~ Функције - обнављање ~

$f: A \to B$ — A домен, B кодомен

$f(x) = \dots \quad x \in \mathbb{R}$

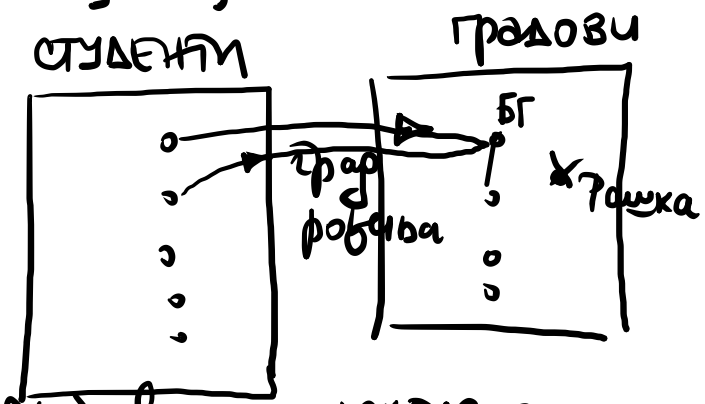
$D_f = \{x \in \mathbb{R} \mid f(x) \text{ дефинисано}\}$

$f: D_f \to \mathbb{R}$

$a \in A$

$a \longmapsto f(a) \in B$

a — сваком $a \in A$; $f(a)$ — тачно 1



f је НА $\iff (\forall b \in B)(\exists a \in A)\ f(a) = b$ — сурјективна

f је 1-1 $\iff (\forall a_1, a_2 \in A)\ a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

инјективна $\iff (\forall a_1, a_2 \in A)\ f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

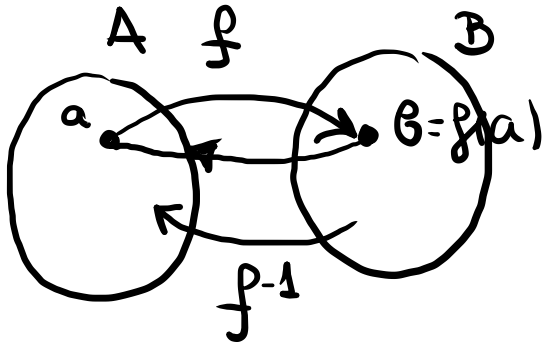
$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$

Закон контрапозиције

f је 1-1 и НА $\to$ f је Бијекција

A и B имају исти елемената $\Leftrightarrow \exists$ бијекција $f: A \to B$

$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| \neq |\mathbb{R}|$



- $f: A \to B$ бијекција

  $\Rightarrow \exists$ инверзна фја

  $f^{-1}: B \to A$

  $B \in B \overset{f \text{ на}}{\Longrightarrow} \exists a \in A \; f(a) = B$ $\quad f^{-1}(B) := a$

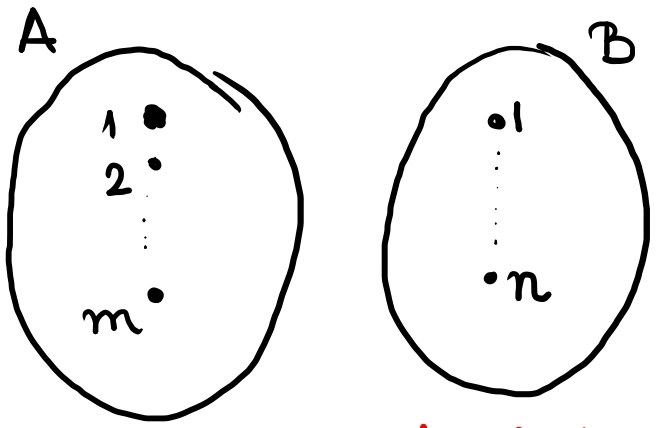
  $\exists 1$ јер f 1-1

  $f^{-1} \circ f(a) = a, \; \forall a \in A \longrightarrow f^{-1} \circ f = \mathbb{1}_A = id_A$

  $f \circ f^{-1} = \mathbb{1}_B = id_B$

1. $|A| = m$, $|B| = n$

(а) Колко има фја $f: A \to B$?

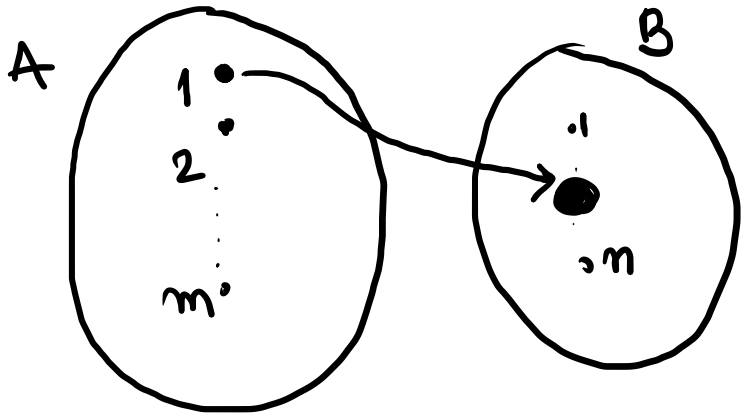


1 - n избора
2 - n -11-
⋮
m - n -11-

$\underbrace{n \cdot n \cdot \ldots \cdot n}_{1.\ 2.\ \ m.} = n^m$

B^A = скуп свих фја $A \to B$

(б) Колко има 1-1 фја $f: A \to B$?



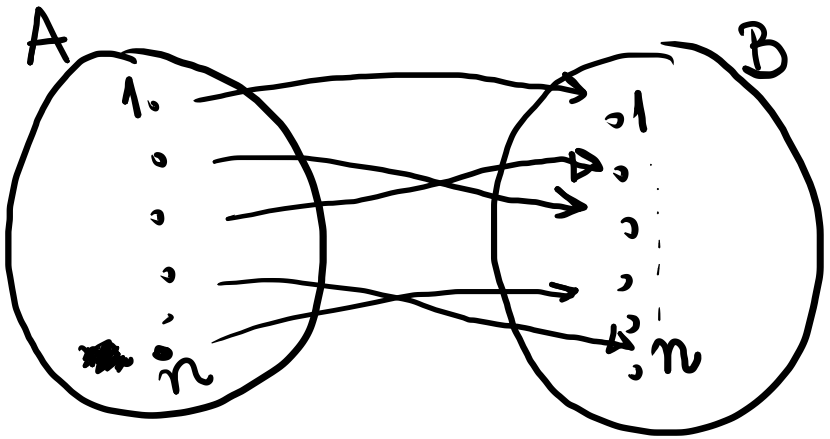
$m > n$ → не постоји 1-1 фја

$m \leq n$:
1. → n избора
2. → n-1 избора
3. → n-2 избора
⋮
m. → $n-(m-1) = n-m+1$

број (n-1) фја

$\underbrace{n \cdot (n-1) \ldots (n-m+1)}_{1\quad 2\quad\quad m}$

(в) Колико има бијекција $A \to B$?



$m \neq n \Rightarrow$ не постоји бијекција

$\boxed{m = n}$

(m>n → није 1-1
n>m → није НА)

$1\text{-}1 \Rightarrow НА$

$НА \Rightarrow 1\text{-}1$

! само за $m = n$

$$\frac{n}{1.} \cdot \frac{(n-1)}{2.} \cdots \frac{(n-(k-1))}{k.} \cdots \frac{\overbrace{(n-(n-1))}^{1}}{n.} = n \cdot (n-1) \cdots 2 \cdot 1$$

$$= n!$$

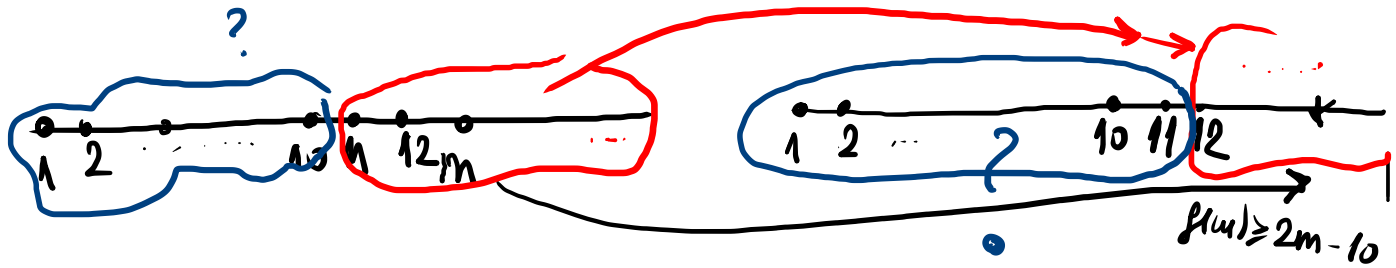
(2) За кои функции $f: \mathbb{N} \to \mathbb{N}$ за които вярно $f(n+5) \geq 2n$, $\forall n \in \mathbb{N}$?

$m = n+5 \geq 6$

$n = m-5$

$m \geq 6$: $f(m) \geq 2(m-5)$

$\circledast$ $\boxed{f(m) \geq 2m-10}$



Предполагаме, че f е НА и $\circledast$

$2m-10 \overset{?}{>} m \iff m > 10 \iff m \geq 11$

За $m \geq 11$: $f(m) \geq 2m-10 \geq m \geq 11$ $\Rightarrow$ $f(m) \geq 12$

Как може да се смята $y \in \{1,2,\ldots,10,11\}$? 11

Само нещо от $\{1,2,\ldots,10\}$ 10

$\Rightarrow$ f не е НА ↯

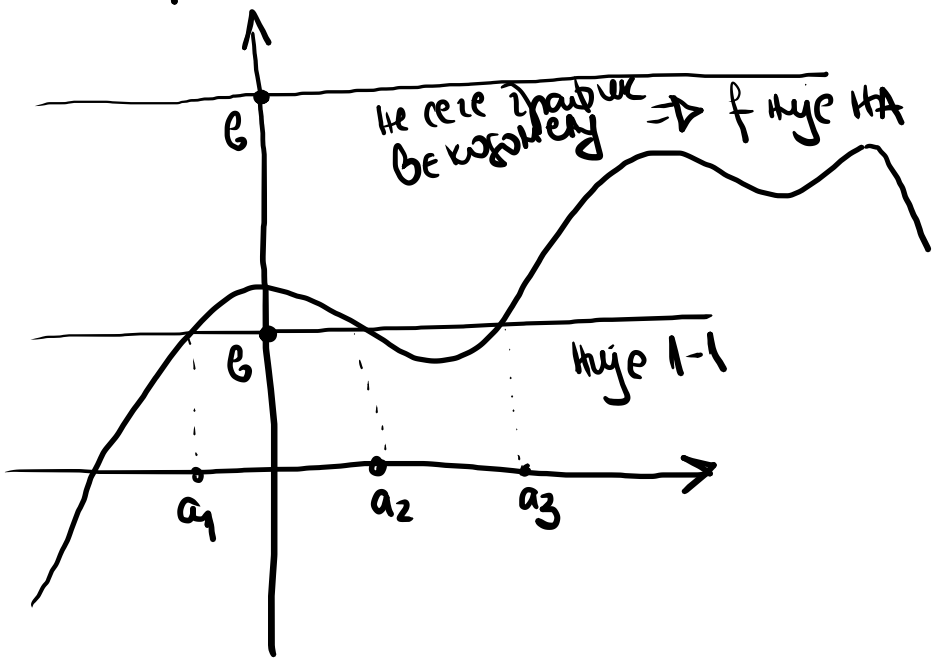
Такова f не съществува

График функције

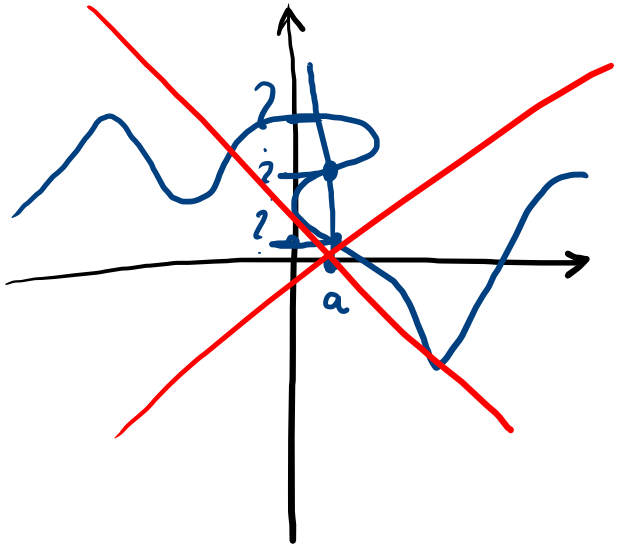
$f: A \to B$ график $\Gamma_f \subset A \times B$

$$\Gamma_f = \{(a,b) \in A \times B \mid b = f(a)\} = \{(a, f(a)) \in A \times B \mid a \in A\}$$

График реалне фје: $f: A \to \mathbb{R}$ $A \subset \mathbb{R}$ $\Gamma_f \subset \mathbb{R}^2$



Није график:

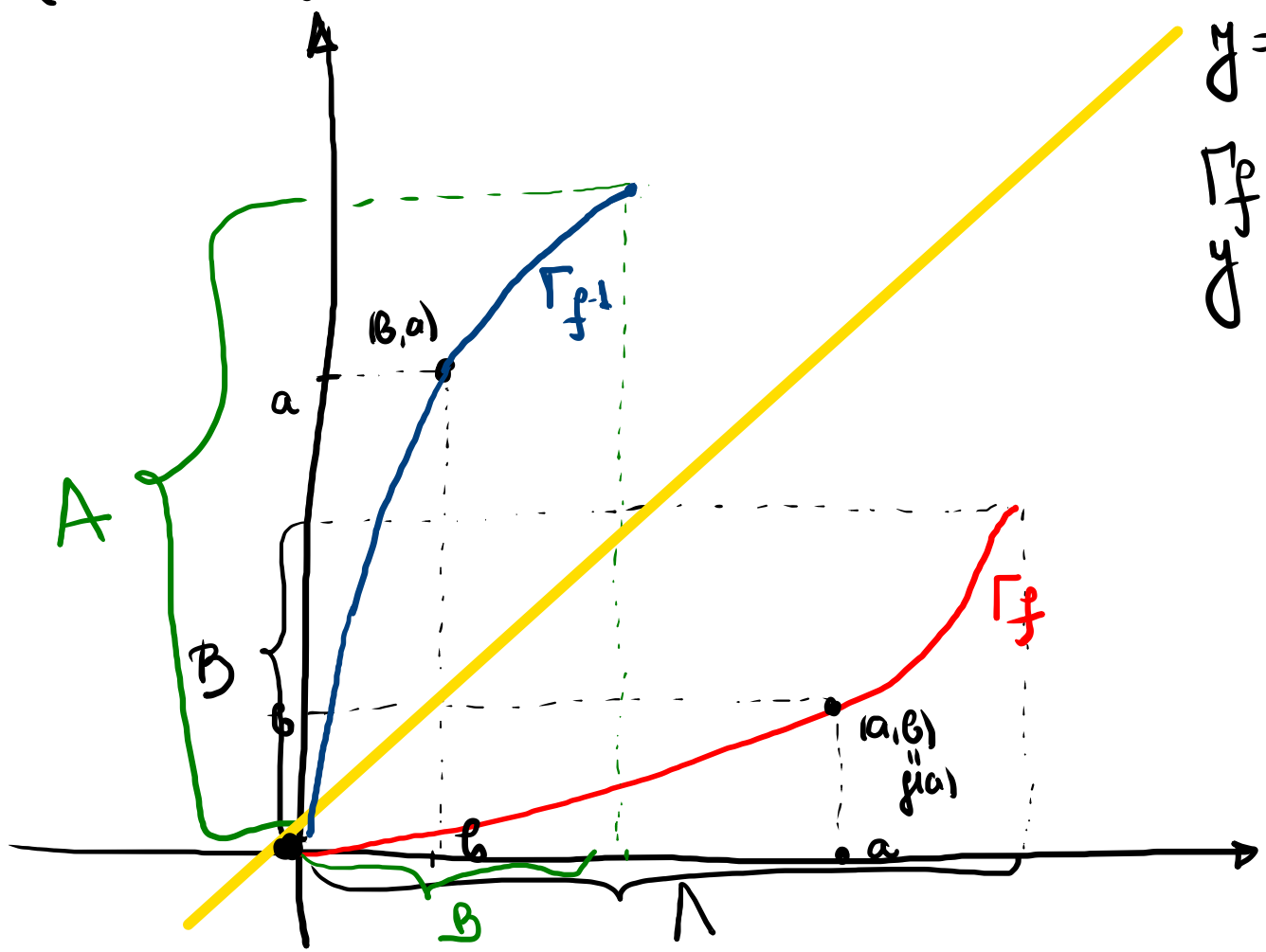


$f: A \to B$ бијекција $\Rightarrow$ $f^{-1}: B \to A$ инверз

$\Gamma_f \subset A \times B$ $\qquad$ $\Gamma_{f^{-1}} \subset B \times A$

$\downarrow \psi$ $\qquad$ $\downarrow \psi$

(a,b) $b = f(a)$ $\rightsquigarrow$ (b,a), $a = f^{-1}(b)$



$y = x$

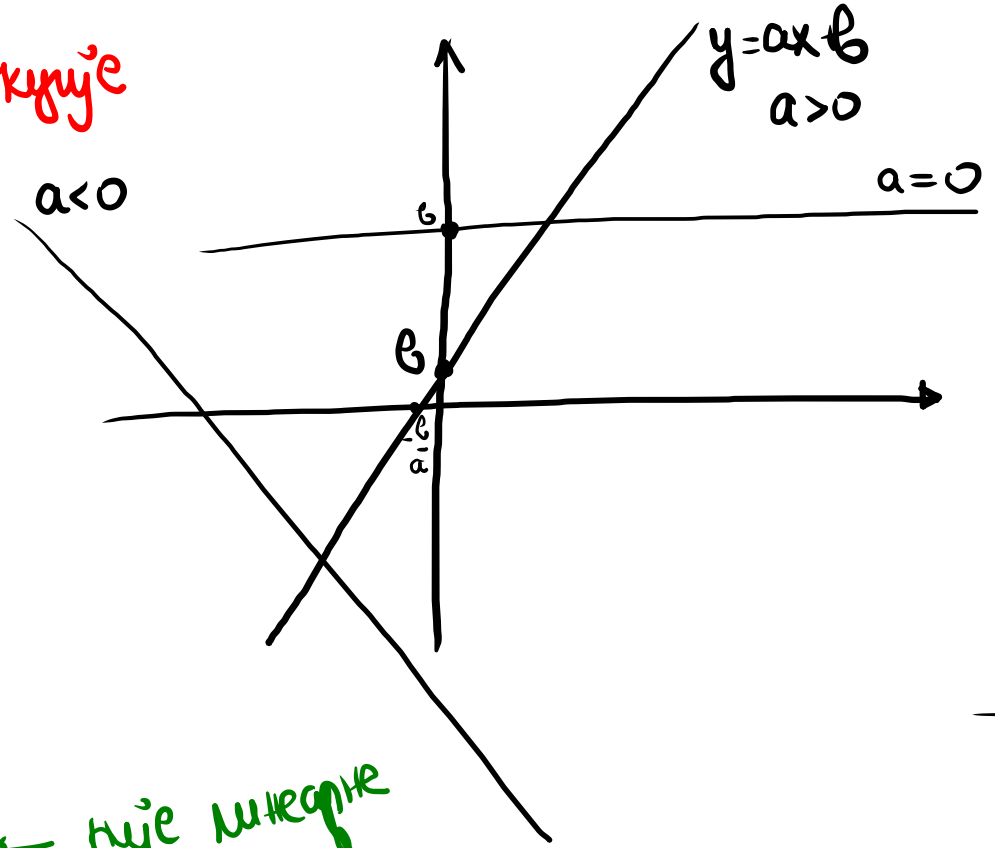
Γ_f и $\Gamma_{f^{-1}}$ су симетрични у односу на праву $y = x$

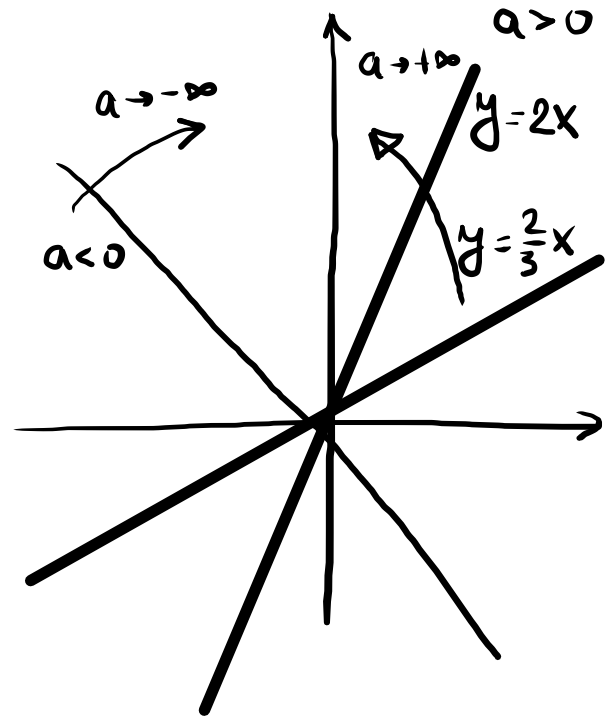
Елементарне функције

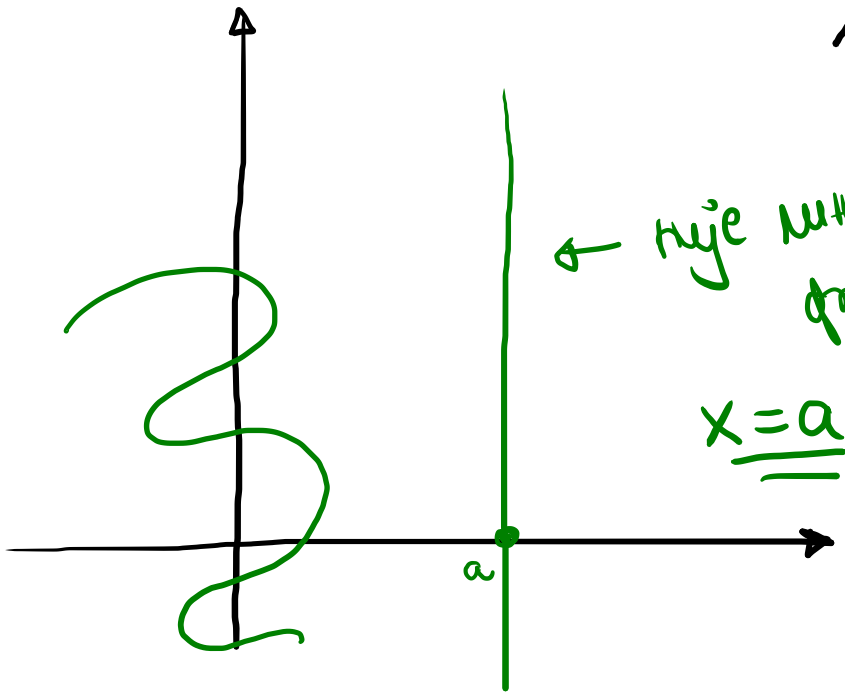
I линеарна функција

$y = ax + b$

$y = f(x)$







← није линеарне фја!

$x = a$

II Квадратна функция

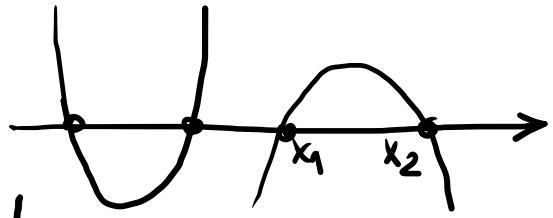
$y = f(x) = ax^2 + bx + c$, $a \neq 0$

$a > 0$: 

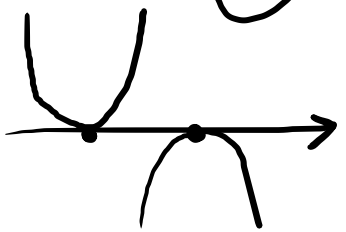
$a < 0$: 

$D = b^2 - 4ac$

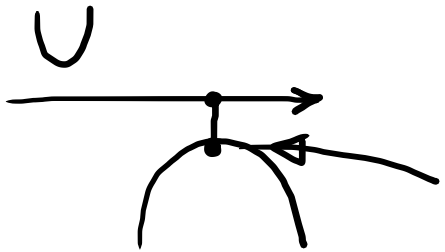
$D > 0$: $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $f(x) = a(x-x_1)(x-x_2)$



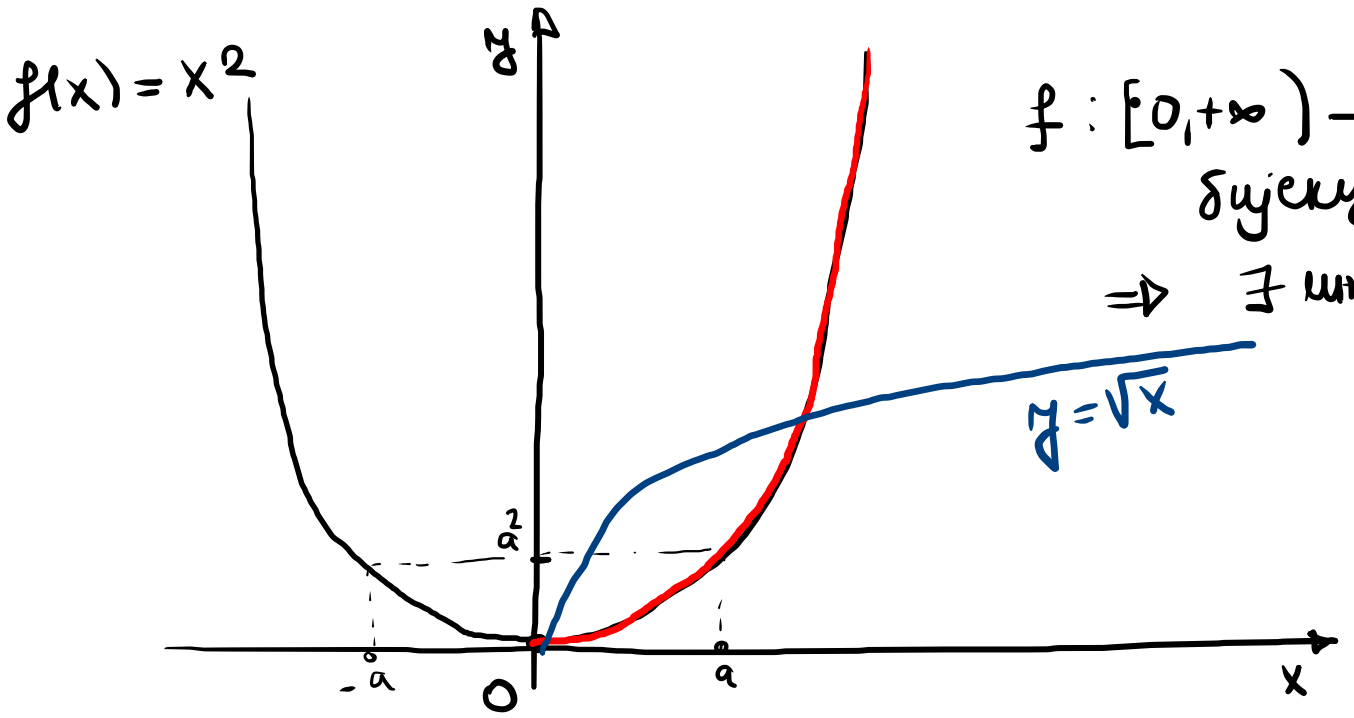
$D = 0$: $x_1 = x_2 = x_0$ $f(x) = a(x-x_0)^2$



$D < 0$: $f(x) \neq 0, \forall x$



Върхът е $\left(-\frac{b}{2a}, \frac{-D}{4a}\right)$



$f : [0,+\infty) \longrightarrow [0,+\infty)$

биекция

$\Rightarrow \exists$ обратная $f^{-1} : [0,+\infty) \to [0,+\infty)$

$f^{-1}(x) = \sqrt{x}$

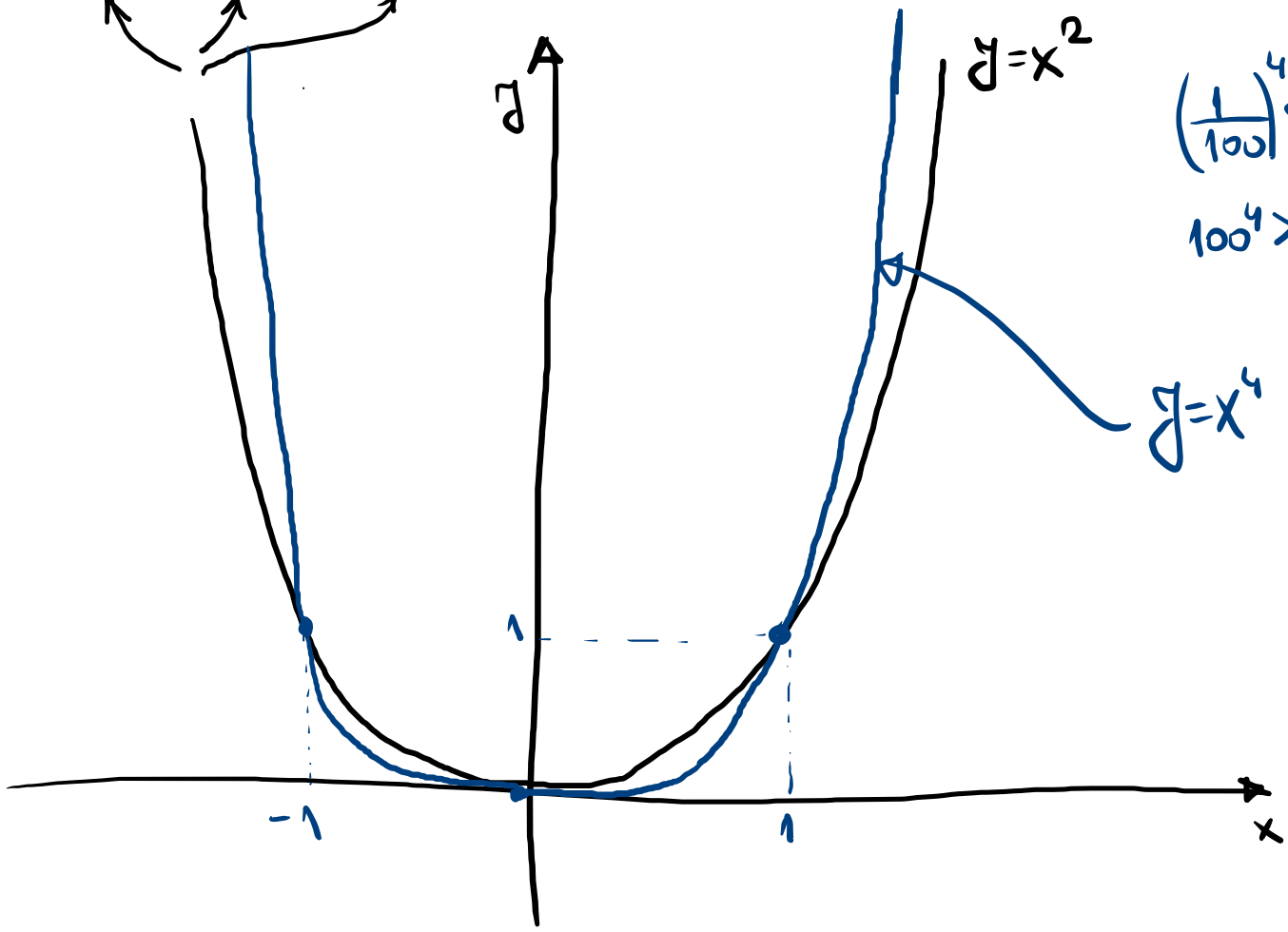
$\sqrt{x} \in [0,+\infty), \forall x$

$\sqrt{4} = 2, \sqrt{4} \neq -2 \quad (-2)^2 = 4$

$\sqrt{f(x)} = g(x) : f(x) \geq 0, g(x) \geq 0 \quad f(x) = (g(x))^2$

III $f(x)=x^n$, $n\in\mathbb{N}$, $n\neq 1$

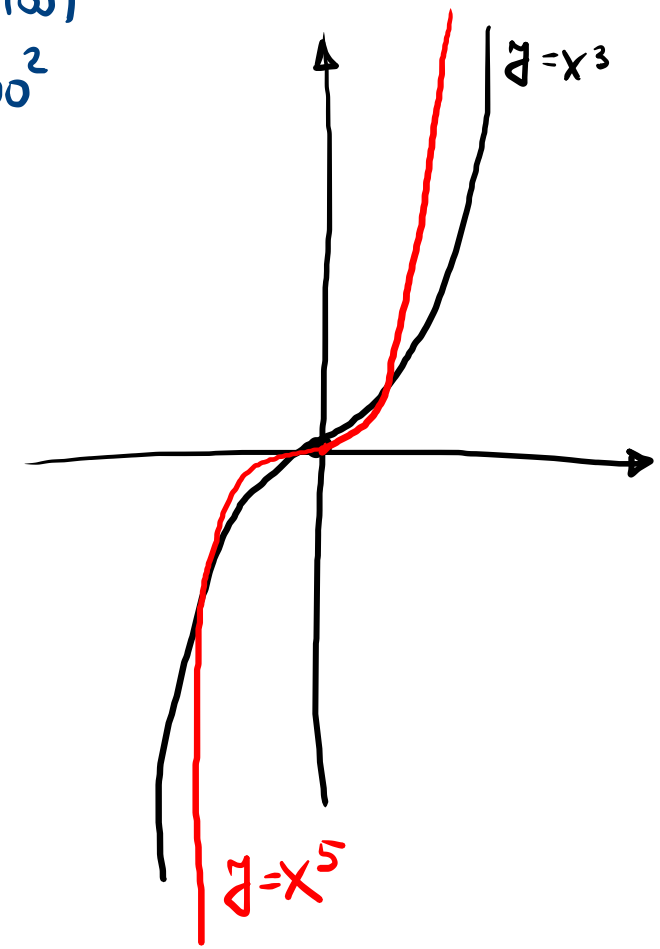
$x^2, x^3, x^4, x^5, x^6, x^7, x^8, \ldots$ $\boxed{n=2k}$



$\left(\frac{1}{100}\right)^4 < \left(\frac{1}{100}\right)^2$

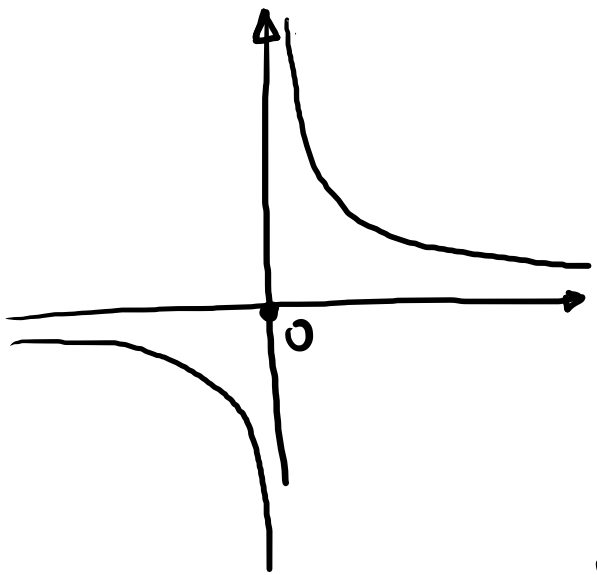
$100^4 > 100^2$

$\boxed{y=2k+1}$

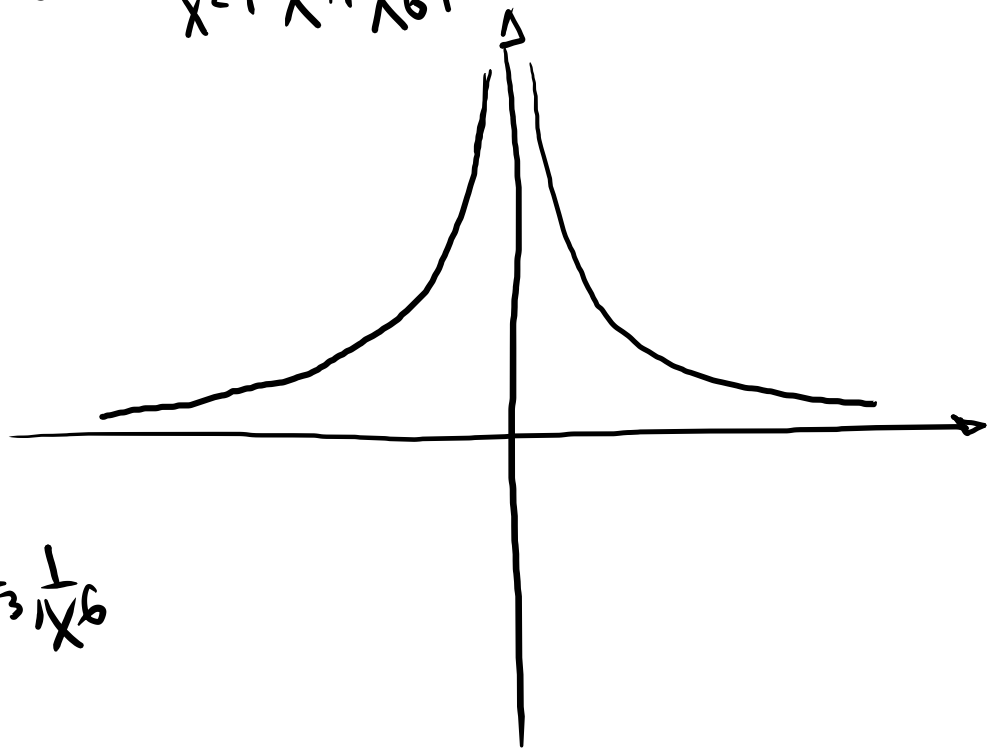


Ⅳ $f(x)=\frac{1}{x^n}$, $n\in\mathbb{N}$

$n=2k+1$, $\frac{1}{x}, \frac{1}{x^3}, \frac{1}{x^5}, \dots$



$n=2k$ $\frac{1}{x^2}, \frac{1}{x^4}, \frac{1}{x^6}, \dots$



За вежбу: $\frac{1}{x}, \frac{1}{x^3}, \frac{1}{x^6}$

⑦ Експоненційна фја

$f(x)=a^x$, $a>0, a\neq 1$

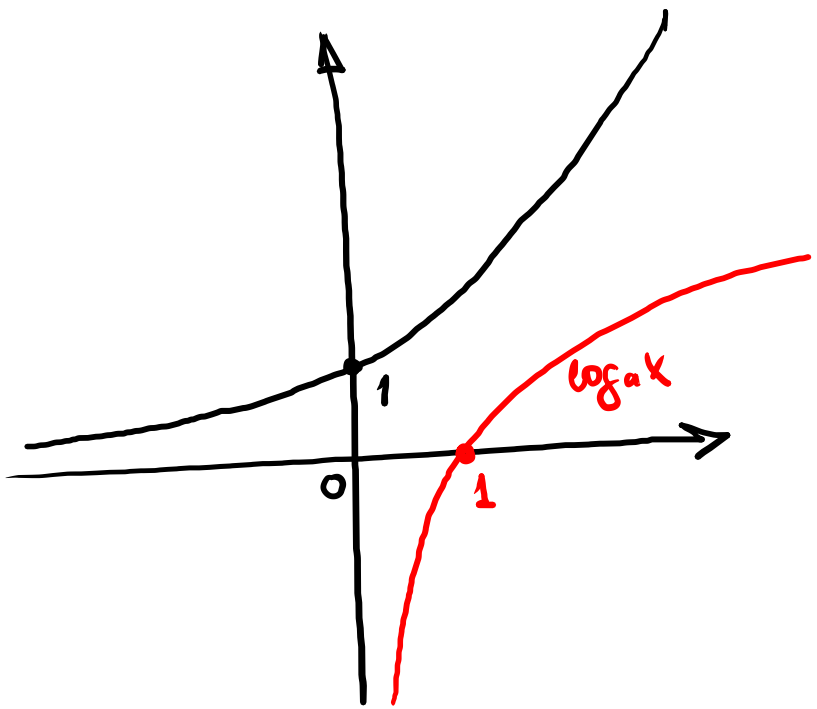
$f: \mathbb{R} \to (0,+\infty)$

бијекција

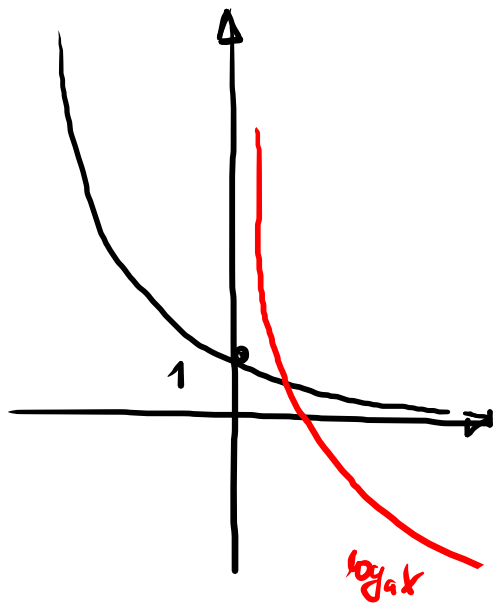
↓ инверз

$\log_a : (0,+\infty) \to \mathbb{R}$

$a>1$



$a\in(0,1)$



(VI) Ciklometrijske funkcije

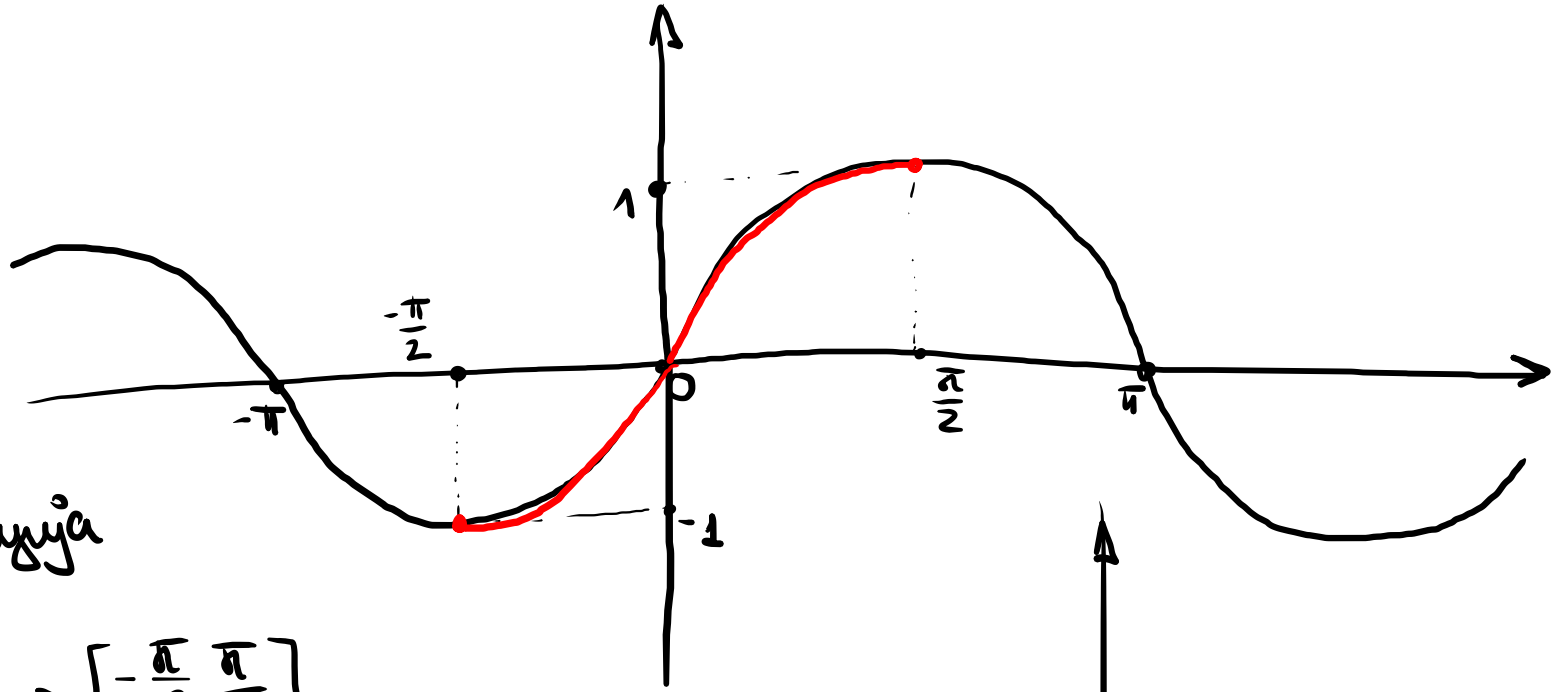
$f(x) = \sin x$

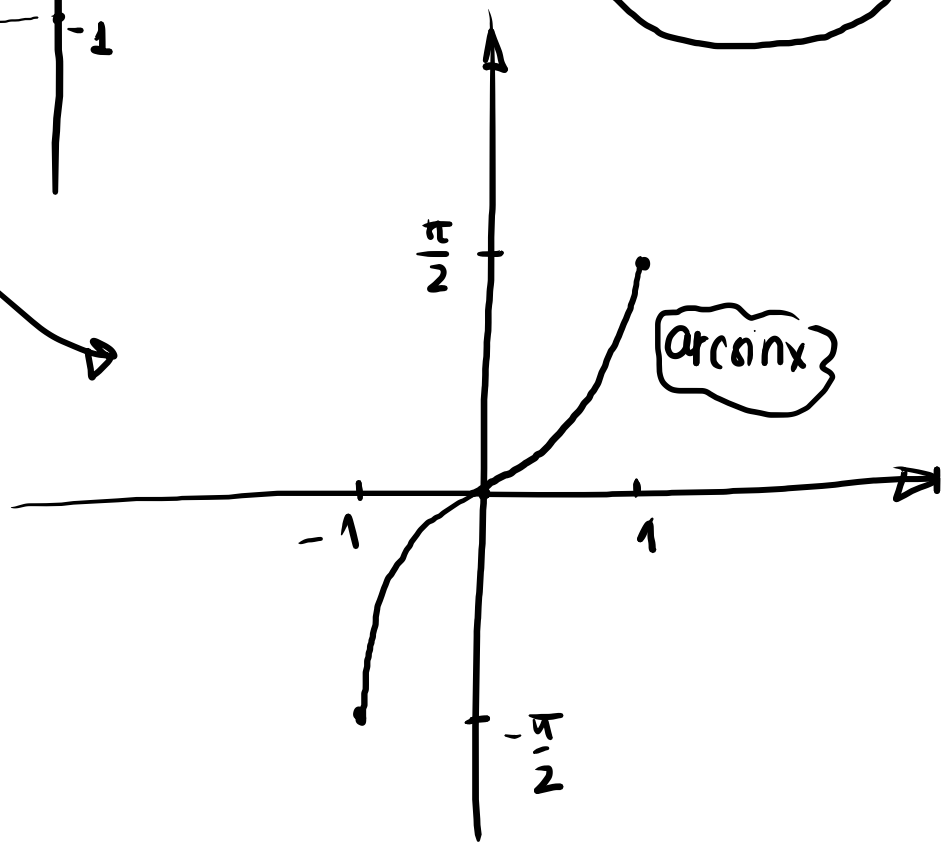
$\mathbb{R} \to \mathbb{R}$ ~~НА~~ ~~1-1~~

$\mathbb{R} \to [-1,1]$ НА ~~1-1~~

$[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ bijekcija

$\Rightarrow \arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$





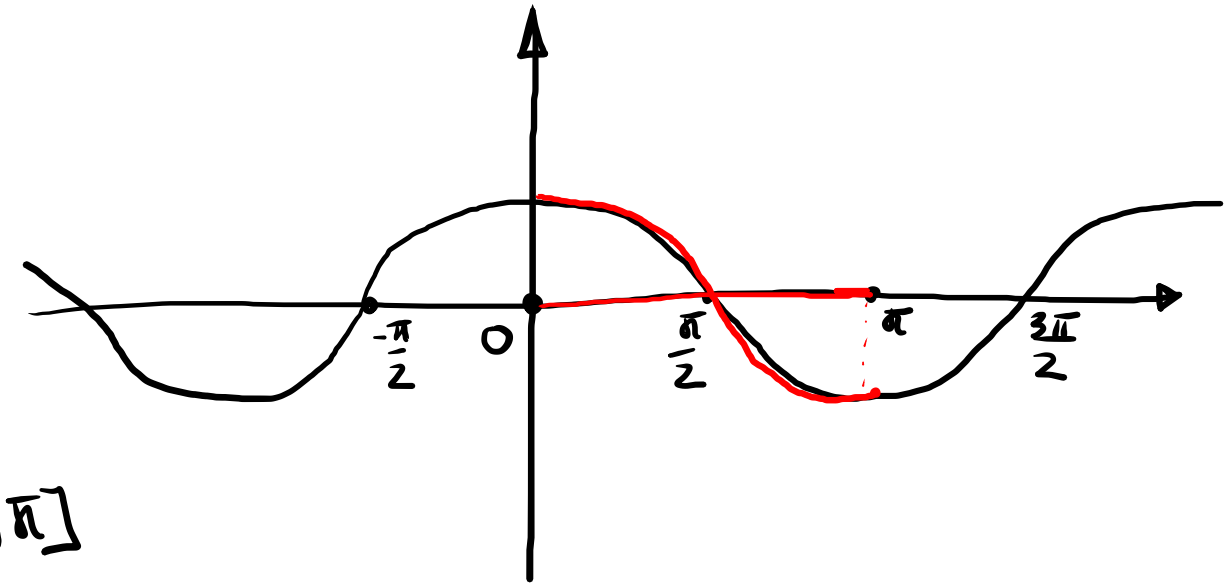
$f(x) = \cos x$

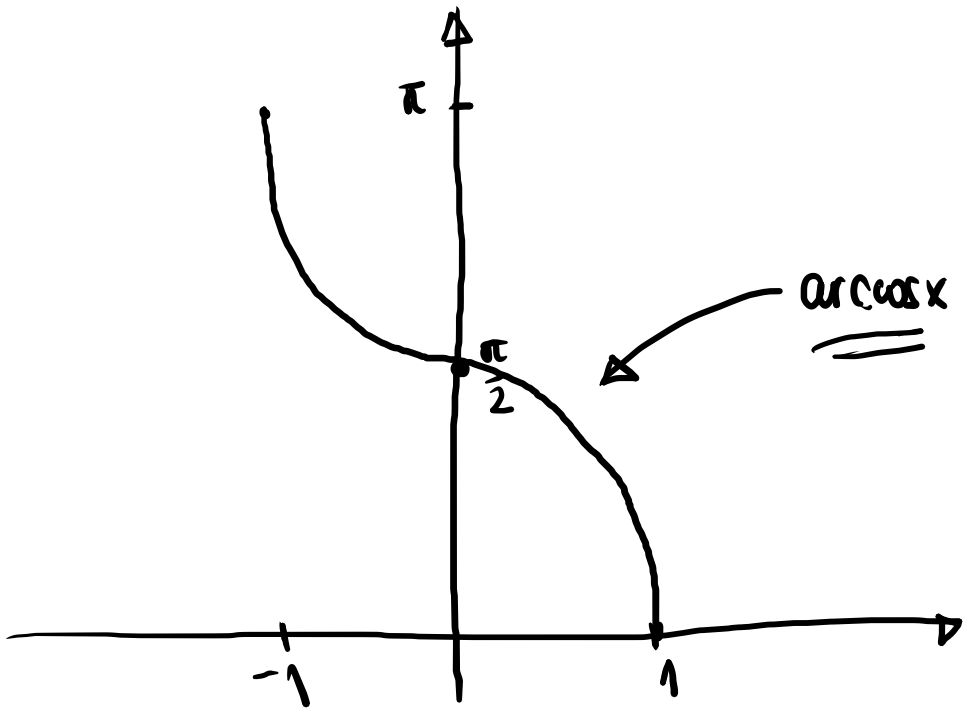
$[0, \pi] \to [-1, 1]$

bijekcija

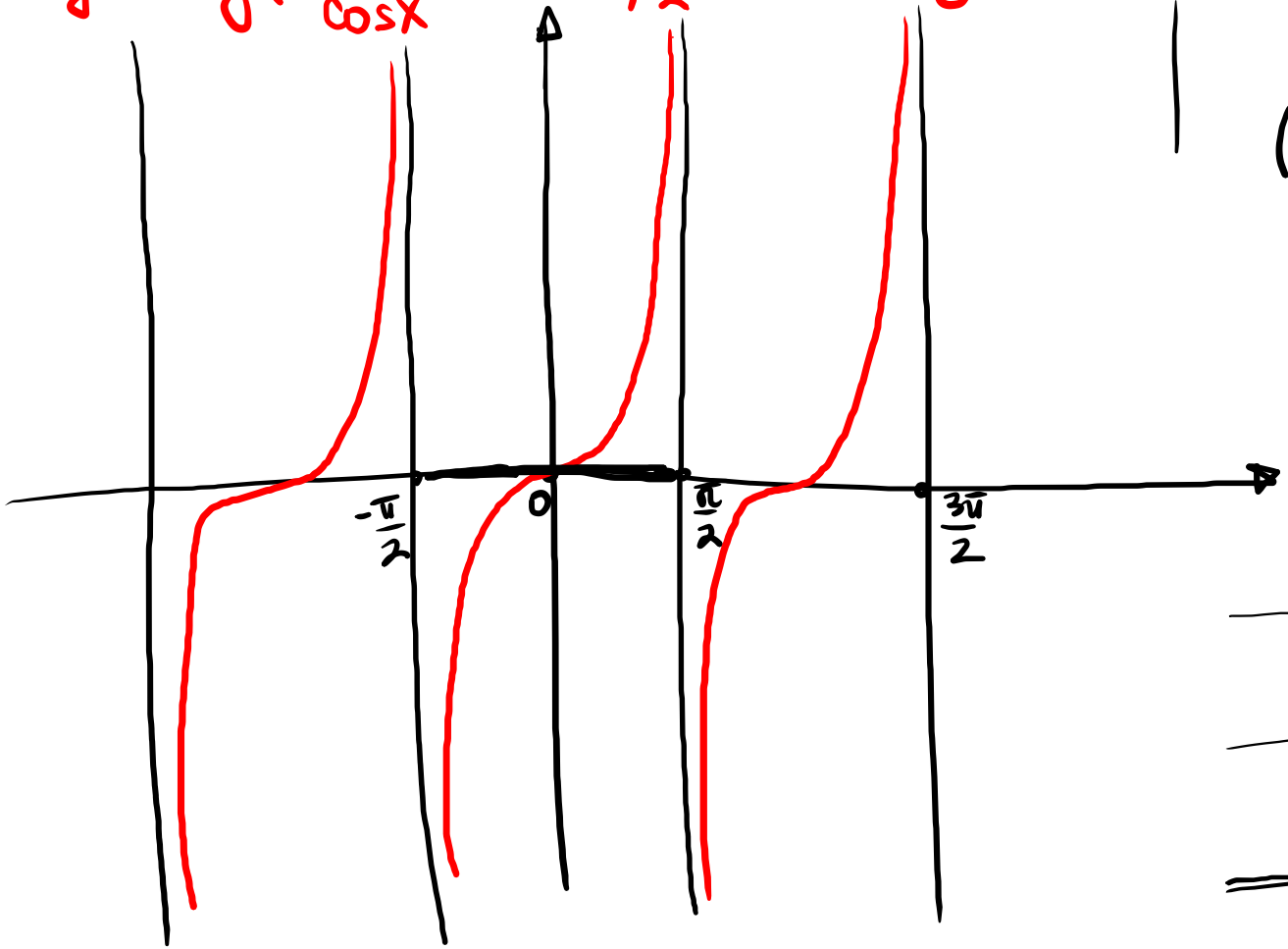
$\to$ inverz

$\arccos : [-1, 1] \to [0, \pi]$





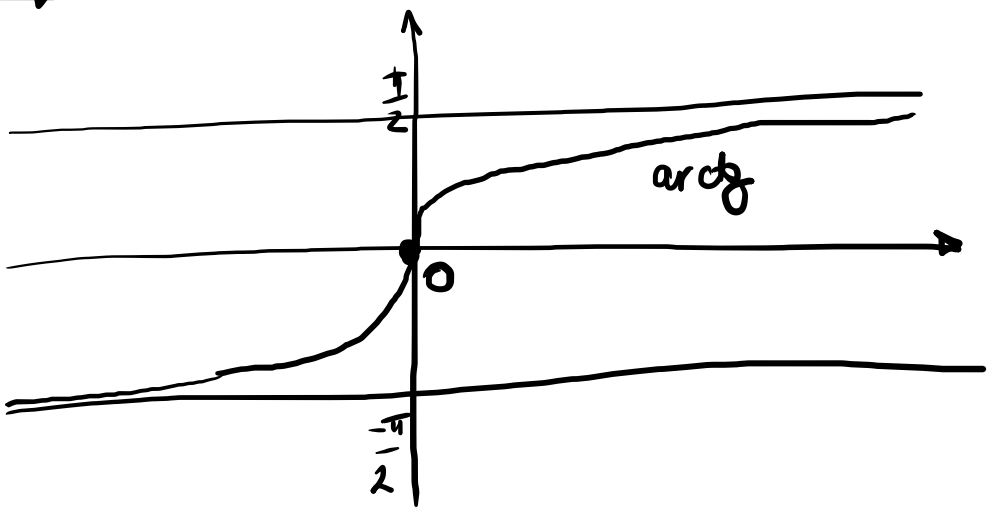
$f(x) = \operatorname{tg} x = \frac{\sin x}{\cos x}$ $\quad \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\} \to \mathbb{R}$



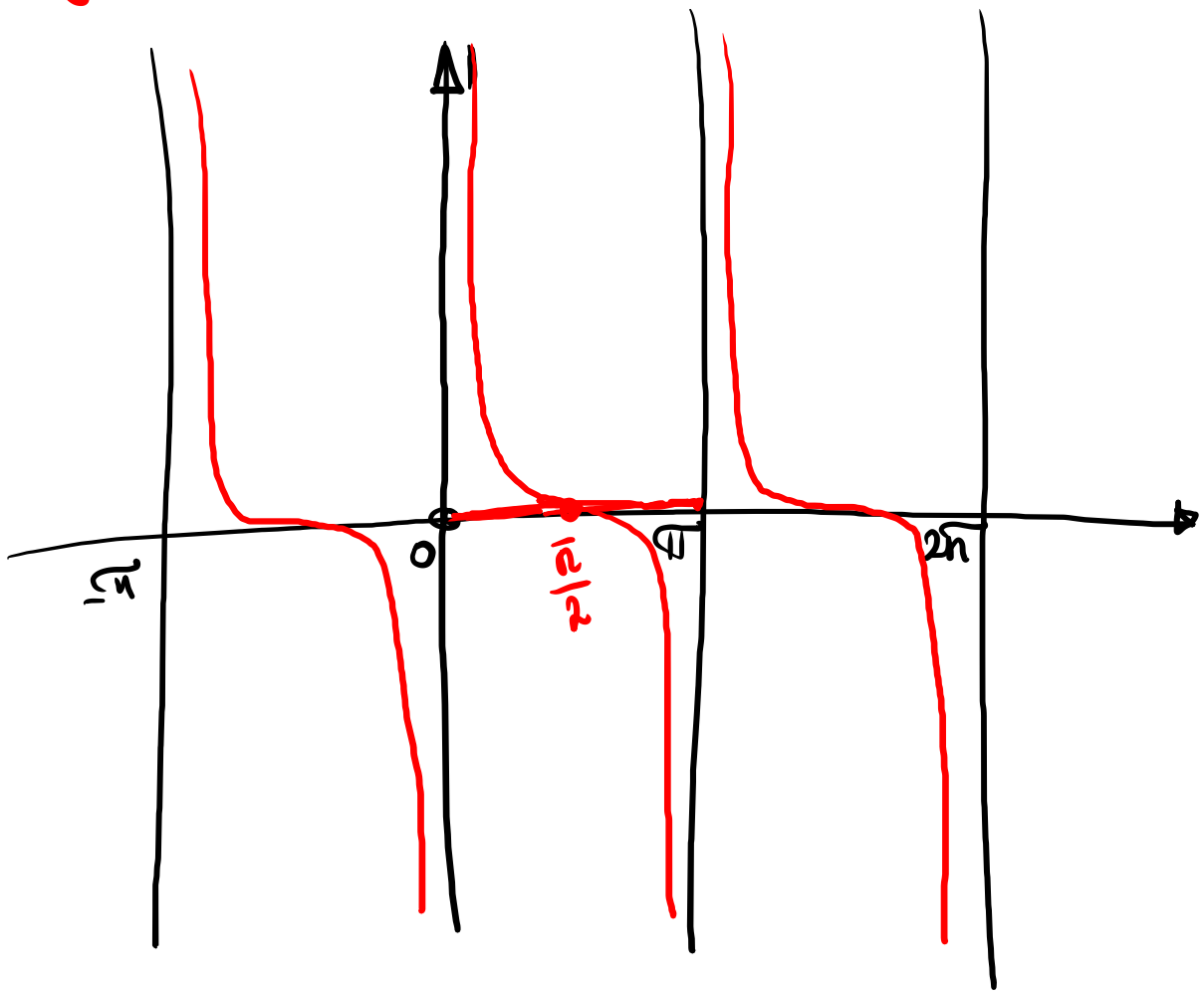
$(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

bijekcja

$\rightsquigarrow \operatorname{arctg} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$



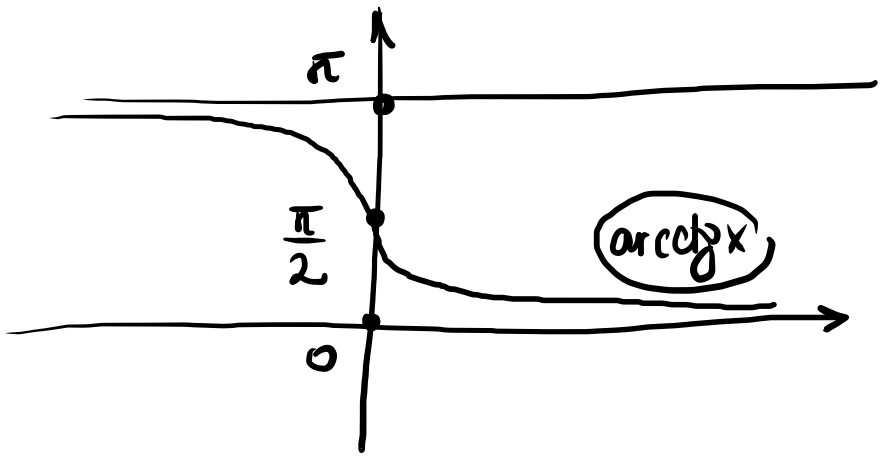
$f(x) = \operatorname{ctg} x = \frac{\cos x}{\sin x}$ $\quad \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\} \to \mathbb{R}$



$(0, \pi) \to \mathbb{R}$

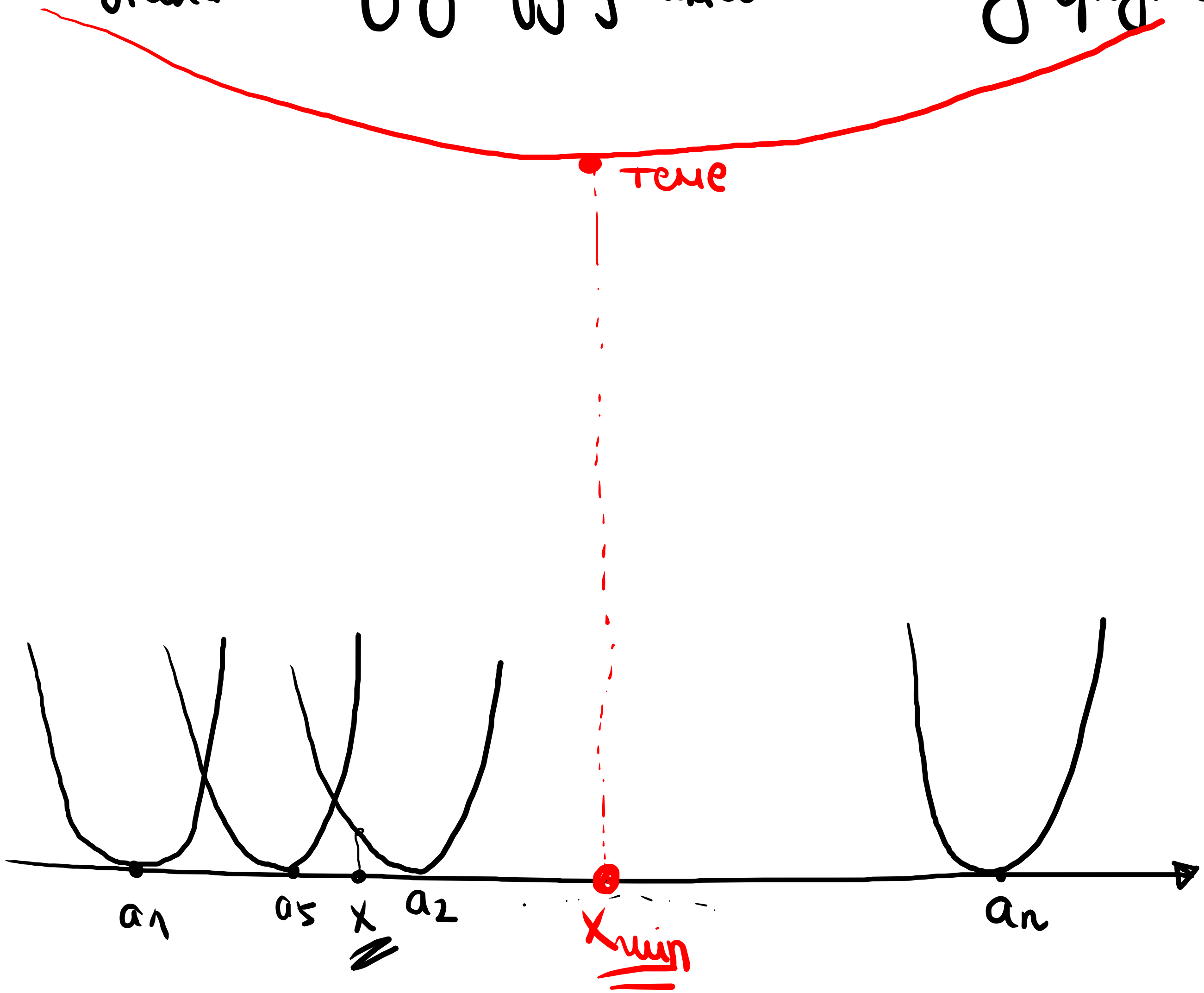
bijekcija

$\rightarrow \operatorname{arcctg} : \mathbb{R} \to (0, \pi)$



③ $f(x) = (x-a_1)^2 + (x-a_2)^2 + \dots + (x-a_n)^2$ $\quad a_1, \dots, a_n \in \mathbb{R}$, $n \in \mathbb{N}$

Нађи тачку у којој f има минималну вредност.



$$f(x) = x^2 - 2a_1x + a_1^2$$
$$+ x^2 - 2a_2x + a_2^2$$
$$\vdots$$
$$+ x^2 - 2a_nx + a_n^2$$

$$\Rightarrow f(x) = n x^2 + (-2)(a_1 + \dots + a_n)\, x + (a_1^2 + \dots + a_n^2)$$

$$y = ax^2 + bx + c \qquad \frac{-b}{2a}$$

$$x_{min} = \frac{-(-2)\sum_{i=1}^{n} a_i}{2n}$$

$$x_{min} = \frac{a_1 + \dots + a_n}{n}$$

аритметичка средина