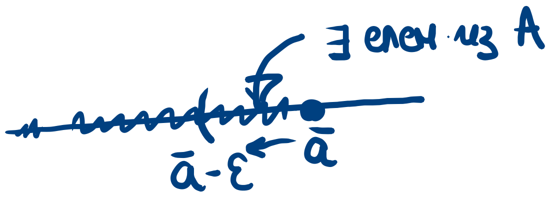
~ Супремум и инфимум - наставак ~

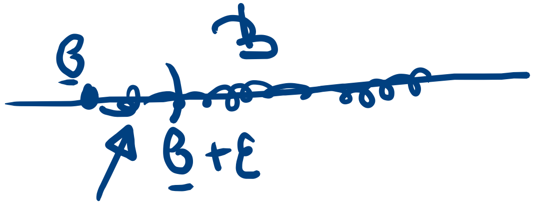
$A \subset \mathbb{R}$ огр. одозго, $A \neq \emptyset$ $\rightarrow$ $\exists \sup A$

$$\bar{a} = \sup A \iff \begin{cases} 1^\circ & (\forall a \in A)\ a \leq \bar{a} \\ 2^\circ & (\forall \varepsilon > 0)(\exists a \in A)\ a > \bar{a} - \varepsilon \end{cases}$$



$B \subset \mathbb{R}$ огр. одоздо, $B \neq \emptyset$ $\rightarrow$ $\exists \inf B$

$$\underline{b} = \inf B \iff \begin{cases} 1^\circ & (\forall b \in B)\ \underline{b} \leq b \\ 2^\circ & (\forall \varepsilon > 0)(\exists b \in B)\ b < \underline{b} + \varepsilon \end{cases}$$



A, B - скупови:
$A, B \subset \mathbb{R}$

$A + B = \{a + b \mid a \in A, b \in B\}$ нпр. $[0,1] + [0,1] = [0,2]$

$(0,1) + (0,1) = (0,2)$

$-A = \{-a \mid a \in A\}$

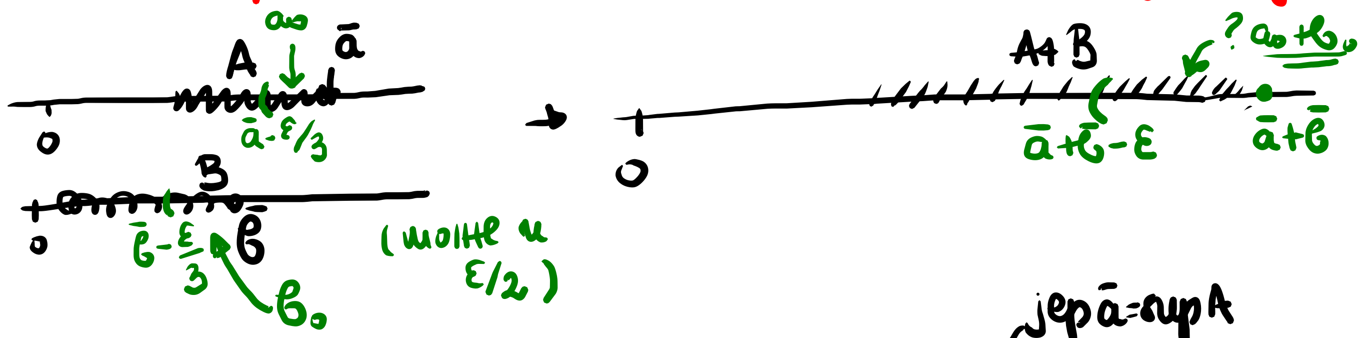
$A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$

① $A, B \neq \emptyset$, $A, B \subset \mathbb{R}$

вежба ☺

а) A, B огр. одозго: $\sup A + \sup B = \sup(A+B)$ б) A, B - огр. одоздо $\inf A + \inf B = \inf(A+B)$

1 а) $\bar{a} = \sup A$, $\bar{b} = \sup B$



Доказујемо:
$\bar{a}+\bar{b} = \sup(A+B)$?

1° $\bar{a}+\bar{b}$ горње огр. скупа $A+B$: произв. ел. из $A+B \ni x$
$\Rightarrow x = a+b$, за неко $a \in A$, $b \in B$

јер $\bar{a} = \sup A$

$$\left.\begin{array}{l} a \le \bar{a} \\ b \le \bar{b} \end{array}\right\} \Rightarrow \boxed{a+b \le \bar{a}+\bar{b}},$$

јер $\bar{b} = \sup B$

2° $\varepsilon > 0$ произвољан

$$\left.\begin{array}{l} \bar{a} = \sup A,\ \varepsilon/3 > 0 \Rightarrow \exists a_0 \in A \quad a_0 > \bar{a} - \varepsilon/3 \\ \bar{b} = \sup B,\ \varepsilon/3 > 0 \Rightarrow \exists b_0 \in B \quad b_0 > \bar{b} - \varepsilon/3 \end{array}\right\} + \Rightarrow \underbrace{a_0+b_0}_{\in A+B} > \bar{a}+\bar{b} - \frac{2\varepsilon}{3} > \bar{a}+\bar{b}-\varepsilon$$

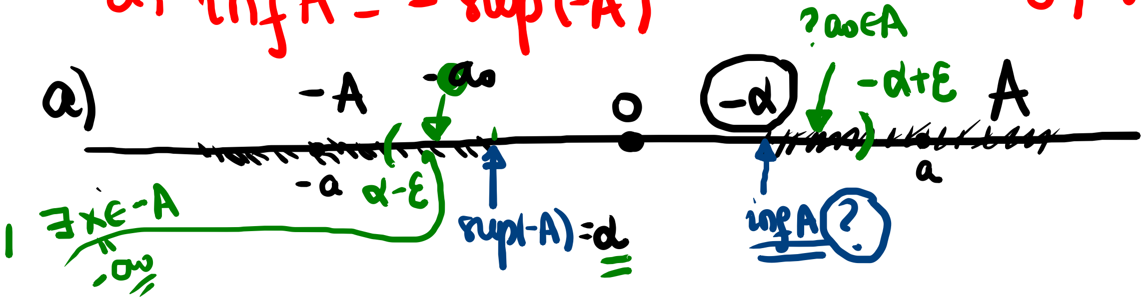
за $\forall \varepsilon > 0 \Rightarrow \bar{a}+\bar{b}$ најмање горње огр.

$$\Rightarrow \boxed{\bar{a}+\bar{b} = \sup(A+B)}$$

2: A-ограничен, $A \subset \mathbb{R}$, $A \neq \emptyset$

а) $\inf A = -\sup(-A)$ б) $\sup A = -\inf(-A)$

а)



$\sup(-A)=\alpha$

доказујемо: $-\alpha = \inf A$?

1° $-\alpha$ доње огр. за A ?

$a \in A$ $\quad a \geq -\alpha$? $\quad / \cdot(-1)$

$\Leftrightarrow -a \leq \alpha$

$\Leftrightarrow$ (Т) јер $\alpha = \sup(-A)$

✓

2° $\varepsilon > 0$ произв.

? $\exists a_0 \in A \quad a_0 < -\alpha + \varepsilon \quad / \cdot(-1)$

$\Leftrightarrow \exists a_0 \in A \quad \underbrace{-a_0}_{\in -A} > \alpha - \varepsilon$

$\Leftrightarrow$ (Т) јер $\alpha = \sup(-A)$ ✓

1°, 2° $\Rightarrow$ $\boxed{-\alpha = \inf A}$

за пример: $A\cdot B=\{a\cdot b \mid a\in A,\ b\in B\}$

$$\underbrace{\sup(0,1)}_{1}\cdot\underbrace{\sup(1,2)}_{2} \stackrel{?}{=} \underbrace{\sup\big(\underbrace{(0,1)\cdot(1,2)}_{(0,2)}\big)}_{=2}$$

✓

<u>али</u>:

$A=(-2,-1)$ $\sup A=-1$

$B=(1,2)$ $\sup B=2$

$$\sup(A\cdot B)=-1\neq(-1)\cdot 2$$

у општем случају

$$\sup A\cdot\sup B\neq\sup(A\cdot B)$$

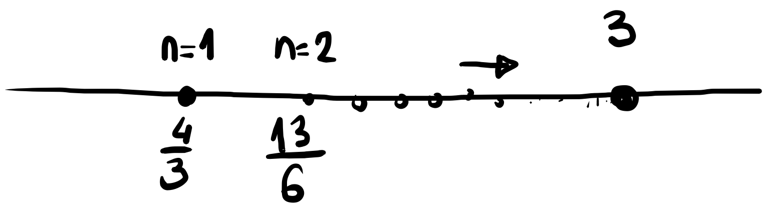
али важи: ако $A,B>0$

$(\forall a\in A\ a>0$ и $\forall b\in B\ b>0)$

$$\sup A\cdot\sup B=\sup(A\cdot B)$$

③ $A = \left\{ \frac{3n^2+1}{n^2+2} \mid n \in \mathbb{N} \right\}$ sup, inf, min, max ? (ако постоје)

$$\frac{3n^2+1}{n^2+2} = \frac{3(n^2+2)-5}{n^2+2} = 3 - \frac{5}{n^2+2}$$



$n=1: \quad 3-\frac{5}{3} = \frac{4}{3} \qquad n=2: \quad 3-\frac{5}{6} = \frac{13}{6}$

$$3 - \frac{5}{n^2+2} \overset{n=1}{\geq} 3 - \frac{5}{1^2+2} = \frac{4}{3} \in A, \quad \forall n \in \mathbb{N} \Rightarrow \boxed{\frac{4}{3} = \min A}, \ \boxed{\frac{4}{3} = \inf A}$$

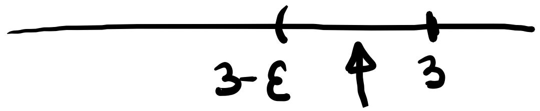
Доказујемо да је $3 = \sup A$:

① $\quad 3 \geq 3 - \frac{5}{n^2+2}, \quad \forall n \in \mathbb{N}$

$(>)$

3 јесте горње огр. за A

2°



A: $3-\frac{5}{n^2+2}$, $n\in\mathbb{N}$

$\varepsilon>0$? $\exists n$ $\quad 3-\frac{5}{n^2+2} > 3-\varepsilon$

$\Leftrightarrow \varepsilon > \frac{5}{n^2+2}$

$\Leftrightarrow n^2+2 > \frac{5}{\varepsilon}$

$\Leftrightarrow n^2 > \frac{5}{\varepsilon}-2$

Можемо изабрати $n_0\in\mathbb{N}$ $n_0 > \frac{5}{\varepsilon}-2$

нпр. $n_0 = \left[\frac{5}{\varepsilon}-2\right] + 3 \in \mathbb{N}$

тада $n_0^2 \geq n_0 > \frac{5}{\varepsilon}-2$ ✓

$1°, 2° \Rightarrow \boxed{3 = \sup A}$

$3 \notin A \Rightarrow \boxed{\nexists \max A}$

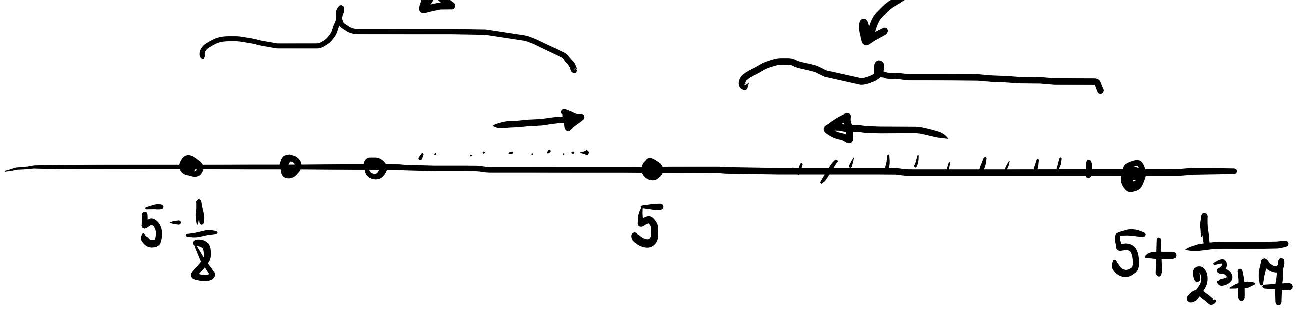
4. Нађи sup, inf, min, max (ако $\exists$), без доказа:

a) $A = \left\{ 5 + \frac{(-1)^n}{n^3+7} \;\middle|\; n \in \mathbb{N} \right\}$

$5 + \frac{(-1)^n}{n^3+7}$

n парно: $5 + \frac{1}{n^3+7}$ 2, 4, 6, ...

n непарно: $5 - \frac{1}{n^3+7}$ 1, 3, 5, 7 ...



$n=1$: $\frac{1}{1^3+7} > \frac{1}{n^3+7}$ за $n>1$

$\underbrace{5 - \frac{1}{1^3+7}} < 5 - \frac{1}{n^3+7}$ за $n>1$

$\boxed{5 - \frac{1}{8} = \min A = \inf A}$

за n парно:

$\underbrace{5 + \frac{1}{2^3+7}} > 5 + \frac{1}{n^3+7}$, $n>2$

$\boxed{5 + \frac{1}{15} = \max A = \sup A}$

δ) $B=\{\underbrace{\frac{n}{4}-\left[\frac{n}{4}\right]}_{\in[0,1)} \mid n\in\mathbb{N}\}$

mod4

$n=4k,\ k\in\mathbb{N}$: $\frac{n}{4}-\left[\frac{n}{4}\right]=k-\underbrace{[k]}_{k}=\{0\}$

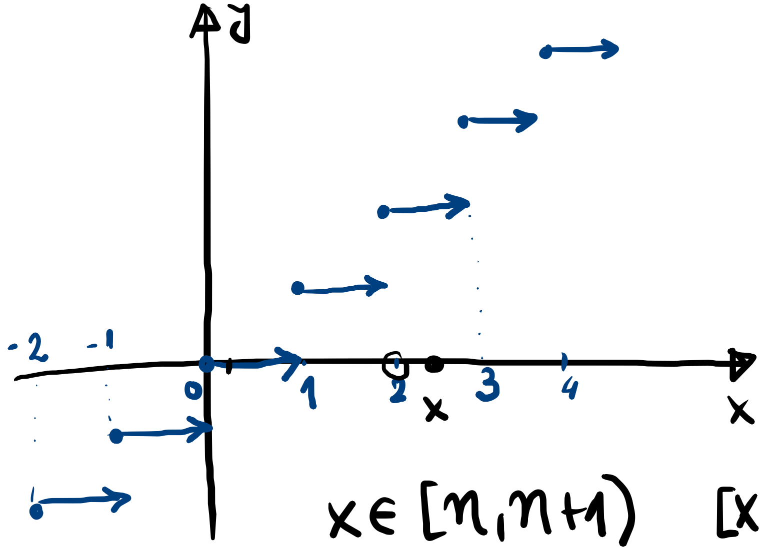
$n=4k+1$: $\frac{n}{4}-\left[\frac{n}{4}\right]=\frac{4k+1}{4}-\left[\frac{4k+1}{4}\right]$

$=k+\frac{1}{4}-\underbrace{\left[k+\frac{1}{4}\right]}_{k}$

$=\{\frac{1}{4}\}$

$n=4k+2$: $\frac{n}{4}-\left[\frac{n}{4}\right]=k+\frac{2}{4}-\underbrace{\left[k+\frac{2}{4}\right]}_{k}=\frac{2}{4}=\{\frac{1}{2}\}$

$n=4k+3$: . . . . $=\{\frac{3}{4}\}$



$x\in[n,n+1)$ $\quad [x]=n$

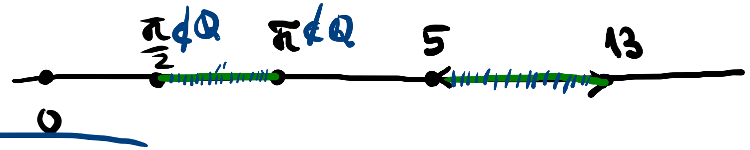
$\boxed{0\le x-[x]<1}$

$B=\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4}\}$

$0=\min B=\inf B$

$\frac{3}{4}=\max B=\sup B.$

в) $C = \underline{\underline{\mathbb{Q}}} \cap \left(\left[\frac{\pi}{2}, \pi\right] \cup (5, \underline{\underline{13}})\right)$



$\boxed{\frac{\pi}{2} = \inf C}$

$\frac{\pi}{2} \notin \mathbb{Q} \Rightarrow \frac{\pi}{2} \notin C \Rightarrow \boxed{\nexists \min C}$

$\boxed{13 = \sup C}$ $\quad \boxed{\nexists \max C}$ зер $13 \notin C$ (зер) затрада)

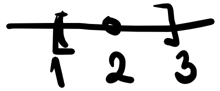
г) $D = \{x \in \mathbb{R} \mid x^2 \cdot \log_2 |x-2| \le 0\}$

$x^2 \cdot \log_2 |x-2| \le 0 \qquad |x-2| \neq 0 \quad \boxed{x \neq 2}$

$\Leftrightarrow x = 0 \ \vee \ \log_2 |x-2| \le 0$

$\Leftrightarrow x = 0 \ \vee \ |x-2| \le 2^0 = 1$

$\Leftrightarrow \underline{x = 0} \ \vee \ \underline{x \in [1, 3]}$



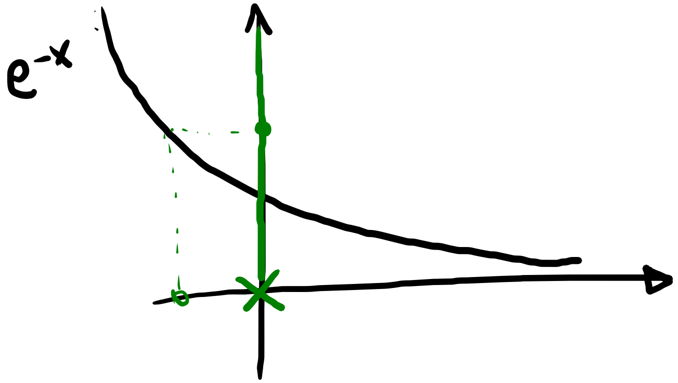
$\boxed{D = \{0\} \cup [1, 2) \cup (2, 3]}$

$0 = \min D = \inf D$

$3 = \max D = \sup D$

f) $E_1 = \{ f(x) \mid x \in \mathbb{R} \}$ $\quad f(x) = e^{-x}$

$$f(x) = e^{-x} = \frac{1}{e^x} = \left(\frac{1}{e}\right)^x \qquad a^x, \; a = \frac{1}{e} < 1$$



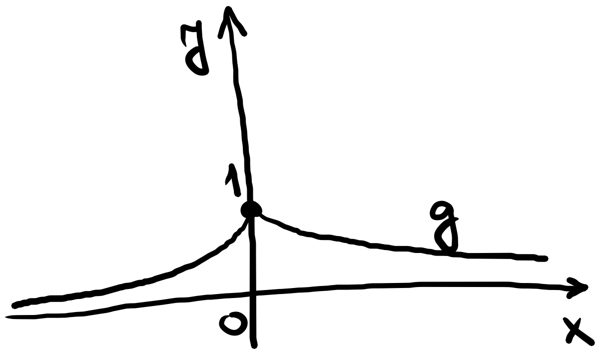
$E_1 = (0, +\infty)$

$0 = \inf E_1$, $\neg\exists \min E_1$ jer $0 \notin E_1$

E_1 nije og. ograđen $\Rightarrow$ $\neg\exists \sup E_1$ i $\neg\exists \max E_1$

$E_2 = \{ g(x) \mid x \in \mathbb{R} \}$, $\quad g(x) = e^{-|x|}$

$$g(x) = e^{-|x|} = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0 \end{cases}$$



$E_2 = (0, 1]$

$\max E_2 = 1 = \sup E_2$

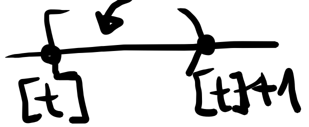
$\inf E_2 = 0 \notin E_2 \Rightarrow \neg\exists \inf E_2.$

5. x-фиксиран $x>0$

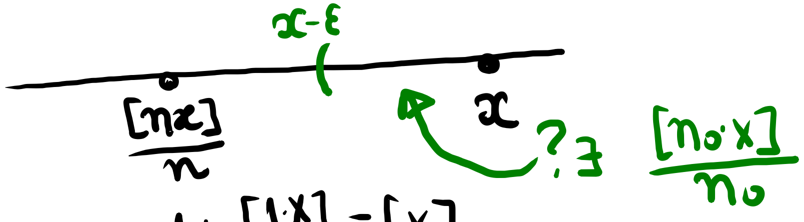
$A=\left\{ \frac{[nx]}{n} \mid n\in\mathbb{N} \right\}$ Доказати да је $x=\sup A$. Да ли је $x=\max A$?

$x \overset{?}{=} \sup A$

☺ $[t]\leq t<[t]+1$



1° $n\in\mathbb{N}$ $\quad \frac{[nx]}{n} \overset{?}{\leq} x \iff [nx]\leq nx \iff$ (+) ✓



$n=1:\ \frac{[1\cdot x]}{1}=[x]$

2° $\varepsilon>0$ произв.

$\overset{?}{\exists}\, n_0\in\mathbb{N}\quad \frac{[n_0\cdot x]}{n_0}>x-\varepsilon$

$\iff \exists n_0\in\mathbb{N}\quad [n_0\cdot x]>n_0\cdot x-n_0\cdot\varepsilon$

$\iff \exists n_0\in\mathbb{N}\quad \underbrace{[n_0\cdot x]}_{[t]}+n_0\cdot\varepsilon>\underbrace{n_0\cdot x}_{t}$

ово је сигурно тачно (из ☺) ако је $n_0\cdot\varepsilon>1$

Архимед $\Rightarrow$ такво n_0 постоји нпр. $n_0=\left[\frac{1}{\varepsilon}\right]+1$

$\exists n_0$ ✓

1°, 2° $\Rightarrow \boxed{x=\sup A}$

? $x \overset{?}{=} \max A \iff \underline{\underline{x}} \in A$

$\iff \exists n \in \mathbb{N} \quad x = \dfrac{[nx]}{n}$

$\iff \exists n \in \mathbb{N} \quad n \cdot x = [n \cdot x]$

$\iff \exists n \in \mathbb{N} \quad n \cdot \underline{\underline{x}} \in \mathbb{N}$

$\iff \boxed{\underline{x \in \mathbb{Q}}}$

$$\boxed{x = \max A \iff x \in \mathbb{Q}}$$

? $\exists n \in \mathbb{N} \quad n \cdot \sqrt{2} \in \mathbb{N}$

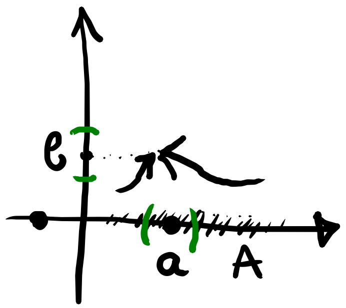
☹ НЕ!

$\dfrac{p}{q} \in \mathbb{Q}_+ \quad q \cdot \dfrac{p}{q} \in \mathbb{N}$

$= p$

"n"

Граничне вредности функција



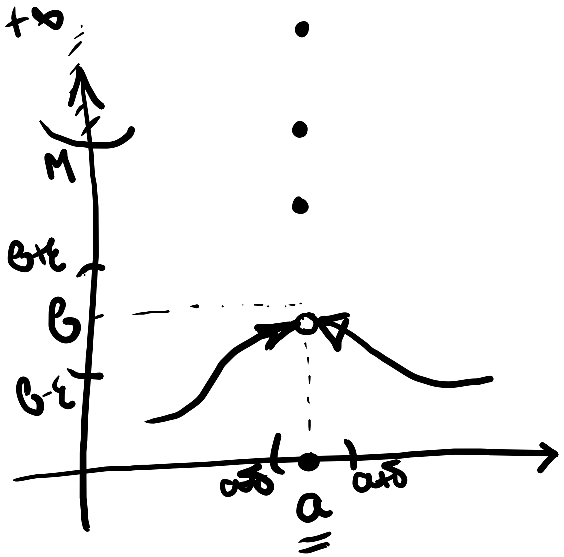
$f: A \to \mathbb{R} \quad A \subset \mathbb{R}$

a – тачка нагомилавања скупа A

$b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

$b = \lim\limits_{x \to a} f(x) \iff$ за $\forall$ околину $V(b)$ тачке b постоји околина $U(a)$ тачке a т.д.

$$(\forall x \in A) \quad x \in \underbrace{U(a) \setminus \{a\}}_{\mathring{U}(a)} \Rightarrow f(x) \in V(b)$$



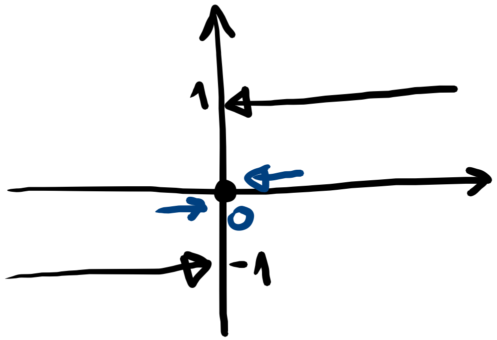
$\boxed{b \in \mathbb{R}}$ $(a \in \mathbb{R})$ $\quad (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A) \; 0 < |x-a| < \delta \Rightarrow |f(x) - b| < \varepsilon$

$\boxed{b = +\infty}$ $(a \in \mathbb{R})$ $\quad (\forall M > 0)(\exists \delta > 0)(\forall x \in A) \; 0 < |x-a| < \delta \Rightarrow f(x) > M$

$a = +\infty / -\infty, \; b = -\infty, \ldots$ ☺ све могућности за домаћи

Пр. $f(x) = \operatorname{sgn} x = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$

$$\boxed{\neg\exists \lim_{x\to 0} \operatorname{sgn} x}$$



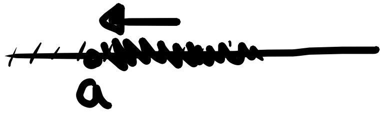
$$\lim_{x\to 0+} \operatorname{sgn} x = 1$$

$$\lim_{x\to 0-} \operatorname{sgn} x = -1$$

<u>ДЕСНИ ЛИМЕС f</u>:

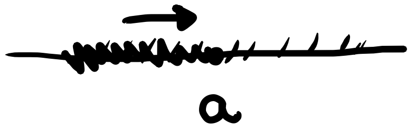
$f: A \to \mathbb{R}$

$$\lim_{\substack{x\to a \\ x\in A\cap(a,+\infty)}} f(x) =: \lim_{x\to a+0} f(x) = \underset{\sim\sim\sim}{\lim_{x\to a+} f(x)} = f(a+0)$$

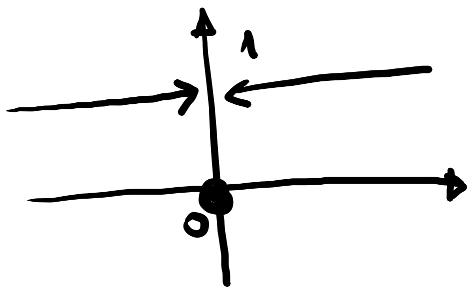


<u>ЛЕВИ ЛИМЕС f</u>

$$\lim_{\substack{x\to a \\ x\in A\cap(-\infty,a)}} f(x) =: \lim_{x\to a-0} f(x) = \underset{\sim\sim\sim}{\lim_{x\to a-} f(x)} = f(a-0)$$

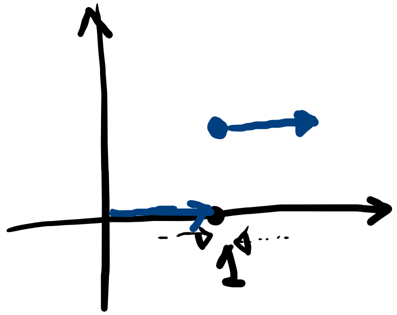


Пр2 $f(x) = |\operatorname{sgn} x|$



$$\lim_{x\to 0} |\operatorname{sgn} x| = 1 \neq f(0)$$

Пр. $f(x) = [x]$



$$\lim_{x\to 1-} \overbrace{[x]}^{0} = 0$$

$$\lim_{x\to 1+} \underbrace{[x]}_{1} = 1$$

домаћи: својства гр. вредности $\left(\begin{array}{ll} f \to a & f+g \to a+b \\ g \to b & \dots \end{array} \right)$