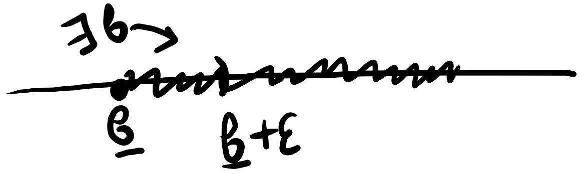
~ Супремум и инфимум - наставак ~

$A\subset\mathbb{R}$ $A\neq\emptyset$ A - ог. одозго

$$\bar{a}=\sup A \iff \begin{cases} 1^\circ & (\forall a\in A)\ a\leq\bar{a} \\ 2^\circ & (\forall\varepsilon>0)(\exists a\in A)\ a>\bar{a}-\varepsilon \end{cases}$$

$B\subset\mathbb{R}$, $B\neq\emptyset$, B - ог. одоздо



$$\underline{b}=\inf B \iff \begin{cases} 1^\circ & (\forall b\in B)\ b\geq\underline{b} \\ 2^\circ & (\forall\varepsilon>0)(\exists b\in B)\ b<\underline{b}+\varepsilon \end{cases}$$

$A, B\subset\mathbb{R}$ скупови

$-A=\{-a \mid a\in A\}$

$A+B:=\{a+b \mid a\in A,\ b\in B\}$

$A\cdot B=\{a\cdot b \mid a\in A,\ b\in B\}$

$[0,1]+(0,1]=(0,2]$

① $A, B \subset \mathbb{R}$, $A, B \neq \emptyset$ ограничене. Тада важи:

(a) $\sup A + \sup B = \sup(A+B)$ (б) $\inf A + \inf B = \inf(A+B)$ ← за вежбу

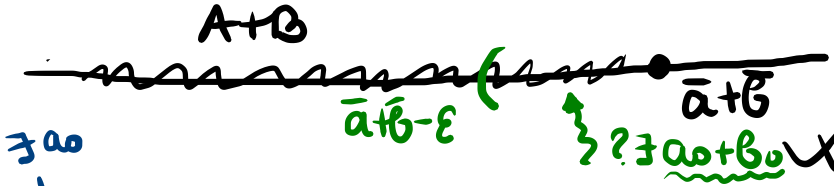
(a) $\bar{a} = \sup A$, $\bar{b} = \sup B$

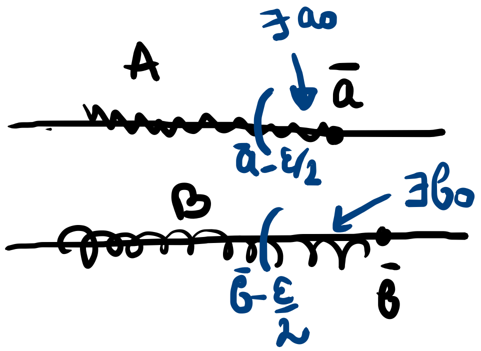
Доказујемо: $\sup(A+B) \overset{?}{=} \bar{a} + \bar{b}$ $\left[A+B = \{a+b \mid a \in A, b \in B\} \right]$

1° произв. елемент из $A+B$: $a+b$, $a \in A$, $b \in B$

$$\left.\begin{array}{ll} a \leq \bar{a} & (\text{јер } \bar{a} = \sup A) \\ b \leq \bar{b} & (\text{јер } \bar{b} = \sup B) \end{array}\right\} + \quad \underline{a+b \leq \bar{a}+\bar{b}} \text{ за } \forall \text{ ел. из } A+B$$

2° $\varepsilon > 0$





$$\left.\begin{array}{l} \bar{a} = \sup A, \ \frac{\varepsilon}{2} > 0 \Rightarrow \exists a_0 \in A \quad a_0 > \bar{a} - \frac{\varepsilon}{2} \\ \bar{b} = \sup B, \ \frac{\varepsilon}{2} > 0 \Rightarrow \exists b_0 \in B \quad b_0 > \bar{b} - \frac{\varepsilon}{2} \end{array}\right\} \overset{+}{\Rightarrow} a_0 + b_0 > \bar{a} + \bar{b} - \varepsilon$$

пронашли

$1°, 2° \Rightarrow \sup(A+B) = \bar{a} + \bar{b}$ □

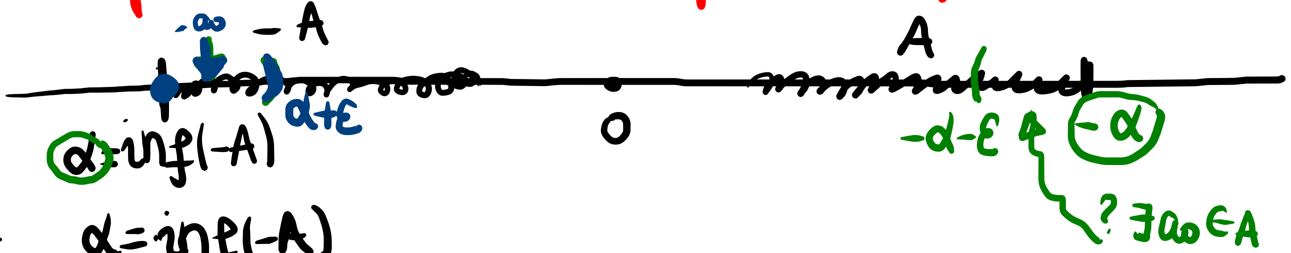
② $A \subset \mathbb{R}$, $A \neq \emptyset$ ограничен доказати:

(а) $\inf A = -\sup(-A)$ (б) $\sup A = -\inf(-A)$

$-A = \{-a \mid a \in A\}$

за вежбу

(б)



Означимо: $\alpha = \inf(-A)$

докажемо да је $\sup A = -\alpha$?

1° $-\alpha$ горње ог. за A:

? $(\forall a \in A)\ a \leq -\alpha \quad / \cdot(-1)$

$\Leftrightarrow (\forall a \in A)\ \underbrace{-a}_{\in -A} \geq \alpha$

$\Leftrightarrow$ Т јер $\alpha = \inf(-A)$ ✓

2° $\varepsilon > 0$ произв.

? $\exists a_0 \in A \quad a_0 > -\alpha - \varepsilon \quad / \cdot(-1)$

$\Leftrightarrow \exists a_0 \in A \quad \underbrace{-a_0}_{\in -A} < \alpha + \varepsilon$

$\Leftrightarrow$ Т такво $-a_0$ постоји јер је $\alpha = \inf(-A)$ ✓

$1°, 2° \Rightarrow \sup A = -\alpha$
$= -\inf(-A)$

* за производ:

$A=(1,2)$ $\sup A = 2$ $A\cdot B=(7,20)$

$B=(7,10)$ $\sup B = 10$

$20=\sup(A\cdot B)=2\cdot 10=\sup A\cdot \sup B$

$\sup(A\cdot B) \overset{?}{=} \sup A\cdot\sup B$

не важи увек

пример:

$A=(1,2)$ $\sup A=2$

$B=(-10,-7)$ $\sup B=-7$

$2\cdot(-7)=-14$

$\sup(A\cdot B)=-7 \neq \sup A\cdot\sup B$

Али важи: ако $A,B>0$ (тј. $(\forall a\in A)\,a>0$ и $(\forall b\in B)\,b>0$)

$\sup(A\cdot B)=\sup A\cdot\sup B.$

③ $A=\left\{ \frac{3n^3+1}{2+n^3} \mid n\in\mathbb{N} \right\}$ sup, inf, min, max (ако $\exists$):

$$\frac{3n^3+1}{n^3+2} = \frac{3(n^3+2)-5}{n^3+2} = 3-\frac{5}{n^3+2}$$

$$\frac{5}{1^3+2} > \frac{5}{n^3+2},\ n>1 \Rightarrow \underbrace{3-\frac{5}{1^3+2}}_{3-\frac{5}{3}=\frac{4}{3}=\min A} < 3-\frac{5}{n^3+2},\ \forall n>1$$

$$\boxed{\min A=\frac{4}{3}} \Rightarrow \boxed{\inf A = 4/3}$$

Доказујемо да је $3=\sup A$:

① $3-\frac{5}{n^3+2}<3,\ \forall n\in\mathbb{N}$ ✓

3 јесте горње ог.

② $\varepsilon>0$? $\exists n_0\in\mathbb{N}$ $3-\frac{5}{n_0^3+2}>3-\varepsilon$

$\Leftrightarrow \varepsilon > \frac{5}{n_0^3+2}$

$\Leftrightarrow n_0^3+2>\frac{5}{\varepsilon}$

$\Leftrightarrow n_0^3 > \frac{5}{\varepsilon}-2$

Постоји такво n_0:

нпр. $n_0>\frac{5}{\varepsilon}-2$

$n_0=\left[\frac{5}{\varepsilon}-2\right]+3$

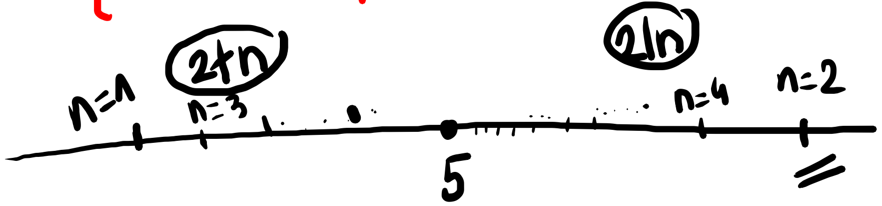
$n_0^3 \geq n_0 > \frac{5}{\varepsilon}-2$ ✓

$1^\circ, 2^\circ \Rightarrow \boxed{3=\sup A}$

$3\notin A \Rightarrow \boxed{\nexists \max A}$

④ Одредити sup, inf, min, max (ако $\exists$):

a) $A = \left\{ 5 + \frac{(-1)^n}{n^2+7} \;\middle|\; n \in \mathbb{N} \right\}$



парност n:

$2|n$: $\quad 5+\frac{1}{n^2+7} > 5 \qquad n=2,4,6,\dots$

$2\nmid n$: $\quad 5+\frac{-1}{n^2+7} = 5-\frac{1}{n^2+7} < 5 \qquad n=1,3,5,7,\dots$

$\boxed{2|n}$ $\quad \underbrace{5+\frac{1}{2^2+7}}_{\substack{=\max A\\ =\sup A}} > 5+\frac{1}{n^2+7}, \; \underline{\underline{n>2}}$

$\boxed{2\nmid n}$ $\quad \underbrace{5-\frac{1}{1^2+7}}_{\substack{\min A\\ =\inf A}} < 5-\frac{1}{n^2+7}, \; \underline{\underline{n>1}}$

(δ) $B = \left\{ \frac{n}{4} - \left[\frac{n}{4}\right] \mid n \in \mathbb{N} \right\}$

$t = \frac{n}{4}$ $\qquad$ $t - [t] < 1$

$t - [t] \geq 0$

$\boxed{\text{mod } 4}$

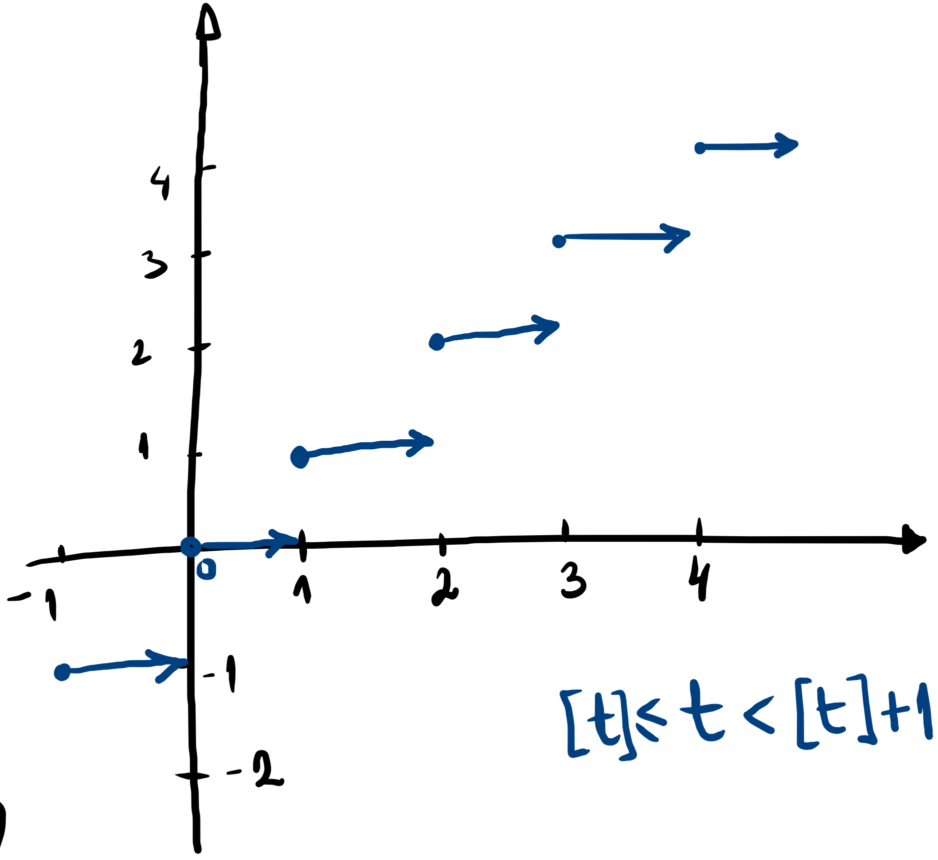
$4 \mid n$ $\quad n = 4k, k \in \mathbb{N}$

$\frac{n}{4} - \left[\frac{n}{4}\right] = k - \underbrace{[k]}_{k} = 0$

$\underline{n = 4k+1}$: $\frac{n}{4} - \left[\frac{n}{4}\right] = \underbrace{\frac{4k+1}{4}}_{k + 1/4} - \underbrace{\left[k + \frac{1}{4}\right]}_{k} = \frac{1}{4}$

$\underline{n = 4k+2}$: $\frac{4k+2}{4} - [\text{-}\|\text{-}] = k + \frac{1}{2} - \underbrace{\left[k + \frac{1}{2}\right]}_{k} = \frac{1}{2}$

$\underline{n = 4k+3}$: $\cdots\cdots = 3/4$



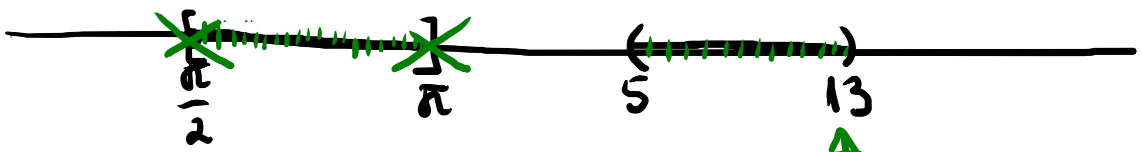
$[t] \leq t < [t] + 1$

$B = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$

$0 = \inf B = \min B$

$\frac{3}{4} = \sup B = \max B$

(б) $C = \mathbb{Q} \cap \left(\left[\frac{\pi}{2}, \pi \right] \cup (5, 13) \right)$



$\inf C = \frac{\pi}{2} \notin \mathbb{Q}$

$\Rightarrow \neg\exists \min C$

$13 = \sup C$

$\neg\exists \max C$ јер $13 \notin C$

(заграда))

в) $D = \{ x \in \mathbb{R} \mid x^2 \cdot \log_2 |x-2| \leq 0 \}$

$x^2 \cdot \log_2 |x-2| \leq 0$ услов: $|x-2| > 0$, $\boxed{x \neq 2}$

$\Leftrightarrow x = 0 \lor \log_2 |x-2| \leq 0$

$\Leftrightarrow x = 0 \lor |x-2| \leq 2^0 = 1$

$\Leftrightarrow x = 0 \lor x \in [1, 3]$

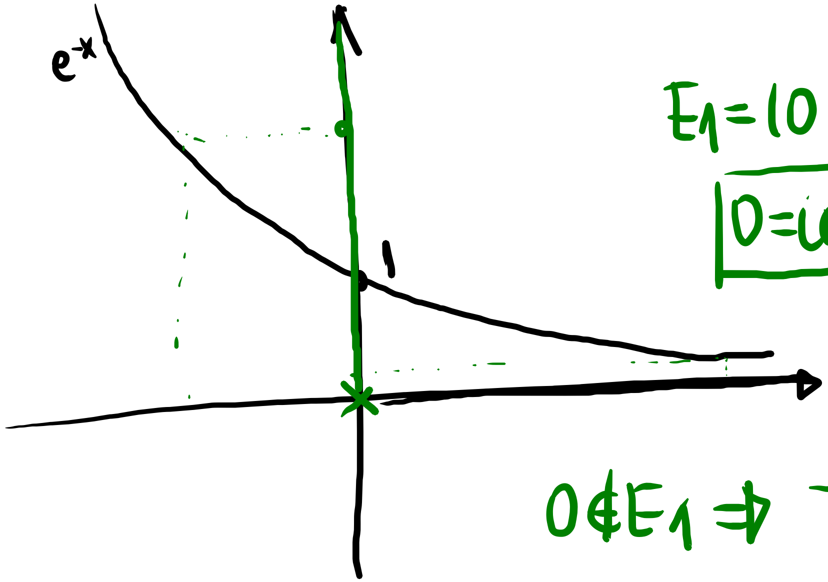
$$\boxed{D = \{0\} \cup [1, 2) \cup (2, 3]}$$

$0 = \min D = \inf D$

$3 = \max D = \sup D$

g) $E_1=\{f(x)\mid x\in\mathbb{R}\}$ $f(x)=e^{-x}$

$$f(x)=e^{-x}=\frac{1}{e^x}=\left(\frac{1}{e}\right)^x \qquad a^x,\ a<1$$



$E_1=(0,+\infty)$

$\boxed{0=\inf E_1}$

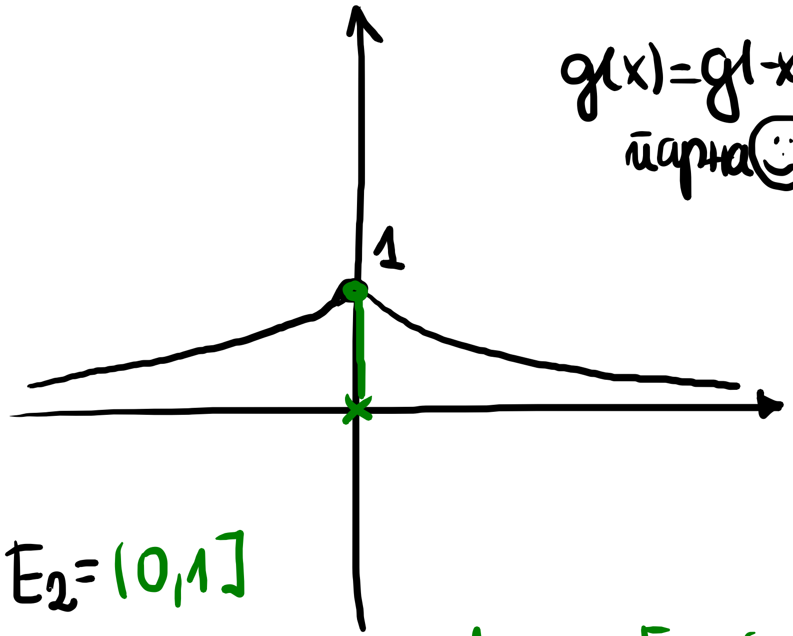
$0\notin E_1 \Rightarrow \neg\exists \min E_1$

E_1 nije ogr. odozgo

$\Rightarrow \neg\exists \sup E_1,\ \neg\exists \max E_1$

$E_2=\{g(x)\mid x\in\mathbb{R}\}$ $g(x)=e^{-|x|}$

$$g(x)=e^{-|x|}=\begin{cases} e^{-x}, & x\geq 0 \\ e^{x}, & x<0 \end{cases}$$



$g(x)=g(-x)$

parna ☺

$E_2=(0,1]$

$0=\inf E_2$

$0\notin E_2$

$\Rightarrow \neg\exists \min E_2$

$1=\max E_2=\sup E_2$

□

5) $x\in\mathbb{R}$, $x>0$ фиксиран

$A=\left\{ \frac{[nx]}{n} \mid n\in\mathbb{N} \right\}$. Доказати да важи: $x=\sup A$. Да ли је x и $\max A$?

$n=1$: $\frac{[1x]}{1}=[x]$

$n=2$: $\frac{[2x]}{2}$

$\vdots$

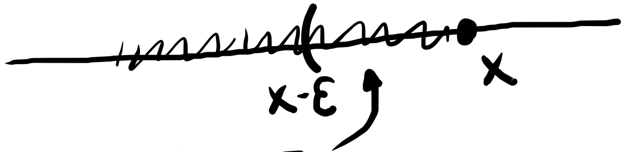
Доказујемо да $x=\sup A$

1° $n\in\mathbb{N}$ $\quad \frac{[nx]}{n} \overset{?}{\leq} x \iff [nx]\leq nx \Longleftarrow$ (T)

$$\boxed{[t]\leq t<[t]+1}$$

x јесте горње огр. A

2° $\varepsilon>0$



$?\exists n_0\in\mathbb{N}\quad \frac{[n_0 x]}{n_0} > x-\varepsilon$

$\iff \exists n_0\in\mathbb{N}\quad [n_0\cdot x] > n_0\cdot x - n_0\cdot\varepsilon$

$\iff \exists n_0\in\mathbb{N}\quad \underbrace{[n_0\cdot x]}_{[t]} + n_0\cdot\varepsilon > \underbrace{n_0\cdot x}_{t}$ (*)

ако је $n_0\cdot\varepsilon$ веће од 1, онда је (*) тачна

Довољно: $n_0\cdot\varepsilon>1 \iff n_0>\frac{1}{\varepsilon}$

такво n_0 постоји $\Rightarrow$ тада важи (*)

нпр. $n_0=\left[\frac{1}{\varepsilon}\right]+1$

1° 2° $\Rightarrow$ $\boxed{x=\sup A}$

da li je $x = \max A$?

$$x = \max A \iff x \in A$$

$$\iff \exists n_0 \in \mathbb{N} \quad x = \frac{[n_0 x]}{n_0}$$

$$\iff \exists n_0 \in \mathbb{N} \quad n_0 \cdot x = [n_0 \cdot x]$$

$$\iff \exists n_0 \in \mathbb{N} \quad n_0 \cdot x \in \mathbb{Z} \qquad (\mathbb{N})$$

$$\iff \boxed{x \in \mathbb{Q}}$$

$$\ulcorner A := \left\{ \frac{[nx]}{n} \;\middle|\; n \in \mathbb{N} \right\} \lrcorner$$

$\ulcorner x = \sqrt{2}$? $\exists n \in \mathbb{N}$

$$\underbrace{n \cdot \sqrt{2}}_{\text{ирац}} = \underbrace{[n \cdot \sqrt{2}]}_{\text{цео}}$$

НЕ ПОСТОЈИ

(!) $t = [t] \iff t \in \mathbb{Z}$

$$x = \frac{5}{100} \qquad 100 \cdot \frac{5}{100} = 5 = \left[100 \cdot \frac{5}{100}\right]$$

$n = 100$

$$2 \cdot \frac{p}{2}$$

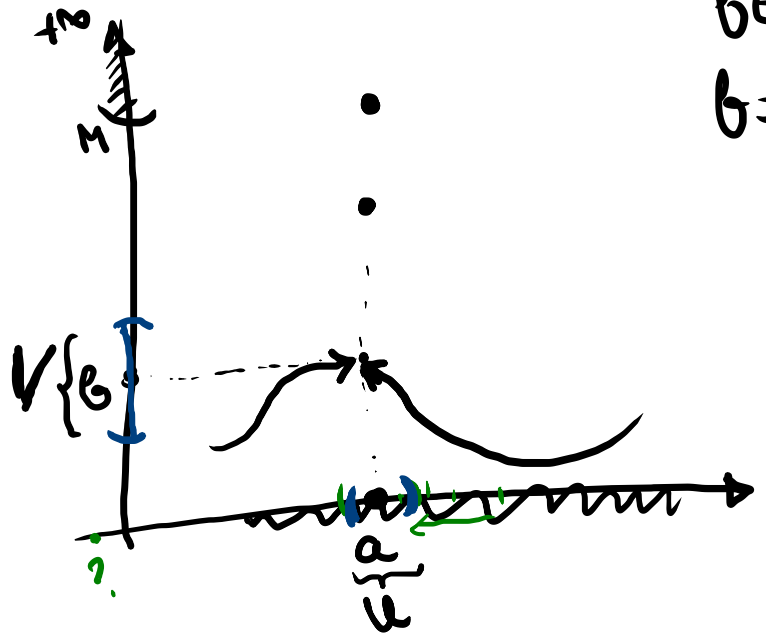
Граничне вредности функција

$f: A \to \mathbb{R}$ $\quad A \subset \mathbb{R}$ $\quad a \in \overline{\mathbb{R}}$ тачка нагомилавања скупа A $\quad a \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

$b \in \overline{\mathbb{R}}$

$b = \lim\limits_{x \to a} f(x) \Leftrightarrow$ за $\forall$ околину $V(b)$ тачке b $\exists$ околина $U(a)$ тачке a тд важи

$\forall x \in A \quad \underbrace{x \in U(a) \setminus \{a\}}_{\mathring{U}(a)} \Rightarrow f(x) \in V(b)$



$a \in \mathbb{R}, b \in \mathbb{R}$:

$b = \lim\limits_{x \to a} f(x) \Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)$

$0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon$

$a \in \mathbb{R}, b = +\infty$:

$+\infty = b = \lim\limits_{x \to a} f(x) \Leftrightarrow (\forall M > 0)(\exists \delta > 0)(\forall x \in A)$

$0 < |x - a| < \delta \Rightarrow f(x) > M$

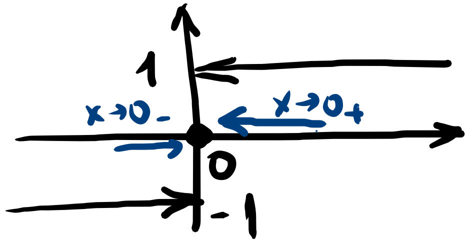
$a \in \mathbb{R}, b = -\infty$

$a = \pm\infty, b \ldots$

домаћи

све дефиниције

Пр. $f(x) = \operatorname{sgn} x = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$



$$\lim_{x\to 0+} \operatorname{sgn} x = 1$$

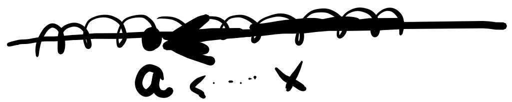
$$\lim_{x\to 0-} \operatorname{sgn} x = -1$$

$$\neg\exists \lim_{x\to 0} (\operatorname{sgn} x)$$

ЈЕДНОСТРАНИ ЛИМЕСИ:

ДЕСНИ ЛИМЕС:

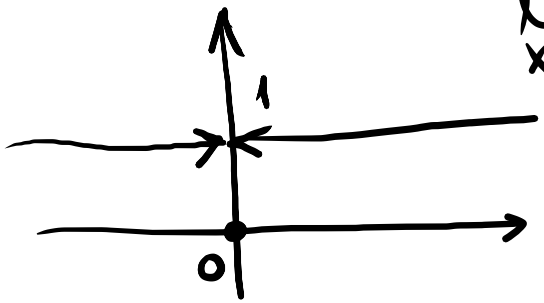
$$\lim_{\substack{x\to a \\ x\in A\cap(a,+\infty)}} f(x) =: \lim_{x\to a+0} f(x) = \lim_{x\to a+} f(x) = f(a+0)$$



ЛЕВИ ЛИМЕС:

$$\lim_{\substack{x\to a \\ x\in A\cap(-\infty,a)}} f(x) =: \lim_{x\to a-0} f(x) = \lim_{x\to a-} f(x) = f(a-0)$$

Пр. $f(x) = |\operatorname{sgn} x|$



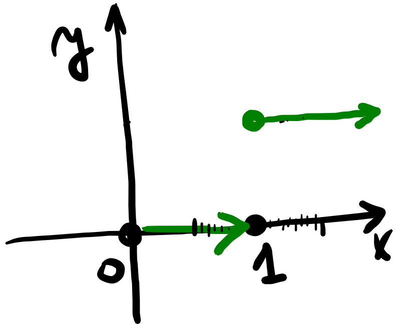
$$\lim_{x\to 0} |\operatorname{sgn} x| = 1$$

$$|\operatorname{sgn} 0| = 0$$

Пр. $f(x) = [x]$

$$\lim_{x\to 1+} [x] = 1$$

$$\lim_{x\to 1-} [x] = 0$$



Научимся !!

свойства пр. пред.

$$\lim_{x\to a} f_1(x) = \alpha$$

$$\lim_{x\to a} f_2(x) = \beta$$

$$\lim_{x\to a} (f_1(x) + f_2(x)) = \alpha + \beta$$

...