~ Неједнакости ~

Бернулијева неједнакост:

$x > -1,\ n \in \mathbb{N}:\quad (1+x)^n \geq 1+nx \quad (*)$

$\left[\ (1+x)^n = 1^n + \binom{n}{1} 1^{n-1} \cdot x + \binom{n}{2} \cdot x^2 + \dots = 1 + n \cdot x + \sqcup\ \right]$

индукција по n:

1) база $n=1$: $(1+x)^1 \geq 1+1 \cdot x$ ✓

2) $n \mapsto n+1$ инд. хип. важи за n $(*)$

за $n+1$: $(1+x)^{n+1} = \underbrace{(1+x)^n}_{} \cdot (1+x)$

$\geq (1+n \cdot x) \cdot (1+x)$

$(1+x)^n \geq 1+nx \quad /\cdot\ (1+x) > 0$

$x > -1$

$\sqcup\sqcup \geq \sqcup\sqcup$

$= 1 + \underbrace{x + nx}_{(n+1)x} + \underbrace{nx^2}_{\geq 0}$

$\geq 1 + (n+1)x$ за $n+1$ ✓

1), 2) $\Rightarrow$ тм.

$\left[\ x=-5,\ n=3 \right.$

$Л = (-4)^3 = -64$

$Д = 1+3 \cdot (-5) = -14$

$\left. Л \ngeq Д\ \right]$

$\boxed{x^2 \geq 0, \forall x \in \mathbb{R}}$ једнакост важи акко $x=0$

① Доказати:

а) $\forall a,b \in \mathbb{R}$ $a^2+b^2 \geq 2ab$

$\iff a^2+b^2-2ab \geq 0$

$\iff (a-b)^2 \geq 0$

$\iff$ T

„=": $x^2 \geq 0$ $x=a-b$

једн. важи акко $a-b=0$

$\boxed{a=b}$

б) $a,b,c \in \mathbb{R}$: $a^2+b^2+c^2 \geq ab+bc+ca$

$$
\begin{aligned}
a^2+b^2 &\geq 2ab \\
b^2+c^2 &\geq 2bc \\
+ \quad a^2+c^2 &\geq 2ac \\
\hline
2(a^2+b^2+c^2) &\geq 2\cdot(ab+bc+ac) \quad /:2
\end{aligned}
$$

„=": ако важи на сва 3 места

$$\left.\begin{aligned} a&=b \\ b&=c \\ a&=c \end{aligned}\right\} \underline{\underline{a=b=c}}$$

в) $a>0$: $a+\frac{1}{a}\geq 2$

$$a+\frac{1}{a}=(\sqrt{a})^2+\left(\frac{1}{\sqrt{a}}\right)^2 \overset{\text{geoal}}{\geq} 2\cdot\sqrt{a}\cdot\frac{1}{\sqrt{a}}=2 \quad \checkmark$$

г) $a,b\in\mathbb{R}$, $a+b\geq 0$ док: $a^3+b^3\geq a^2b+b^2a$

$$\Leftrightarrow a^3+b^3-a^2b-b^2a\geq 0$$

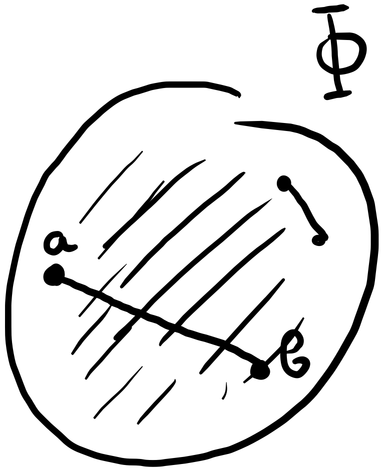
$$\Leftrightarrow a^2(a-b)+b^2\cdot(b-a)\geq 0$$

$$\Leftrightarrow (a^2-b^2)(a-b)\geq 0$$

$$\Leftrightarrow \underbrace{(a+b)}_{\substack{\text{услов}\\ \geq 0}}\underbrace{(a-b)(a-b)}_{(a-b)^2\geq 0}\geq 0$$

$$\Leftrightarrow T$$

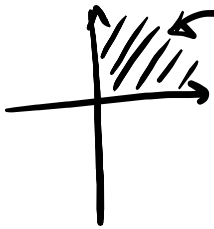
КОНВЕКСАН СКУП

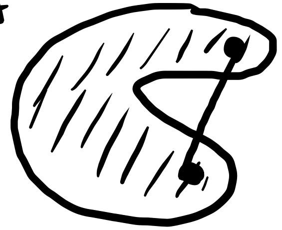


$\Phi \subset \mathbb{R}^2$

Φ конвексан скуп ако

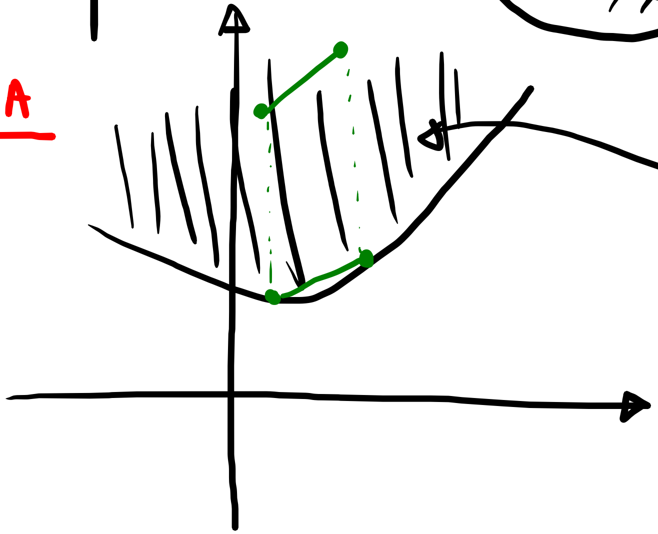
за $\forall a,b \in \Phi$ важи $[a,b] \subset \Phi$

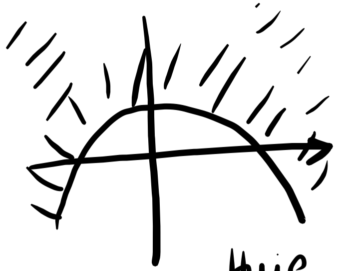




КОНВЕКСНА ФУНКЦИЈА

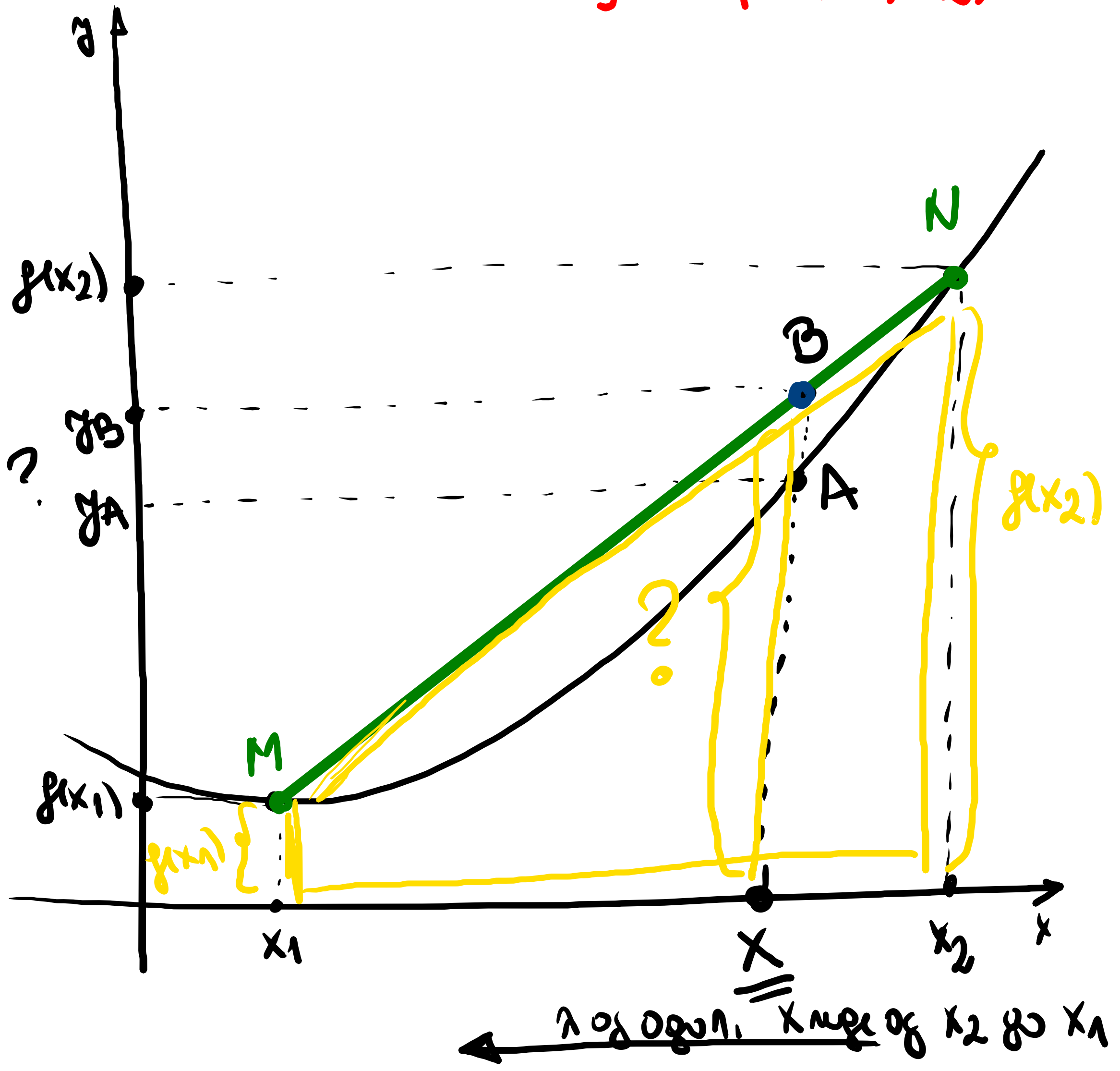
„неформално":





ДЕФ. $f:(a,b)\to\mathbb{R}$ је конвексна ако: $\forall x_1,x_2\in(a,b)$, $\forall\lambda\in[0,1)$ важи

$$f(\lambda x_1+(1-\lambda)x_2)\le\lambda\cdot f(x_1)+(1-\lambda)f(x_2).$$



Желимо да закључимо да је дуж MN изнад графика

за $\forall x\in(x_1,x_2)$ $\uparrow$ $\underline{\underline{A,B}}$

тачка A испод B ? $\boxed{y_A \overset{?}{\le} y_B}$

$x\in(x_1,x_2)$: $x=\lambda x_1+(1-\lambda)x_2$

$\lambda=0$: $x=x_2$; $\lambda=1$: $x=x_1$

$$\underbrace{f(\lambda x_1+(1-\lambda)x_2)}_{}=f(x)=y_A\overset{?}{\le}y_B=\lambda\cdot f(x_1)+(1-\lambda)\cdot f(x_2)$$

y_B ← Талесова теорема

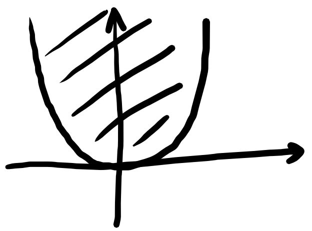
из деф: $f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$ $\quad (\lambda=\frac{1}{2})$

Јенсенова неједнакост: f конвексна на $(a,b)\subset\mathbb{R}$ $\quad(\mathbb{R})\quad$ $\forall x_1,\dots,x_n\in(a,b)$

$$f\left(\frac{x_1+x_2+\dots+x_n}{n}\right) \le \frac{f(x_1)+\dots+f(x_n)}{n}$$

Неједнакости средина

$f(x)=x^2$ конвексна на $\mathbb{R}$ $\quad x_1,\dots,x_n\in\mathbb{R}$



Јенсен

$$\left(\frac{x_1+\dots+x_n}{n}\right)^2 \le \frac{x_1^2+\dots+x_n^2}{n}$$

$$\frac{x_1+\dots+x_n}{n} \le \sqrt{\frac{x_1^2+\dots+x_n^2}{n}}$$

A_n аритметичка средина $\quad$ K_n квадратна средина

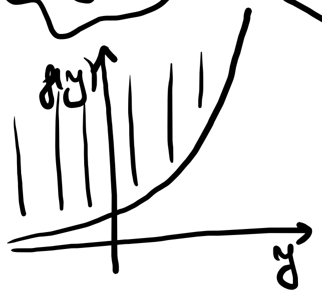
$\sqrt{a^2}=?$

$\sqrt{a^2}=|a|$

$a^2\le x$

$\underline{a}\le|a|\le\underline{\sqrt{x}}$

$f(y)=e^y$ конв на $\mathbb{R}$ $y_1,\dots,y_n\in\mathbb{R}$



Јенсен

$$e^{\frac{y_1+\dots+y_n}{n}} \le \frac{e^{y_1}+\dots+e^{y_n}}{n}$$

$$e^{\frac{y_1+\dots+y_n}{n}} = \sqrt[n]{e^{y_1}\cdot e^{y_2}\cdots e^{y_n}}$$

$$a^{\frac{m}{n}}=\sqrt[n]{a^m}$$

$x_i=e^{y_i}$ y_i - произв $\mathbb{R}$

$x_i \to$ прои. $(0,+\infty)$

$f(\text{прои})=e^{\text{прои}}$

G_n геометриска средина

$$\boxed{\sqrt[n]{x_1\cdot x_2\cdots x_n}} \le \boxed{\frac{x_1+\dots+x_n}{n}}, \quad \forall x_1,\dots,x_n>0$$

A_n

- $H_n=\dfrac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_n}}$

$H_n \le G_n$, $x_1,\dots,x_n>0$

$$\Leftrightarrow \frac{n}{\frac{1}{x_1}+\dots+\frac{1}{x_n}} \le \sqrt[n]{x_1\cdots x_n}$$ сл. доз.

$$\Leftrightarrow \boxed{\frac{\frac{1}{x_1}+\dots+\frac{1}{x_n}}{n}} \ge \frac{1}{\sqrt[n]{x_1\cdots x_n}} = \boxed{\sqrt[n]{\frac{1}{x_1}\cdots\frac{1}{x_n}}} \Leftrightarrow (T) \checkmark$$

Ар.ср. за $\frac{1}{x_1},\dots,\frac{1}{x_n}$ → Геом. ср.

За все $x_1, \dots, x_n > 0$:

$$\underbrace{\frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}}}_{H_n} \le \underbrace{\sqrt[n]{x_1 \cdot \ldots \cdot x_n}}_{G_n} \le \underbrace{\frac{x_1 + \dots + x_n}{n}}_{A_n} \le \underbrace{\sqrt{\frac{x_1^2 + \dots + x_n^2}{n}}}_{K_n}$$

"=":

$x_1 = \dots = x_n = t$

$H_n = G_n = A_n = K_n = \underline{\underline{t}}$

$$M_s = \left(\frac{x_1^s + x_2^s + \dots + x_n^s}{n}\right)^{\frac{1}{s}}$$

$s \le t \Rightarrow M_s \le M_t$

$$\begin{array}{ccccccc} H_n & \le & G_n & \le & A_n & \le & K_n \\ \| & & \text{!!} & & \| & & \| \\ M_{-1} & & \underline{\underline{M_0}} & & M_1 & & M_2 \end{array}$$

Пр: $x_1, \dots, x_n > 0$: $(x_1 + \dots + x_n) \cdot \left(\frac{1}{x_1} + \dots + \frac{1}{x_n}\right) \ge n^2$

$H_n \le A_n \quad \frac{n}{\sqcup} \le \frac{\sqcup}{n}$

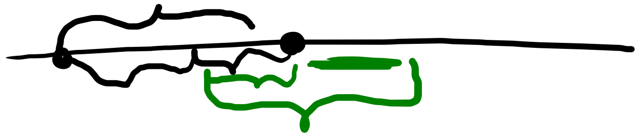
За вкъщи: Док:

$$\left(\frac{2n+1}{3}\right)^{\frac{n(n+1)}{2}} \ge \underbrace{1 \cdot 2^2 \cdot 3^3 \cdot \ldots \cdot n^n}_{\text{ср } G_n} \ge \left(\frac{n+1}{2}\right)^{\frac{n(n+1)}{2}}$$

$$\underbrace{1, 2, 2, 3, 3, 3, \dots, \underbrace{n, \dots, n}_{n}}_{\frac{n(n+1)}{2}}$$

Коши-Шварц: $a_1,\dots,a_n,b_1,\dots,b_n\in\mathbb{R}$: $\quad (a_1b_1+\dots+a_nb_n)^2\le(a_1^2+\dots+a_n^2)\cdot(b_1^2+\dots+b_n^2)$

Неједнакост Δ: $x,y\in\mathbb{R}$: $\quad |x+y|\le|x|+|y|$



* $x_1,\dots,x_n\in\mathbb{R}$ $\quad |x_1+\dots+x_n|\le|x_1|+\dots+|x_n|$

доказ: инд. по n

1) $n=2$: $|x_1+x_2|\le|x_1|+|x_2|$ ✓ неј Δ

2) $n\mapsto n+1$ $\quad$ И.Х. важи за n $\quad \left|\sum_{i=1}^{n}x_i\right|\le\sum_{i=1}^{n}|x_i|$

за $n+1$:

$$\left|\sum_{i=1}^{n+1}x_i\right| = \left|\underbrace{\sum_{i=1}^{n}x_i}_{I}+\underbrace{x_{n+1}}_{II}\right| \overset{\Delta}{\le} \left|\sum_{i=1}^{n}x_i\right|+|x_{n+1}| \overset{\text{И.Х.}}{\le} \sum_{i=1}^{n}|x_i|+|x_{n+1}| = \sum_{i=1}^{n+1}|x_i| \quad ✓$$

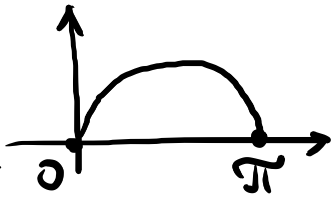
1), 2) $\Rightarrow$ важи за $\forall n$ □

• $x_1, \dots, x_n \in [0, \pi]$ док: $\left|\sin\left(\sum_{i=1}^{n} x_i\right)\right| \leq \sin x_1 + \dots + \sin x_n$ ⊛

1) $n=1$: $|\sin x_1| \leq \sin x_1$

$x_1 \in [0, \pi]$

$\sin x_1 \geq 0$ → важи $\boxed{=}$



2) $n \mapsto n+1$ н.х. важи за n ⊛

за $n+1$: $|\sin(\underbrace{x_1 + \dots + x_n} + \underbrace{x_{n+1}})| = |\underbrace{\sin(x_1 + \dots + x_n) \cdot \cos x_{n+1}} + \underbrace{\cos(x_1 + \dots + x_n) \cdot \sin x_{n+1}}|$

$\overset{\text{нер.}\Delta}{\leq} |\sin(x_1 + \dots + x_n)| \cdot \underbrace{|\cos x_{n+1}|}_{\leq 1} + \underbrace{|\cos(x_1 + \dots + x_n)|}_{\leq 1} \cdot |\sin x_{n+1}|$

$|a \cdot b| = |a| \cdot |b|$

$|\cos t| \leq 1$

$a > 0 \quad a \cdot |\cos t| \leq a$

$\leq |\sin(x_1 + \dots + x_n)| + \underbrace{|\sin x_{n+1}|}_{= \, \leftarrow \, x_{n+1} \in [0, \pi], \, \sin \geq 0}$

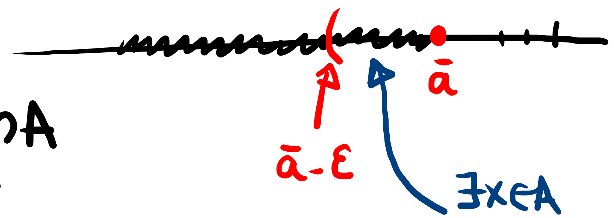
$\overset{\text{н.х.}}{\leq} \sin x_1 + \dots + \sin x_n + \sin x_{n+1}$

$\Rightarrow$ важи за $n+1$

1), 2) $\Rightarrow$ важи за $\forall n$ □

~ Супремум и инфимум ~

$A \subset \mathbb{R},\ A \neq \emptyset$



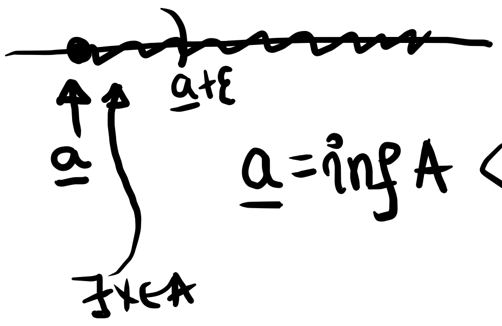
СУПРЕМУМ

A - ограничен одозго, $A \neq \emptyset$ $\xrightarrow[\text{супремума}]{\text{аксиома}}$ $\exists\ \sup A$

$\sup A$ = најмање горње ограничење скупа A

$$\bar{a} = \sup A \iff \begin{cases} 1^\circ \text{ горње ор. } (\forall a \in A)\ a \leq \bar{a} \\ 2^\circ \text{ најмање: } (\forall \varepsilon > 0)(\exists x \in A)\ x > \bar{a} - \varepsilon \end{cases}$$

ИНФИМУМ



$\inf A$ = највеће доње ограничење

$$\underline{a} = \inf A \iff \begin{cases} 1^\circ \text{ доње ор. } (\forall a \in A)\ \underline{a} \leq a \\ 2^\circ\ (\forall \varepsilon > 0)(\exists x \in A)\ x < \underline{a} + \varepsilon \end{cases}$$

- $\alpha = \max A \iff \alpha \in A \wedge (\forall a \in A)\, a \leq \alpha$

max & sup

$$\alpha = \max A \Rightarrow \sup A = \alpha$$
$$\not\Leftarrow$$

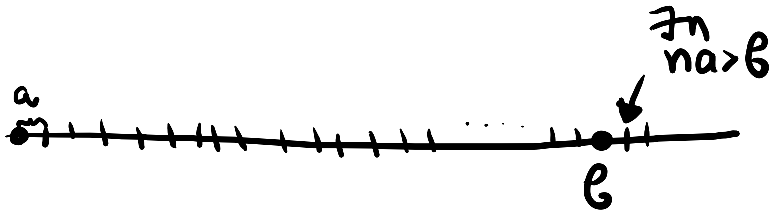
inf & min

за вежбу

$$\alpha = \sup A,\ \alpha \in A \Rightarrow \alpha = \max A$$
$$\alpha = \sup A,\ \alpha \notin A \Rightarrow \neg\exists \max A$$

Архимедова аксиома (теорема):

$\forall a, b \in \mathbb{R},\ a > 0\quad \exists n \in \mathbb{N}\quad \underline{\underline{n \cdot a > b}}$



① Одредити sup, inf, min, max скупа A, ако постоје.

a) $A = \{2 - \frac{1}{n} \mid n \in \mathbb{N}\}$ $2-\frac{1}{1}=1,\ 2-\frac{1}{2}=\frac{3}{2},\ 2-\frac{1}{3}, \dots$

inf A: $\boxed{1 = \min A}$ јер $1 \le 2 - \frac{1}{n}, \forall n \in \mathbb{N}$

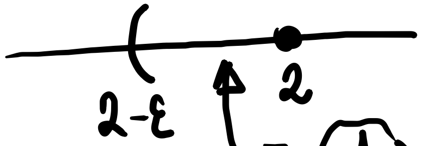
$\Downarrow$

$\boxed{1 = \inf A}$

sup A: Доказујемо: $2 = \sup A$

1° горње огр.: $2 - \frac{1}{n} \le 2, \forall n \in \mathbb{N}$ ✓

2° $\varepsilon > 0$ фиксирано → тражимо $n_0 \in \mathbb{N}$ тд. $2 - \frac{1}{n_0} > 2 - \varepsilon$



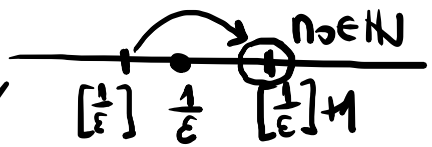
$\exists\ 2 - \frac{1}{n_0}$ за неко n_0 $= n_0(\varepsilon)$

$\iff -\frac{1}{n_0} > -\varepsilon \iff \varepsilon > \frac{1}{n_0} \iff n_0 \cdot \varepsilon > 1$

∃ n_0 засигурно из Арх. акс.

једно такво n_0:

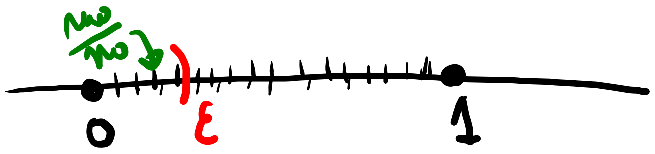
$n_0 \cdot \varepsilon > 1 \iff n_0 > \frac{1}{\varepsilon}$ $\boxed{n_0 := [1/\varepsilon] + 1}$ ✓



Тако за $\forall \varepsilon \Rightarrow$ 2 јесте најмање горње огр., $\boxed{2 = \sup A}$

$2 = \sup A \notin A \Rightarrow \boxed{\neg\exists \max A}$

б) $A = \left\{ \frac{m}{n} \mid m,n \in \mathbb{N},\ m<n \right\} = (0,1) \cap \mathbb{Q}$



$\boxed{\inf A}$ 1° $0 < \frac{m}{n}$, $\forall m,n \in \mathbb{N}$ ✓

2° $\varepsilon > 0$ фикс.

$? \exists \frac{m_0}{n_0} \in A$ тд. $\frac{m_0}{n_0} < \varepsilon$

узимо: $\underline{m_0 = 1}$

$? \exists n_0$ тд. $\frac{1}{n_0} < \varepsilon$, $n_0 > 1$

$\iff 1 < \varepsilon \cdot n_0$

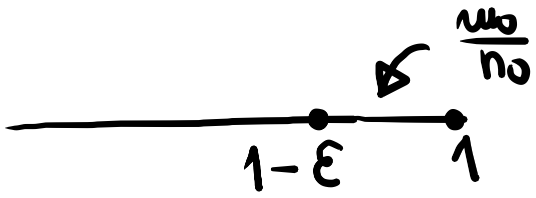
$\exists$ такво n_0 из Арх. аксиоме ✓

1°, 2° $\Rightarrow \boxed{0 = \inf A} \notin A \Rightarrow \boxed{\neg\exists \min A}$

$\boxed{\sup A} = 1$ (за вежбу)

1° ....

2°



$\varepsilon > 0$ фиксирано

$? \exists m_0, n_0 \in \mathbb{N}$ $m_0 < n_0$

тд. $\frac{m_0}{n_0} > 1 - \varepsilon$

„што ближе 1"

$\rightsquigarrow m_0$ што ближе n_0

узимо: $\boxed{m_0 = n_0 - 1}$

$? \exists n_0 : \frac{n_0 - 1}{n_0} > 1 - \varepsilon$

$\iff 1 - \frac{1}{n_0} > 1 - \varepsilon \iff \varepsilon > \frac{1}{n_0}$

...

$1 \notin A \Rightarrow \neg\exists \max A$.

в) $A=\left\{\frac{m}{n}+\frac{n}{m} \mid m,n\in\mathbb{N}\right\}$

A није ограничен одозго $\Rightarrow$ $\boxed{\neg\exists \sup A,\ \neg\exists \max A}$

за свако $N\in\mathbb{N}$: $\frac{N}{1}+\frac{1}{N}\in A$, $\frac{N}{1}+\frac{1}{N}>N$

inf & min

за $m=n$: $\frac{m}{n}+\frac{n}{m}=1+1=2\in A$

$m,n\in\mathbb{N}$: $\frac{m}{n}+\frac{n}{m}\geq 2$ тј. сви елементи A су ≥ 2

$\Rightarrow$ $\boxed{2=\min A=\inf A}$

$a+\frac{1}{a}\geq 2$, $a>0$ (први час)

□