x\to 0} \frac{\underbrace{\frac{\sin 5x}{5x}}_{\to 1}\cdot 5 - \underbrace{\frac{\sin 3x}{3x}}_{\to 1}\cdot 3}{\underbrace{\frac{\sin x}{x}}_{\to 1}} = \frac{5-3}{1} = \boxed{2}$$

$t = 3x \to 0, x \to 0$

$\frac{\sin 3x}{3x} \to 1$, $3x = t$

④ $\lim\limits_{x\to a} \frac{\sin x - \sin a}{x-a}$, $a \in \mathbb{R}$ $„\frac{0}{0}"$

$$= \lim_{x\to a} \frac{2\sin\frac{x-a}{2}\cdot\cos\frac{x+a}{2}}{x-a} = \lim_{x\to a} \underbrace{\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}}_{\to 1} \cdot \underbrace{\cos\underbrace{\frac{x+a}{2}}_{\to a}}_{\cos a} = 1\cdot\cos a = \boxed{\cos a}$$

$t \to a, \cos t \to \cos a$

непрекидност $\cos$!

⑤ $\lim\limits_{x\to 5} \frac{\operatorname{tg}(\pi\cdot x)}{x-\sqrt{7}} = \frac{0}{5-\sqrt{7}} = \boxed{0}$

$\to \operatorname{tg} 5\pi = 0$

$\to 5-\sqrt{7} \neq 0$

није неодређен!

⑥ $$\lim_{x\to 0}\frac{e^{a\cdot x}-e^{b\cdot x}}{x} = \lim_{x\to 0}\Big(\underbrace{\frac{e^{a\cdot x}-1}{a\cdot x}}_{\frac{e^{ax}}{x}-\frac{1}{\cancel{x}}}\cdot a - \underbrace{\frac{e^{b\cdot x}-1}{b\cdot x}}_{\frac{e^{bx}}{x}-\frac{1}{\cancel{x}}}\cdot b\Big) = 1\cdot a - 1\cdot b = \boxed{a-b}$$

$a, b \neq 0$

$$\left[\frac{e^t-1}{t}\to 1,\ t\to 0\right]$$

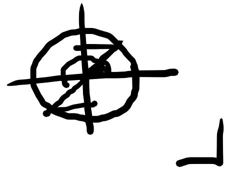
⑦ $$\lim_{x\to 1}\frac{1-x^2}{\sin \pi x} \ \frac{\text{смена } x=1+t}{t\to 0}\ \lim_{t\to 0}\frac{1-(1+t)^2}{\sin\pi(1+t)} = \lim_{t\to 0}\frac{1-(1+t)^2}{-\sin\pi t} =$$

$$\text{„}\frac{0}{0}\text{"}$$

$$= \lim_{t\to 0}\frac{(1+t)^2-1}{\sin \pi t} = \lim_{t\to 0}\frac{(1+t)^2-1}{t}\cdot\frac{\pi t}{\sin\pi t}\cdot\frac{1}{\pi} = \boxed{\frac{2}{\pi}}$$

$$\frac{(1+t)^2-1}{t}\to 2,\quad \pi t\to 0,\quad \frac{\pi t}{\sin \pi t}\to 1$$

$$\left[\sin(\pi+\pi t) = -\sin\pi t\right]$$



$$\left[\frac{(1+t)^\alpha-1}{t}\to\alpha,\ t\to 0\right]$$

⑧ $$\lim_{x\to+\infty} x\cdot(\ln(1+x)-\ln x) = \lim_{x\to+\infty} x\cdot\ln\left(1+\frac{1}{x}\right) = \lim_{x\to+\infty}\ln\Big(\underbrace{\left(1+\frac{1}{x}\right)^x}_{\to e}\Big) = \ln e = \boxed{1}$$

$\infty\cdot(\infty-\infty)$

$+\infty\cdot 0$

непр. ф-је ln

9. $\lim_{x\to 0} \frac{1}{x}\cdot \ln\sqrt{\frac{1+x}{1-x}} = \lim_{x\to 0} \frac{1}{x}\cdot\frac{1}{2}\cdot \ln\frac{1+x}{1-x} = \lim_{x\to 0} \frac{1}{2x}\cdot \ln\left(1+\overset{\to 0}{\frac{2x}{1-x}}\right) =$

$= \lim_{x\to 0} \frac{1}{\cancel{2x}}\cdot \frac{\ln\left(1+\overset{t}{\frac{2x}{1-x}}\right)}{\frac{2x}{1-x}\to t\to 0}\cdot \frac{\cancel{2x}}{\underbrace{1-x}_{\to 1}} = \boxed{1}$

$\to 1$

$\frac{1+x}{1-x} \to 1,\ x\to 0 \qquad "\frac{0}{0}"$

$1+\frac{2x}{1-x}\to 0$

10. $\lim_{x\to 0} (\cos x)^{\frac{1}{3x^2}} = \lim_{x\to 0} e^{\frac{1}{3x^2}\cdot \ln(\overset{\to 1}{\cos x})} = \lim_{x\to 0} e^{\frac{1}{3x^2}\cdot \frac{\ln(1+(\overset{\to 0}{\cos x-1}))}{\cos x-1}\cdot(\cos x-1)}$

$"1^{+\infty}"$

$= \lim_{x\to 0} e^{\frac{1}{3}\cdot \overset{\to -\frac{1}{2}}{\frac{\cos x-1}{x^2}}\cdot \overset{\to 1}{\frac{\ln(1+(\overset{\to 0}{\cos x-1}))}{\cos x-1}}} \qquad \left[\frac{1-\cos x}{x^2}\to\frac{1}{2},\ x\to 0\right]$

$\left(1+\frac{1}{t}\right)^t \to e,\ t\to+\infty$

$\to 1$

1^{∞}

непр. e^t $= e^{\frac{1}{3}\cdot\frac{-1}{2}} = e^{-\frac{1}{6}} = \frac{1}{\sqrt[6]{e}}$

(11) $L=\lim_{x\to 0}\dfrac{\sqrt[m]{\cos\alpha\cdot x}-\sqrt[n]{\cos\beta x}}{x^2}=\lim_{x\to 0}\dfrac{(1+(\cos\alpha x-1))^{1/m}-(1+(\cos\beta x-1))^{1/n}}{x^2}$

$\dfrac{(1+t)^{\alpha}-1}{t}\to\alpha,\ t\to 0$

$m,n\in\mathbb{N}$

$\alpha,\beta\neq 0$

" $\dfrac{0}{0}$ "

$=\lim_{x\to 0}\left[\dfrac{(1+(\cos\alpha x-1))^{\frac{1}{m}}-1}{\underbrace{\cos\alpha x-1}_{\to 0}}\xrightarrow{}\frac{1}{m}\cdot\underbrace{\dfrac{\cos\alpha x-1}{x^2}}_{\to-\frac{1}{2}\alpha^2}+\cancel{\dfrac{1}{x^2}}-\dfrac{(1+(\cos\beta x-1))^{\frac{1}{n}}-1}{\cos\beta x-1}\xrightarrow{}\frac{1}{n}\cdot\underbrace{\dfrac{\cos\beta x-1}{x^2}}_{\to-\frac{1}{2}\beta^2}-\cancel{\dfrac{1}{x^2}}\right]$

$\dfrac{\cos\alpha x-1}{x^2}=-\boxed{\dfrac{1-\cos\alpha x}{(\alpha x)^2}}\cdot\alpha^2$ $\quad(\to 1/2)$

$\to-\dfrac{1}{2}\cdot\alpha^2,\ x\to 0$

$=\dfrac{1}{m}\cdot-\dfrac{1}{2}\alpha^2-\left(\dfrac{1}{n}\cdot-\dfrac{1}{2}\beta^2\right)=\boxed{\dfrac{1}{2}\left(\dfrac{\beta^2}{n}-\dfrac{\alpha^2}{m}\right)}$

(12.) $\dfrac{\text{полином}}{\text{полином}}$:

$$\lim_{x\to+\infty}\frac{a_m\cdot x^m+\dots+a_1x+a_0}{b_k\cdot x^k+\dots+b_0}=\begin{cases}+\infty, & m>k \text{ и } a_m, b_k \text{ одн. знака}\\ -\infty, & m>k \text{ и } a_m, b_k \text{ розл. знака}\\ 0, & m<k\\ \dfrac{a_m}{b_k}, & m=k\end{cases}$$

$a_m, b_k \neq 0$

(13.) $$\lim_{x\to 1}\frac{x^m-1}{x^n-1}\overset{x=1+t}{=\!=}\lim_{t\to0}\frac{(1+t)^m-1}{(1+t)^n-1}=\lim_{t\to0}\underbrace{\frac{(1+t)^m-1}{t}}_{\to m}\cdot\underbrace{\frac{t}{(1+t)^n-1}}_{\to\frac{1}{n}}=\boxed{\frac{m}{n}}$$

$m,n\in\mathbb{N}$ "$\frac{0}{0}$"

II спосіб: $x^m-1=(x-1)(x^{m-1}+x^{m-2}+\dots+x+1)$

$$\lim_{x\to1}\frac{x^m-1}{x^n-1}=\lim_{x\to1}\frac{\cancel{(x-1)}\overbrace{(\underset{\to1}{x^{m-1}}+\underset{\to1}{x^{m-2}}+\dots+\underset{\to1}{x}+1)}^{\text{зб.}\to m}}{\cancel{(x-1)}\underbrace{(\underset{\to1}{x^{n-1}}+\underset{\to1}{x^{n-2}}+\dots+\underset{\to1}{x}+1)}_{\to n}}=\boxed{\frac{m}{n}}$$

[$\lim\limits_{x\to a}f_1(x)=\alpha$

$\lim\limits_{x\to a}f_2(x)=\beta$

$\lim\limits_{x\to a}\dfrac{f_1(x)}{f_2(x)}=\dfrac{\alpha}{\beta}$

$f_1\equiv const=1=\alpha$

$\lim\limits_{x\to a}\dfrac{1}{f_2(x)}=\dfrac{1}{\beta}$]

$$⑬\quad \lim_{x\to+\infty}\left(\frac{x+a}{x-a}\right)^x = \lim_{x\to+\infty}\left(1+\frac{2a}{x-a}\right)^x = \lim_{x\to+\infty}\left(1+\frac{1}{\underbrace{\frac{x-a}{2a}}_{\to+\infty}}\right)^x$$

$a\neq 0$

$$\frac{x+a}{x-a} \xrightarrow{\text{под.}/\text{под.}} \frac{1}{1}=1$$

1^{∞}

$$\left(1+\frac{1}{t}\right)^t \to e,\ t\to+\infty,\ t\to-\infty$$

$$(1+t)^{1/t}\to e,\ t\to 0$$

$$= \lim_{x\to+\infty} \underbrace{\left(1+\frac{1}{\underbrace{\frac{x-a}{2a}}_{t\to+\infty}}\right)^{\frac{x-a}{2a}}}_{\to e}{}^{\frac{2a}{x-a}\cdot x}$$

$$\frac{2a}{x-a}\cdot x = \frac{2ax}{x-a}\to 2a,\ x\to+\infty$$

$$= \boxed{e^{2a}}$$

$$⑭\quad \lim_{x\to-\infty}\left(\frac{3x^2-x+1}{3x^2+x+1}\right)^{\frac{x^2+1}{x-1}} = \lim_{x\to-\infty}\left(1+\frac{-2x}{3x^2+x+1}\right)^{\frac{x^2+1}{x-1}}$$

$1^{-\infty}$

$$= \lim_{x\to-\infty} \underbrace{\left(1+\frac{1}{\underbrace{-\frac{3x^2+x+1}{2x}}_{\to+\infty}}\right)^{-\frac{3x^2+x+1}{2x}}}_{\to e}{}^{\cdot\left(-\frac{2x}{3x^2+x+1}\cdot\frac{x^2+1}{x-1}\right)}$$

$$-\frac{2x}{3x^2+x+1}\cdot\frac{x^2+1}{x-1} = -2/3$$

$$\lim_{x\to-\infty}\frac{-2x^3-2x}{3x^3+\text{малк. степ.}} = -\frac{2}{3}$$

$$= e^{-2/3}$$

15.

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+b^2}+x}\ \underset{t\to+\infty}{\overset{t=-x}{=}}\ \lim_{t\to+\infty}\frac{\sqrt{t^2+a^2}-t}{\sqrt{t^2+b^2}-t}\ \overset{:t>0}{=}\ \lim_{t\to+\infty}\frac{\sqrt{1+\frac{a^2}{t^2}}-1}{\sqrt{1+\frac{b^2}{t^2}}-1}$$

$$\frac{\sqrt{t^2}}{t}\overset{t>0}{=}\sqrt{\frac{t^2}{t^2}}=1$$

$a,b\neq0$

$$\text{"}\frac{+\infty+-\infty}{+\infty+-\infty}\text{"}$$

$$=\lim_{t\to+\infty}\underbrace{\frac{\left(1+\left(\frac{a}{t}\right)^2\right)^{1/2}-1}{\left(\frac{a}{t}\right)^2}}_{\to 1/2}\cdot\left(\frac{a}{\cancel{t}}\right)^2\cdot\underbrace{\frac{\left(b/t\right)^2}{\left(1+\left(\frac{b}{t}\right)^2\right)^{\frac12}-1}}_{\to\frac{1}{1/2}=2}\cdot\frac{\cancel{t^2}}{b^2}$$

$\left(\frac{a}{t}\right)^2\to0$, $\left(b/t\right)^2\to0$

$$\left[\ \frac{(1+x)^\alpha-1}{x}\xrightarrow[x\to0]{}\alpha\ \right]$$

$$=\frac12\cdot a^2\cdot2\cdot\frac{1}{b^2}=\boxed{a^2/b^2}$$

16.

$$\lim_{x\to\pi}\frac{\sqrt{1-tgx}-\sqrt{1+tgx}}{\sin2x}\qquad\text{"}\frac{0}{0}\text{"}$$

$$\left[\cdot\sqrt{\ }+\sqrt{\ }\right]$$

$$=\lim_{x\to\pi}\frac{(\cancel{1}-tgx)-(\cancel{1}+tgx)}{\sqrt{1-tgx}+\sqrt{1+tgx}}\cdot\frac{1}{2\cdot\sin x\cdot\cos x}$$

$$=\lim_{x\to\pi}\frac{-\cancel{2}\cdot\frac{\cancel{\sin x}}{\cos x}}{\sqrt{1-tgx}+\sqrt{1+tgx}}\cdot\frac{1}{\cancel{2}\,\cancel{\sin x}\cdot\cos x}=\lim_{x\to\pi}\frac{-1}{\underbrace{\cos^2x}_{\to1}\cdot\underbrace{\left(\sqrt{1-tgx}+\sqrt{1+tgx}\right)}_{\to2}}$$

$$=\boxed{-1/2}$$

17) Лево/десно лимес:

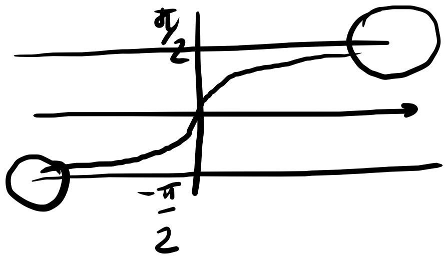
(a) $\lim\limits_{x\to 1+} \operatorname{arctg}\frac{1}{1-x} = L_1$, $\lim\limits_{x\to 1-} \operatorname{arctg}\frac{1}{1-x} = L_2$

$\underbrace{\operatorname{arctg}\underbrace{\frac{1}{\underbrace{1-x}_{\to 0_-}}}_{\to -\infty}}$ $\to -\frac{\pi}{2}$

$x\to 1+$ $x = 1+\varepsilon,\ \varepsilon > 0$

$1-x = -\varepsilon < 0$

$\boxed{L_1 = -\frac{\pi}{2}}$



$L_2 = ?$ $x\to 1-$ $x = 1-\varepsilon,\ \varepsilon > 0$ $1-x = \varepsilon > 0$

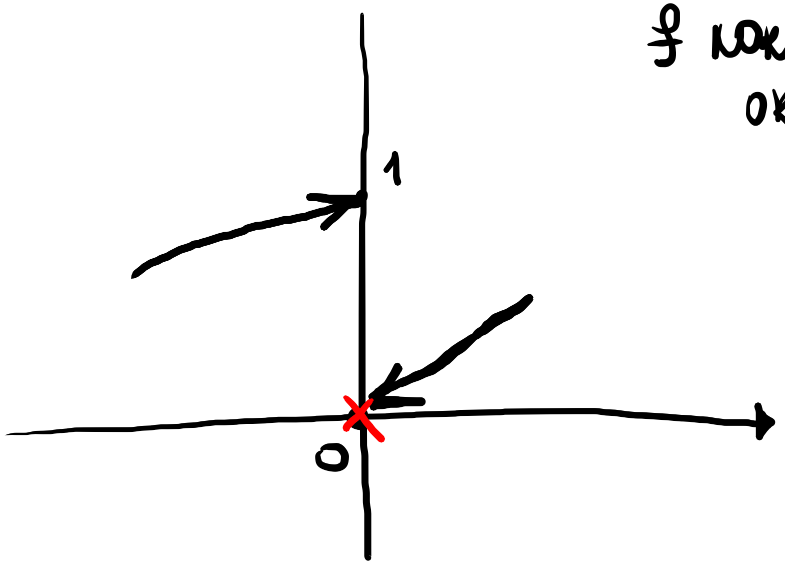
$\operatorname{arctg}\underbrace{\frac{1}{\underbrace{1-x}_{\to 0_+}}}_{\to +\infty} \to \frac{\pi}{2}$

$\boxed{L_2 = \frac{\pi}{2}}$

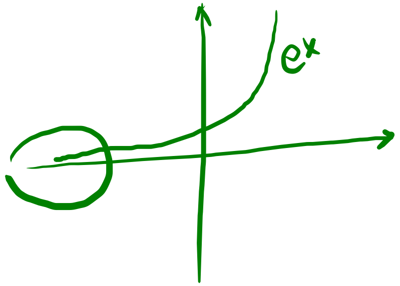
(б) $f(x) = \dfrac{1}{1+e^{1/x}}$ $\quad x \to 0+$? $\quad x \to 0-$?

$$\lim_{x\to 0+} \frac{1}{\underbrace{1+e^{\overbrace{1/x}^{\to +\infty}}}_{\to +\infty}} \overset{\text{"}\frac{1}{+\infty}\text{"}}{=\!=} 0$$

$$\lim_{x\to 0-} \frac{1}{\underbrace{1+e^{\overbrace{1/x}^{\to -\infty}}}_{\to 0}} = \frac{1}{1+0} = 1$$

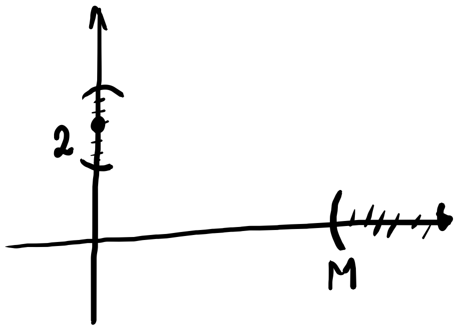


f скача
около 0



17. По дефиницији доказати: $\lim\limits_{x\to+\infty} \underbrace{\frac{2x+7}{x-3}}_{f(x)} = 2$

$$\lim_{x\to+\infty} f(x) = 2 \iff (\forall \varepsilon>0)(\exists M>0)\ \forall x>M\ \ |f(x)-2|<\varepsilon$$



$\varepsilon>0$ фиксирано ? M

$$|f(x)-2|<\varepsilon \iff \left|\frac{2x+7}{x-3}-2\right|<\varepsilon$$

$$\iff \left|\frac{-13}{x-3}\right|<\varepsilon$$

$$\iff \frac{13}{\varepsilon}<|x-3|$$

за $x>3+\frac{13}{\varepsilon}$ важи

$$\boxed{M=3+\frac{13}{\varepsilon}}$$

$$\xrightarrow[\text{за } \forall\varepsilon>0]{\text{тако}} \boxed{\lim_{x\to+\infty} f(x)=2}$$

За вежбу:

* $\lim\limits_{x\to 0} \dfrac{1-\sqrt[3]{1+x}}{\ln(1-x)}$

* $\lim\limits_{x\to 0} \dfrac{\ln(\cos\alpha x)}{\ln(\cos\beta x)}$ $\quad \alpha, \beta \neq 0$

* $\lim\limits_{x\to 0} \dfrac{\log_{10}(1+\sin x)}{\operatorname{tg} 2x}$

* $\lim\limits_{x\to 1} \dfrac{(1+x)^x-2}{x-1}$

* $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}$

* $\lim\limits_{x\to 1} (1-x)\operatorname{tg}\dfrac{\pi x}{2}$

* $\lim\limits_{x\to 0} \dfrac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}$

* $\lim\limits_{x\to 4} \dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$