Математичка индукција

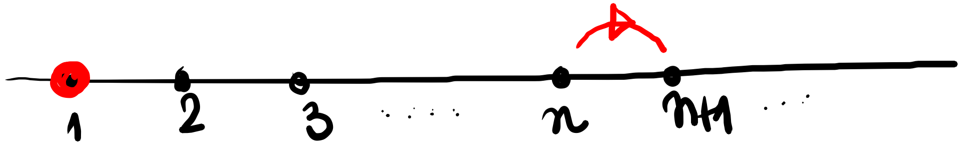
$P(n)$ - својство

1) БАЗА важи $P(1)$

2) ИНДУКТИВНИ КОРАК:

$$\underbrace{P(n)}_{\text{ИНДУКТИВНА ХИПОТЕЗА}} \Rightarrow P(n+1) \quad , \forall n \in \mathbb{N}$$

1), 2) $\Rightarrow$ важи за све $n \in \mathbb{N}$ $P(n)$



1) $P(n_0)$ ✓

2) $\forall n \geq n_0 : P(n) \Rightarrow P(n+1)$

1), 2) $\Rightarrow$ $P(n)$ важи за све $n \geq \underline{\underline{n_0}}$

① Докажете:

а) $1+2+\dots+n=\frac{n(n+1)}{2}$

б) $1^2+2^2+\dots+n^2=\frac{n(n+1)(2n+1)}{6}$

IB. в) $1^3+2^3+\dots+n^3=\left(\frac{n(n+1)}{2}\right)^3$

г) $1+3+5+\dots+(2n-1)=n^2$

д) $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\dots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

☺ мали Гаус

$$S=1+2+\dots+(n-1)+n$$
$$S=n+(n-1)+\dots+2+1 \quad +$$

k

$n+1-k$

$$2S=n(n+1) \Rightarrow S=\frac{n(n+1)}{2}$$

а) $P(n):\ 1+2+\dots+n=\frac{n(n+1)}{2}$

1) база $n=1$: $1=\frac{1(1+1)}{2}$ ✓

2) $\boxed{n\mapsto n+1}$

И.Х. важи $P(n)$

за $n+1$: $\underbrace{1+2+\dots+n}_{\text{за } n}+(n+1)\overset{?}{=}\frac{(n+1)(n+2)}{2}$

$$\underbrace{1+2+\dots+n}+(n+1)\overset{\text{И.Х.}}{=}\frac{n(n+1)}{2}+(n+1)$$
$$=\frac{1}{2}(n+1)(n+2)$$ ✓✓

важи за $n+1$ ако важи за n

1),2) $\Rightarrow P(n)$ важи за $\forall n\in\mathbb{N}$

в) $P(n): \; 1^3+2^3+\dots+n^3=\left(\frac{n(n+1)}{2}\right)^2$

1) $n=1: \; 1^3=\left(\frac{1(1+1)}{2}\right)^2$ ✓

2) $\boxed{n\mapsto n+1}$ (ИХ) $P(n)$ важи за n

за $n+1$:

$$\underbrace{1^3+2^3+\dots+n^3}_{\text{за } n}+\underline{\underline{(n+1)^3}}\overset{\text{ИХ}}{=}\left(\frac{n(n+1)}{2}\right)^2+(n+1)^3$$

$$=\frac{1}{4}(n+1)^2\underbrace{\left(n^2+4(n+1)\right)}_{(n+2)^2}$$

$$=\left(\frac{(n+1)(n+2)}{2}\right)^2$$

✓ за $n+1$

1), 2) $\Rightarrow P(n)$ важи за $\forall n\in\mathbb{N}$

д) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

I начин: индукција

II начин:

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}=$$

$$=\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\dots+\cancel{\frac{1}{n}}-\frac{1}{n+1}$$

$$=1-\frac{1}{n+1}$$

$$=\frac{n}{n+1}$$

② Доказати:

а) $7 \mid 2^{n+1}+3^{2n-1}$

б) $133 \mid 11^{n+2}+12^{2n+1}$

за вежбу

а) I начин: $2^1 \equiv ? \mod 7$
$2^2 \equiv$
$\vdots$
} период

II начин: индукција:

① $n=1$: $2^{1+1}+3^{2-1}=4+3=7$ $\quad 7 \mid 7$ ✓

② $n \mapsto n+1$ $\quad$ (И.Х.) $7 \mid A_n = 2^{n+1}+3^{2n-1}$

за $n+1$: $A_{n+1} = 2^{n+2}+3^{2(n+1)-1} = 2^{n+2}+3^{2n+1}$

$$= 2\cdot 2^{n+1} + \underset{2+7}{\underset{\|}{9}}\cdot 3^{2n-1} = 2\cdot(\underbrace{2^{n+1}+3^{2n-1}}_{A_n\ \text{(И.Х.) дељ. са } 7}) + \underbrace{7\cdot 3^{2n-1}}_{\text{дељ. са } 7}$$

$7 \mid A_{n+1}$ ✓✓

$\boxed{1),2) \Rightarrow \forall n\in\mathbb{N}}$

③ Док:

а) $\forall n \in \mathbb{N}$: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$

б) $\forall n \geq 2$: $\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$

в) $\forall n \geq 5$: $n^2 < 2^n$

г) $\forall n \in \mathbb{N}$: $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \leq \sqrt{\frac{1}{3n+1}}$

вешба

а) 1) $n=1$: $\frac{1}{\sqrt{1}} \overset{?}{>} 2(\sqrt{2}-1)$

$\Leftrightarrow \frac{1}{2} > \sqrt{2} - 1$

$\Leftrightarrow \frac{3}{2} > \sqrt{2} \quad |^2$ две >0

$\Leftrightarrow \frac{9}{4} > 2$

$\Leftrightarrow$ Ⓣ

2) $n \longmapsto n+1$

И.Х.: $\underbrace{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}}_{A_n} > \underbrace{2(\sqrt{n+1} - 1)}_{B_n}$

За $n+1$:

$$\underbrace{\overbrace{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}}^{A_n} + \frac{1}{\sqrt{n+1}}}_{A_{n+1}} \overset{\boxed{?}}{>} \underbrace{2(\sqrt{n+2} - 1)}_{B_{n+1}}$$

$$A_{n+1} = A_n + \frac{1}{\sqrt{n+1}} \overset{\text{И.Х.}}{>} \underbrace{2(\sqrt{n+1} - 1) + \frac{1}{\sqrt{n+1}}}_{C}$$

Било би лесно да видим $C > B_{n+1}$

тогава би: $A_{n+1} > C > B_{n+1} \Rightarrow A_{n+1} > B_{n+1}$

$a > b$

✓ c $\boxed{?}$

Искаме да докажем: $C > B_{n+1}$ ⓘ?

$$2(\sqrt{n+1}-\cancel{1})+\frac{1}{\sqrt{n+1}} \overset{?}{>} 2(\sqrt{n+2}-\cancel{1})$$

$\Leftrightarrow 2\sqrt{n+1}+\frac{1}{\sqrt{n+1}} > 2\sqrt{n+2} \quad / \cdot\sqrt{n+1}$

$\Leftrightarrow 2(n+1)+1 > 2\sqrt{(n+1)(n+2)} \quad /^2$ дч>0

$\Leftrightarrow 4n^2+12n+9 > \underbrace{4(n^2+3n+2)}_{4n^2+12n+8}$

$\Leftrightarrow$ Ⓣ

$\Rightarrow A_{n+1} > B_{n+1}$ т.е. вярно за $n+1$ ✓

1), 2) $\Rightarrow \forall n \in \mathbb{N}$ ✓

б) $\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$, $n \geq 2$

1) $n=2$: $\frac{4^2}{3} < \frac{4!}{(2!)^2} = 6$ ✓

2) $\boxed{n \mapsto n+1}$ и.х. $\underline{4^n (n!)^2} < (n+1)\cdot(2n)!$

за $n+1$: $\underbrace{4^{n+1}\cdot((n+1)!)^2}_{L} \overset{?}{<} \underbrace{(n+2)\cdot(2(n+1))!}_{D}$ ⊛

$L = 4\cdot\underline{4^n\cdot(n!)^2}\cdot(n+1)^2 \overset{и.х.}{<} \underbrace{4\cdot(n+1)\cdot(2n)!\cdot(n+1)^2}_{C}$

Искаме да док: $C < D$

$\Leftrightarrow 4\cdot(n+1)^3\cdot(2n)! < (n+2)(2n+2)! \quad /:(2n)!$

$\Leftrightarrow 4\cdot(n+1)^3 < (n+2)\cdot\cancel{(2n+2)}(2n+1)$

$\Leftrightarrow 2(n+1)^2 < (n+2)(2n+1)$

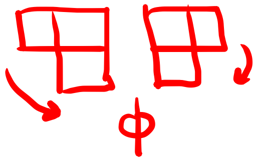
$\Leftrightarrow 2n^2+4n+2 < 2n^2+5n+2 \Leftrightarrow$ Ⓣ

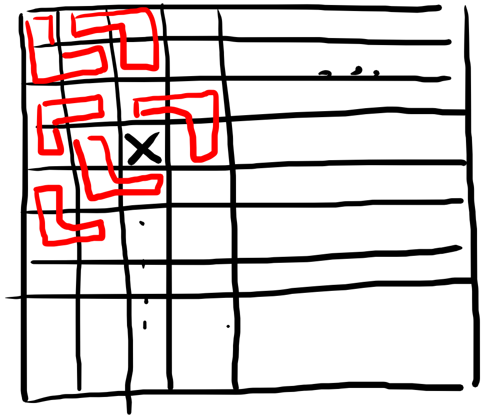
$L<C,\ C<D \Rightarrow L<D$ ⊛ $\Rightarrow$ вярно за $n+1$ ✓

1),2) $\Rightarrow \forall n.$

④ Табла $2^n \times 2^n$ без 1 поља

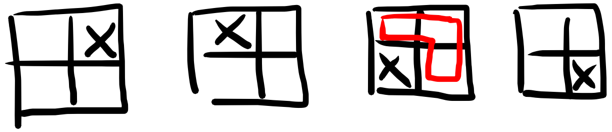
Док. да се може поплочати, без преклапања, фигурама Ф.





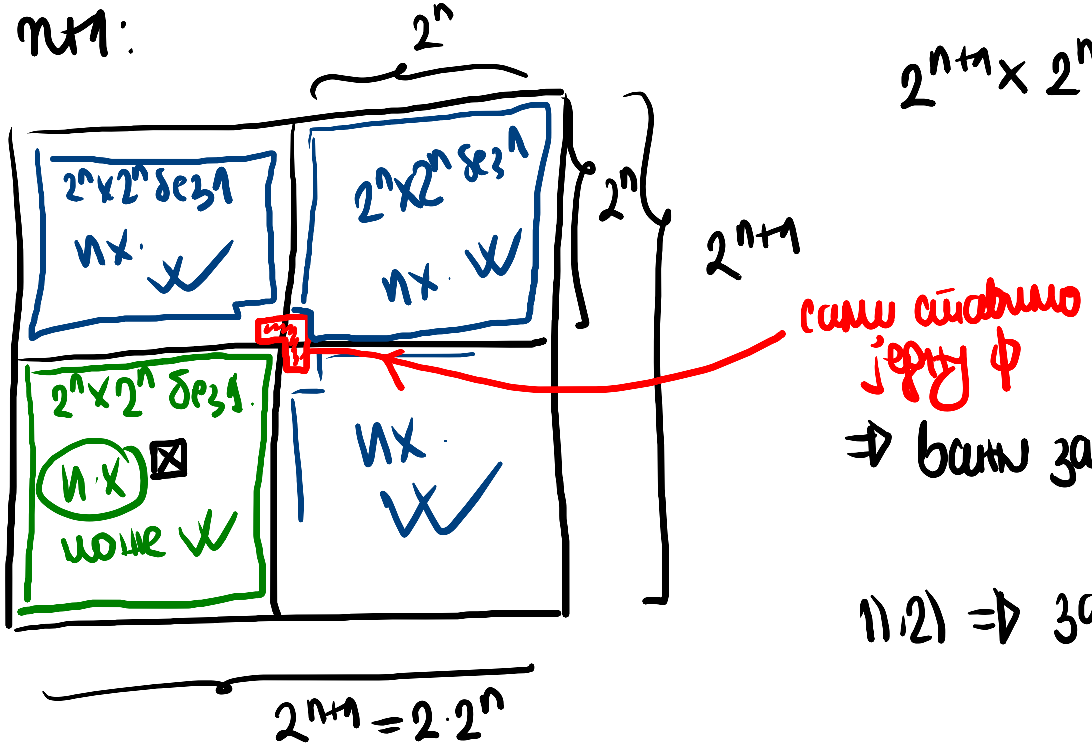
1) $n=1$ 2×2 без 1 поља

тачно 1 фигура Ф ✓



2) $n \mapsto n+1$ (И.Х.) важи за n: $2^n \times 2^n \setminus 1$ поља

за $n+1$:



⇒ важи за $n+1$

1),2) ⇒ за $\forall n \in \mathbb{N}$.

⑤ Док. да за $\forall n\in\mathbb{N}$ важи да је број $A_n=(n+1)(n+2)\cdots(n+n)$ дељив са 2^n, а није дељив са 2^{n+1}

1) $\boxed{\text{База}}$: $n=1$ $A_1=1+1=2$ $2\mid A_n$, $2^2\nmid A_n$ ✓

2) $\boxed{n\mapsto n+1}$

(И.Х.) $2^n\mid A_n$ и $2^{n+1}\nmid A_n$

за A_{n+1}:

$$A_{n+1}=(n+2)(n+3)\cdots(\underbrace{n+1+n-1}_{n+n})(n+1+n)(\underbrace{n+1+n+1}_{2(n+1)})$$

$$=2\cdot\underbrace{(n+1)(n+2)\cdots\cdots(n+n)}_{A_n}\cdot(2n+1)=2\cdot A_n\cdot(2n+1)$$

(И.Х.) $2^n\mid A_n$, $2^{n+1}\nmid A_n$ → $A_n=2^n\cdot q$, $2\nmid q$

$$\Rightarrow A_{n+1}=2\cdot2^n\cdot q\cdot(2n+1)=2^{n+1}\cdot \underbrace{q\cdot(2n+1)}_{\text{непаран}}$$

$\Rightarrow$ $2^{n+1}\mid A_{n+1}$
$2^{n+2}\nmid A_{n+1}$ ✓

1),2) $\Rightarrow \forall n\in\mathbb{N}$

⌜ $k=\underbrace{p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_m^{\alpha_m}}_{\text{прости}}$ $p_i\neq p_j$, $i\neq j$

$p_1^{\alpha_1}\mid k$, $p_1^{\alpha_1+1}\nmid k$

$p_1^{\alpha_1}\parallel k$ „тачно дели".

$36=2^2\cdot3^2$ $2^2\mid 36$, $2^3\nmid 36$
$2^2\parallel 36$ ⌟

„Доказ" да су сви природни бројеви једнаки

$\max(1,2)=2$ /-1
$\max(\underline{0},1)=1$

$\boxed{\text{ЛЕМА}}$ $k,l,n\in\mathbb{N}$ $\max(k,l)=n \Rightarrow k=l=n$

$\boxed{\text{ДОКАЗ}}$ 1) $n=1$ $k,l\in\mathbb{N}$ $\max(k,l)=1 \Rightarrow k,l=1$ ✓

2) $\boxed{n\mapsto n+1}$ (И.Х.) за n важи: $k,l\in\mathbb{N}$ $\max(k,l)=n \Rightarrow k=l=n$

за $n+1$: $k,l\in\mathbb{N}$

$$\max(k,l)=n+1$$

$$\overset{-1}{\Rightarrow} \max(k-1,l-1)=\underline{\underline{n}}$$

$$\overset{\text{(И.Х.)}}{\cancel{\Rightarrow}} \quad k-1=l-1=n \quad /+1$$

$$\Rightarrow k=l=n+1 \quad \text{важи за } n+1 \ ✓$$

$k-1$ и $l-1$ не морају бити $\in\mathbb{N}$

1), 2) $\Rightarrow \forall n\in\mathbb{N}$. ∎

Примени лему:

$\max(1,2)=2 \Rightarrow 1=2$
$\max(2,3)=3 \Rightarrow 2=3$
$\vdots$
$\max(n,n+1)=n+1 \Rightarrow n=n+1$ …

$\Rightarrow 1=2=3=\ldots=n=n+1=\ldots$

Биномни коефицијенти

$$\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!} = \frac{n\cdot(n-1)\cdots(n-k+1)}{k!}, \quad \binom{n}{0}=1=\binom{n}{n} \quad 0!=1$$

$n,k\in\mathbb{N}$

$$\binom{n}{k}=\binom{n}{n-k}$$

$\binom{n}{k}$ = бр. начина да од n елемената изаберемо k одједном

$$\| $$

$$\frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$\underbrace{n}_{1.}\cdot\underbrace{(n-1)}_{2.}\cdot\ldots\cdot\underbrace{(n-(k-1))}_{k.}$$

делимо са бројем начина да k елемената ставимо у ред:

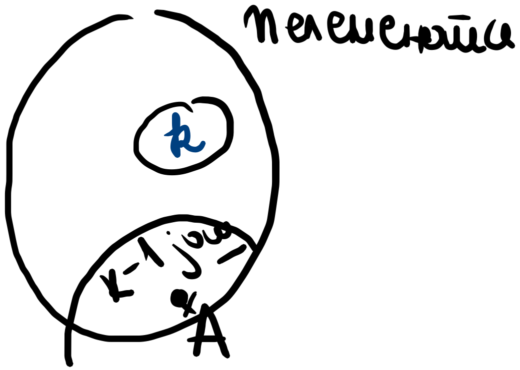
$$\frac{k}{1.}\cdot\frac{(k-1)}{2.}\cdot\ldots\cdot\frac{k-(k-2)=2}{k-1}\cdot\frac{1}{k}=k!$$

$$\text{(I)}\quad \underbrace{\binom{n}{k}}_{k\text{ бирамо}}=\underbrace{\binom{n}{n-k}}_{(n-k)\text{ остаје}}$$

② $\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k}$, $\forall k\in\mathbb{N}, n\geq 2$

1. начин → аналитички...

2. начин: комбинаторно:



$$\underbrace{\boxed{\binom{n}{k}}}_{\text{од } n \text{ бирамо } k} =$$

бр. начина да k изаб. од n тд. изабрано A + бр. нач. k од n тд A није изабрано

|| ||

од n-1 осталих бирамо k-1 | од n-1 бирамо сад k

|| ||

$$\boxed{\binom{n-1}{k-1}} + \boxed{\binom{n-1}{k}}$$

✓

Њутнова биномна формула

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$\vdots$

$$(a+b)^n = \underbrace{\binom{n}{0}a^n}_{a^n} + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \underline{\underline{\binom{n}{k}a^{n-k}b^k}} + \dots + \binom{n}{n-1}ab^{n-1} + \underbrace{\binom{n}{n}b^n}_{b^n}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Доказ: индукција по n

1) $n=1$: $(a+b)^1 = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = a^1 + b^1 = a+b$ ✓

2) $n \mapsto n+1$ (И.Х) НБФ. вярна за n

за $n+1$: $(a+b)^{n+1} = \underline{(a+b)^n}\cdot(a+b)$

$$\overset{\text{(И.Х)}}{=} \left(\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{k}a^{n-k}b^k + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n\right)(a+b) = \dots$$

$$= \binom{n}{0}a^{n+1} + \binom{n}{1}a^n b + \binom{n}{2}a^{n-1}b^2 + \dots + \binom{n}{k}a^{n+1-k}b^k + \dots + \binom{n}{n-1}a^2b^{n-1} + \binom{n}{n}ab^n$$

$$+ \binom{n}{0}a^n b + \binom{n}{1}a^{n-1}b^2 + \dots + \binom{n}{k-1}a^{n+1-k}b^k + \dots + \binom{n}{n-2}a^2b^{n-1} + \binom{n}{n-1}ab^n + \binom{n}{n}b^{n+1}$$

$\binom{n+1}{0} = 1$ $\qquad \binom{n}{n} = 1$

$$= a^{n+1} + \binom{n+1}{1}a^n b + \binom{n+1}{2}a^{n-1}b^2 + \dots + \binom{n+1}{k}a^{n+1-k}b^k + \dots + \binom{n+1}{n}ab^n + b^{n+1}$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^k$$

II доказ НБФ. - комбинаторно

збир=n

$$(a+b)^n = (a+b)(a+b)\cdots(a+b) \rightsquigarrow a^{\square}\cdot b^{\square}$$

$$= \sum_{k=0}^{n} \boxed{?}\, a^{n-k}\cdot b^{k}$$

бр. начина да од $\boxed{n}$ заграда изаберемо $\boxed{k}$ из којих узимамо b (а из осталих узимамо a)

$$\boxed{?} = \binom{n}{k}$$

∎

Паскалов троугао:

1 ← $(a+b)^0 = 1$

1 1 ← $(a+b)^1 = a+b$

1 2 1 ← $(a+b)^2 = \underline{\ }a^2+\underline{2}ab+\underline{\ }b^2$

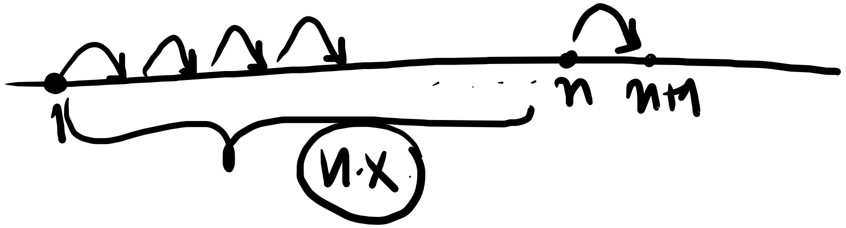
$1 \overset{+}{\searrow} 3 \swarrow 2$

1 3 3 1

1 4 6 4 1 ← $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

⋮

ТРАНСФИНИТНА ИНДУКЦИЈА



1.) База $P(1)$ ✓

2) $\underbrace{P(1), P(2), \dots, P(n)}_{\text{И.Х.}} \Rightarrow P(n+1)$, $\forall n \in \mathbb{N}$

1), 2) $\Rightarrow$ $P(n)$ важи за све $n \in \mathbb{N}$

⑥ Низ: $a_0 = 1$, $a_n = a_{n-1} + \dots + a_2 + a_1 + 2 \cdot a_0$. Доказати: $a_n = 2^n$, $\forall n \in \mathbb{N}_0$

1) $n = 0$ $a_0 = 1 = 2^0$ ✓

2) $\boxed{\text{И.Х.}}$ важи $a_i = 2^i$ $i = 0, 1, 2, \dots, n-1$

доказујемо за $\underline{\underline{n}}$

$$\boxed{a_n} = 2^{n-1} + 2^{n-2} + \dots + 2^2 + 2^1 + 2 \cdot \underline{\underline{1}}$$

$$= (2^{n-1} + \dots + 2^1 + 1) + 1 = \frac{2^n - 1}{2 - 1} + 1 = 2^n - 1 + 1 = \boxed{2^n}$$ ✓

$\ulcorner$ $1 + q + q^2 + \dots + q^n = \frac{q^{n+1} - 1}{q - 1}$, $q \neq 1$

$$\left.\begin{aligned} S &= 1 + q + \dots + q^n \\ qS &= q + \dots q^n + q^{n+1} \end{aligned}\right\} - \lrcorner$$

1), 2) $\Rightarrow$ $\forall n$ ✓

Индукција са више претпоставки

$$\left.\begin{array}{l} 1)\ P(1), P(2)\ ✓ \\ 2)\ P(n), P(n+1) \Rightarrow P(n+2),\ \forall n\in\mathbb{N} \end{array}\right\} \Rightarrow \forall n\in\mathbb{N} \text{ важи } P(n)$$

За вежбу:

① Низ: $x_1=1$, $x_2=2$

$x_n=(n-1)\cdot(x_{n-1}+x_{n-2})$, $\forall n\geq 3$

Док. да $\forall n\in\mathbb{N}$ важи $x_n=n!$

② Док. да за $\forall n\geq 2$ важи да је $\cos\frac{\pi}{2^n}$ ирационалан.

③ Док. да n правих у равни дели раван на највише 2^n делова.