Са прошлог часа: • $\forall n \in \mathbb{N}: \ \frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$ 24.5.2021.

• Низ $a_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n$ конвергира

1 $L = \lim\limits_{n\to\infty} \underbrace{\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right)}_{x_n}$ n чланова

Пробамо т. о 3 лимеса:

$n \cdot$ најмањи $\leq x_n \leq n \cdot$ највећи

$$\underbrace{n \cdot \frac{1}{2n}}_{\frac{1}{2}} \leq x_n \leq n \cdot \frac{1}{n+1}, \ \forall n$$

$$n \cdot \frac{1}{n+1} \xrightarrow{n\to\infty} 1$$

не може т. о 3 лимеса

Знамо: $a_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n$ $\quad \exists \lim\limits_{n\to\infty} a_n = \gamma \in \mathbb{R}$

$$x_n = \frac{1}{n+1}+\dots+\frac{1}{2n} = \left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)$$

$$= \underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{2n}-\ln 2n\right)}_{a_{2n}} - \underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n\right)}_{a_n} + \underbrace{\ln 2n - \ln n}_{\ln\frac{2n}{n} = \ln 2}$$

$$= a_{2n} - a_n + \ln 2$$

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty}(a_{2n} - a_n + \ln 2) = \gamma - \gamma + \ln 2 = \boxed{\ln 2}$$

$a_{2n} \to \gamma, \quad a_n \to \gamma$

Штолцова теорема

(x_n) - низ $\quad (y_n)$ - растући низ, $\lim\limits_{n\to\infty} y_n = +\infty$

<u>Ако постоји</u> $\lim\limits_{n\to\infty} \dfrac{x_n - x_{n-1}}{y_n - y_{n-1}} \in \overline{\mathbb{R}}$, онда $\exists$ и $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n}$ и важи

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} \longleftarrow \lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

$\boxed{2.}$ $k\in\mathbb{N}$ дато $\quad L = \lim\limits_{n\to\infty} \dfrac{1^k + 2^k + \dots + n^k}{n^{k+1}}$

$x_n = 1^k + 2^k + \dots + n^k$

$y_n = n^{k+1} \to +\infty$, (y_n) растући

$$\lim_{n\to\infty} \frac{\overbrace{1^k + 2^k + \dots + n^k}^{x_n}}{\underbrace{n^{k+1}}_{y_n}} = \lim_{n\to\infty} \frac{x_n}{y_n} \underset{\text{ако } \exists \,\to}{\overset{\text{штолцова т.}}{=}} \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} \frac{(\cancel{1^k + 2^k + \dots +}\, n^k) - (\cancel{1^k + 2^k + \dots + (n-1)^k})}{n^{k+1} - \underbrace{(n-1)^{k+1}}_{\text{из Н.Б. формуле}}}$$

$$= \lim_{n\to\infty} \frac{n^k}{\cancel{n^{k+1}} - \left(\cancel{n^{k+1}} + \underbrace{\binom{k+1}{1} \cdot n^k (-1)^1 + \binom{k+1}{2} \cdot n^{k-1} (-1)^2 + \dots + (-1)^{k+1}}_{\text{полином степена } k}\right)}$$

$$= \lim_{n\to\infty} \frac{n^k}{(k+1) \cdot n^k + \text{полином степена } (k-1)}$$

$$= \boxed{\frac{1}{k+1}} \quad \checkmark$$

За вежбу: $L=\lim\limits_{n\to\infty}\dfrac{1^k+3^k+5^k+\cdots+(2n+1)^k}{n^{k+1}}$

$\boxed{3}$ $$L=\lim_{n\to\infty}\left(\frac{1^k+2^k+\dots+n^k}{n^k}-\frac{n}{k+1}\right)=\lim_{n\to\infty}\frac{(k+1)(1^k+\dots+n^k)-n^{k+1}}{(k+1)\cdot n^k}=$$

$$=\frac{1}{k+1}\cdot\lim_{n\to\infty}\frac{(k+1)(1^k+\dots+n^k)-n^{k+1}}{n^k}$$

x_n

$n^k=y_n\uparrow,\to+\infty$

$$\underset{\text{ако } \exists \to}{\overset{\text{Ш.Т.}}{=}}\frac{1}{k+1}\cdot\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}$$

$$=\frac{1}{k+1}\cdot\lim_{n\to\infty}\frac{(k+1)\cdot(\cancel{1^k+\dots+}n^k)-n^{k+1}-(k+1)\cdot\cancel{(1^k+\dots+(n-1)^k)}+(n-1)^{k+1}}{n^k-(n-1)^k}$$

$$=\frac{1}{k+1}\cdot\lim_{n\to\infty}\frac{(k+1)\,n^k-n^{k+1}+(n-1)^{k+1}}{n^k-(n-1)^k}$$

$$=\frac{1}{k+1}\cdot\lim_{n\to\infty}\frac{\cancel{(k+1)n^k}-\cancel{n^{k+1}}+\cancel{n^{k+1}}+\cancel{\binom{k+1}{1}n^k(-1)}+\overbrace{\binom{k+1}{2}n^{k-1}\cdot(-1)^2+\text{помали степени }(k-2)}}{\cancel{n^k}-\left(\cancel{n^k}+\underbrace{\binom{k}{1}n^{k-1}(-1)}+\text{пом. степ. } k-2\right)}$$

$$=\frac{1}{k+1}\cdot\lim_{n\to\infty}\frac{\frac{(k+1)\cdot k}{2}n^{k-1}+\text{пом. степ. } k-2}{k\cdot n^{k-1}+\text{пом. степ. } k-2}=\frac{1}{k+1}\cdot\frac{\frac{(k+1)k}{2}}{k}=\boxed{\frac{1}{2}}$$

На предавању: **Кошијева I**: $\lim\limits_{n\to\infty} a_n = a \Rightarrow \lim\limits_{n\to\infty} \underbrace{\frac{a_1+\dots+a_n}{n}}_{A_n} = a$

Тврђење: $a_n>0, \forall n$, $\lim\limits_{n\to\infty} a_n = a \Rightarrow \lim\limits_{n\to\infty} \underbrace{\frac{n}{\frac{1}{a_1}+\dots+\frac{1}{a_n}}}_{H_n} = a = \lim\limits_{n\to\infty} \underbrace{\sqrt[n]{a_1\cdots a_n}}_{G_n}$

$\boxed{\text{Тврђење}}$ $(a_n), (\forall n)\, a_n>0$

Ако $\exists \lim\limits_{n\to\infty} \frac{a_n}{a_{n-1}}$, онда постоји и $\lim\limits_{n\to\infty}\sqrt[n]{a_n}$ и важи: $\lim\limits_{n\to\infty}\sqrt[n]{a_n} = \lim\limits_{n\to\infty}\frac{a_n}{a_{n-1}}$.

Доказ: $x_n = \frac{a_n}{a_{n-1}}, \forall n\geq 2$, $x_1 = \frac{a_1}{1}$ Претпоставка: $\exists \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}\frac{a_n}{a_{n-1}} = X$

За G_n: $\lim\limits_{n\to\infty}\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n} = X$

$$\lim_{n\to\infty}\sqrt[n]{\frac{a_1}{1}\cdot\frac{a_2}{\cancel{a_1}}\cdot\frac{\cancel{a_3}}{\cancel{a_2}}\cdot\ldots\cdot\frac{a_n}{\cancel{a_{n-1}}}} = X$$

$$\Rightarrow \lim_{n\to\infty}\sqrt[n]{a_n} = X \quad ✓ \quad \square$$

4. Одреди $L = \lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n}$

$$L = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} \qquad \frac{n!}{n^n} = a_n$$

Рачунамо: $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}} = \lim\limits_{n\to\infty} \dfrac{\cancel{n!}/n^n}{\cancel{(n+1)!}/(n+1)^{n+1}} = \lim\limits_{n\to\infty} \dfrac{1/n^{n+1}}{1/(n+1)^{n+1}} = \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^{n+1} =$

$$= \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty} \underbrace{\left(1+\frac{1}{\underset{\to -\infty}{-n}}\right)^{-n}}_{\to e}{}^{\overbrace{\frac{n+1}{-n}}^{\to -1}} = e^{-1} = \boxed{\frac{1}{e}}$$

Шторц $\Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{a_n} = \dfrac{1}{e} \Rightarrow \boxed{\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}}$ $\square$

! $\sqrt[n]{\dfrac{n!}{n^n}} \to \dfrac{1}{e},\ n\to\infty$

$\dfrac{n!}{n^n} \to 0,\ n\to\infty$

$n! \ll n^n$

$x_n \underset{n\to\infty}{\to} 0 \not\Rightarrow \sqrt[n]{x_n} \underset{n\to\infty}{\to} 0$

$2^n \to 0 \not\Rightarrow 2 \to 0$

За вежбу:

* $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = ?$

* $\lim\limits_{n\to\infty} \frac{1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \dots + n(n+1)(n+2)}{n^4} = ?$

$\boxed{5}$ (a_n): $a_0>0$ дато, $a_{n+1} = a_n + \frac{1}{3\cdot a_n^2}$, $\forall n\geq 0$

(а) Одредити $\lim\limits_{n\to\infty} a_n$.

(б) $\lim\limits_{n\to\infty} \frac{a_n^3}{n} = ?$

(а) $a_{n+1} = a_n + \underbrace{\frac{1}{3a_n^2}}_{>0} > a_n$

$\boxed{(a_n)\uparrow}$ — ограничен $\Rightarrow \exists \lim\limits_{n\to\infty} a_n = a\in\mathbb{R}$ X

— неограничен $\Rightarrow \lim\limits_{n\to\infty} a_n = +\infty$

Ако $\exists \lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$: $\to a>0$ јер $a_0>0$ и $(a_n)\uparrow$

$a_{n+1} = a_n + \frac{1}{3a_n^2}$ $/\lim\limits_{n\to\infty}$

$a = a + \frac{1}{3a^2}$

$0 = \frac{1}{3a^2}$ ↯ $\Rightarrow$ не конвергира $\boxed{\lim\limits_{n\to\infty} a_n = +\infty}$

(б) $\lim\limits_{n\to\infty} \frac{a_n^3}{n}$ (где $a_n^3 = x_n$, $n = y_n \uparrow, \to +\infty$) $\underset{\text{ако } \exists \to}{\overset{\text{Ш.Т.}}{=\!=}} \lim\limits_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$

$= \lim\limits_{n\to\infty} \frac{a_n^3 - a_{n-1}^3}{n-(n-1)}$ ($n-(n-1) = 1$)

$= \lim\limits_{n\to\infty} (a_n^3 - a_{n-1}^3)$

$= \lim\limits_{n\to\infty} \left(a_{n-1} + \frac{1}{3a_{n-1}^2}\right)^3 - a_{n-1}^3$

$= \lim\limits_{n\to\infty} \left(\cancel{a_{n-1}^3} + 3a_{n-1}^2\cdot\frac{1}{3a_{n-1}^2} + 3\cdot a_{n-1}\cdot\frac{1}{9a_{n-1}^4} + \frac{1}{27a_{n-1}^6} - \cancel{a_{n-1}^3}\right)$

$= \lim\limits_{n\to\infty} \left(1 + \cancel{\frac{1}{3a_{n-1}^3}}^{\to 0} + \cancel{\frac{1}{27a_{n-1}^6}}^{\to 0}\right) = \boxed{1}$ ($a_{n-1} \to +\infty$)

б. (x_n): $x_0>0$

за вкъщи

(а) $\lim\limits_{n\to\infty} x_n = ?$

(б) $\lim\limits_{n\to\infty}(n\cdot x_n^2) = ?$

$\circledast$ $x_{n+1} = \dfrac{x_n}{1+x_n^2}, \ \forall n\geq 0$

(а) $(x_n)\downarrow$, $x_n>0, \forall n$

$\overset{T2}{\Longrightarrow}$ $\exists \lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$

$\circledast / \lim\limits_{n\to\infty} \Rightarrow \dots \boxed{x=0}$

$$\boxed{\lim_{n\to\infty} x_n = 0}$$

(б) $n\cdot x_n^2$, $n \to +\infty$, $x_n^2 \to 0$: $+\infty \cdot 0$

$$\boxed{\lim_{n\to\infty}\frac{1}{n\cdot x_n^2}} = \lim_{n\to\infty}\frac{1/x_n^2}{n}$$

$n = y_n \uparrow, \to +\infty$

$$\overset{\text{Щ.Т.}}{\underset{\text{ако }\exists\to}{=}} \lim_{n\to\infty}\frac{\frac{1}{x_n^2}-\frac{1}{x_{n-1}^2}}{n-(n-1)}$$

$$\overset{\circledast}{=} \lim_{n\to\infty}\frac{\left(\frac{1+x_{n-1}^2}{x_{n-1}}\right)^2-\frac{1}{x_{n-1}^2}}{1}$$

$$= \lim_{n\to\infty}\frac{\cancel{1}+2x_{n-1}^2+x_{n-1}^4-\cancel{1}}{x_{n-1}^2}$$

$$= \lim_{n\to\infty}(2+x_{n-1}^2) = \boxed{2}$$

$$\Rightarrow \lim_{n\to\infty}(n\cdot x_n^2) = \frac{1}{\lim\limits_{n\to\infty}\frac{1}{n\cdot x_n^2}} = \frac{1}{2} \quad \square$$

~ Поднизови и тачке нагомилавања ~

(x_n) $\quad n_1 < n_2 < n_3 < \dots \underline{\underline{n_k}} < \dots$

$x_{n_1}, x_{n_2}, \dots \quad (x_{n_k})_{k \in \mathbb{N}}$

$a \in \overline{\mathbb{R}}$ је тачка нагомилавања (x_n) ако $\exists$ подниз (x_{n_k}) тд. $\lim\limits_{k \to \infty} x_{n_k} = a$

т. наг.

$T(x_n)$ - скуп свих т. наг. низа x_n

најмања т. наг. = $\liminf x_n$, $\underline{\lim}\, x_n$ $\qquad$ највећа т. наг. = $\limsup x_n$, $\overline{\lim}\, x_n$

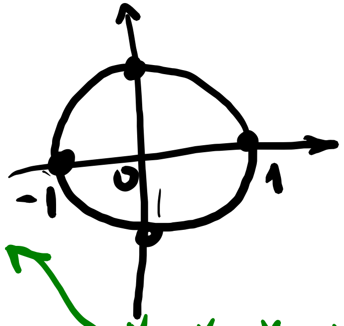
* ако (x_n) конвергентан $\Rightarrow$ сваки подниз конвергира

$\lim\limits_{n \to \infty} x_n = x \qquad \lim\limits_{k \to \infty} x_{n_k} = x$

* (x_n) - ограничен низ $\Rightarrow$ има бар 1 т. наг. у $\mathbb{R}$

* (x_n) - низ $\Rightarrow$ има бар 1 т. наг. у $\overline{\mathbb{R}}$

$\boxed{1.}$ $T(x_n)=?$, $\underline{\lim}\, x_n=?$, $\overline{\lim}\, x_n=?$ $\qquad x_n = 1+\underbrace{\frac{n^2}{n^2+5}}_{\to 1}\cdot\cos\frac{n\pi}{2}$



$$\left[\begin{array}{ll} n=1: & \cos\frac{\pi}{2}=0 \\ n=2: & \cos\frac{2\pi}{2}=-1 \\ n=3: & \cos\frac{3\pi}{2}=0 \\ n=4: & \cos\frac{4\pi}{2}=1 \end{array}\right.$$

$n=5: \quad \cos\frac{5\pi}{2}=0$

$$\cos\frac{n\pi}{2}=\begin{cases} 0, & n\equiv 1 \mod 4 \\ -1, & n\equiv 2 \mod 4 \\ 0, & n\equiv 3 \mod 4 \\ 1, & n\equiv 0 \mod 4 \end{cases}$$

$\to (x_{4k+1})_{k\in\mathbb{N}} \leftarrow x_1, x_5, x_9, x_{13}, \dots$

$\to (x_{4k+2})_{k\in\mathbb{N}}$

- $\lim\limits_{k\to\infty} x_{4k+1} = \lim\limits_{k\to\infty}\left(1+\frac{(4k+1)^2}{(4k+1)^2+5}\cdot 0\right) = 1$
- $\lim\limits_{k\to\infty} x_{4k+2} = \lim\limits_{k\to\infty}\left(1+\underbrace{\frac{(4k+1)^2}{(4k+1)^2+5}}_{\to 1}(-1)\right) = 1+1\cdot(-1) = 0$
- $\lim\limits_{k\to\infty} x_{4k+3} = 1+1\cdot 0 = 1$
- $\lim\limits_{k\to\infty} x_{4k+4} = 1+1\cdot 1 = 2$

$T(x_n)=\{0,1,2\}$

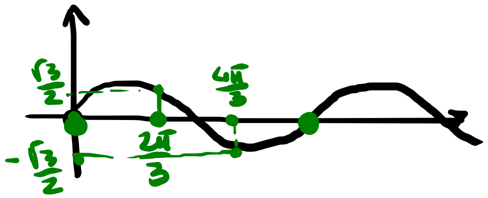
$\underline{\lim}\, x_n = 0$

$\overline{\lim}\, x_n = 2$

$\boxed{2.}$ $x_n = \underbrace{(-1)^n}\cdot\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e} + \underbrace{\frac{n}{n+1}}_{\to 1}\cdot\underbrace{\sin\frac{2n\pi}{3}} + \underbrace{\frac{\ln n}{n!}}_{\to 0,\ n\to\infty}$; $T(x_n)$, $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$?

$(-1)^n = \begin{cases} 1, & n=2k \\ -1, & n=2k+1 \end{cases}$ $\boxed{\text{mod } 2}$

$\sin n\frac{2\pi}{3}$



$\sin\frac{2n\pi}{3} = \begin{cases} 0, & n=3k \\ \frac{\sqrt3}{2}, & n=3k+1 \\ -\frac{\sqrt3}{2}, & n=3k+2 \end{cases}$ $\boxed{\text{mod } 3}$

! mod 2
mod 3 НЗС(2,3)=6 $\boxed{\text{mod } 6}$

$x_{6k}, x_{6k+1}, \dots, x_{6k+5}$

$\boxed{6k}$ $\lim\limits_{k\to\infty} x_{6k} = \lim\limits_{k\to\infty} \underbrace{(-1)^{6k}}_{=1}\cdot\underbrace{\left(1+\frac{1}{6k}\right)^{6k}}_{\to e} + \underbrace{\frac{6k}{6k+1}}_{\to 1}\cdot\underbrace{\sin\frac{2\cdot 6k\pi}{3}}_{=0} + \cancel{\frac{\ln(6k)}{(6k)!}}^{\to 0} = 1\cdot e + 1\cdot 0 + 0 = \boxed{e}$

$\boxed{6k+1}$ $(-1)^{6k+1} = -1$

$\sin\frac{2(6k+1)\pi}{3} = \sin\left(\frac{2\pi}{3}+4k\pi\right) = \frac{\sqrt3}{2}$

$\lim\limits_{k\to\infty} x_{6k+1} = (-1)\cdot e + 1\cdot\frac{\sqrt3}{2} + 0 = \boxed{\frac{\sqrt3}{2} - e}$

За всяко: $x_{6k+2}\to e-\frac{\sqrt3}{2}$ $x_{6k+4}\to e+\frac{\sqrt3}{2}$

$k\to\infty$ $x_{6k+3}\to -e$ $x_{6k+5}\to -e-\frac{\sqrt3}{2}$

$T(x_n) = \{\underbrace{-e-\frac{\sqrt3}{2}}, e-\frac{\sqrt3}{2}, \frac{\sqrt3}{2}-e, -e, e, \underbrace{\frac{\sqrt3}{2}+e}\}$

$\underline{\lim}\, x_n = -e-\frac{\sqrt3}{2}$ $\overline{\lim}\, x_n = \frac{\sqrt3}{2}+e$

$\boxed{3.}$ $T(a_n) = ?$

$$a_n = \sqrt[n]{2^n + 5^{n(-1)^n}}$$

$(-1)^n$ mod 2

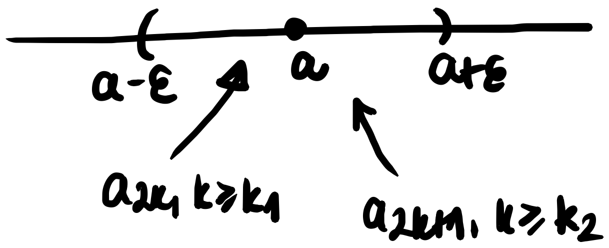
$$a_{2k} = \sqrt[2k]{2^{2k} + 5^{2k}} \xrightarrow[k\to+\infty]{} \max\{2,5\} = 5$$

$$a_{2k+1} = \sqrt[2k+1]{2^{2k+1} + 5^{-(2k+1)}} = \sqrt[2k+1]{2^{2k+1} + \left(\frac{1}{5}\right)^{2k+1}} \xrightarrow[k\to\infty]{} \max\{\tfrac{1}{5}, 2\} = 2$$

$$T(a_n) = \{2, 5\} \qquad \square$$

$\boxed{4.}$ (a_n) : $\lim\limits_{k\to\infty} a_{2k} = a = \lim\limits_{k\to\infty} a_{2k+1}$, $a\in\mathbb{R}$ (за важи $a=\pm\infty$)

Доказати да тада цео низ (a_n) конвергира и $\lim\limits_{n\to\infty} a_n = a$



$\boxed{\varepsilon > 0}$

$\lim\limits_{k\to\infty} a_{2k} = a$: $\exists k_1\in\mathbb{N}\ \ \forall k\geq k_1\ \ |a_{2k}-a|<\varepsilon$ (1)

$\lim\limits_{k\to\infty} a_{2k+1} = a$; $\exists k_2\in\mathbb{N}\ \ \forall k\geq k_2\ \ |a_{2k+1}-a|<\varepsilon$ (2)

$n_0(\varepsilon) = \boxed{n_0} = 2\cdot\max\{k_1,k_2\}+1$

(1),(2) $\Rightarrow$ $\forall n\geq n_0\ \ |a_n-a|<\varepsilon$

$\forall\varepsilon>0$

$\Rightarrow \lim\limits_{n\to\infty} a_n = a$ $\square$

$\boxed{5.}$ (x_n) Ако поднизови $(x_{2k}), (x_{2k+1}), (x_{3k})$ конвергирају доказати да конвергира низ (x_n).

$$\lim_{k\to\infty} x_{2k} = a \qquad \lim_{k\to\infty} x_{3k} = c \qquad a, b, c \in \mathbb{R}$$

$$\lim_{k\to\infty} x_{2k+1} = b$$

$x_{2k} \xrightarrow[k\to\infty]{} a$: $x_2, x_4, x_6, x_8, x_{10}, x_{12}, x_{14}, x_{16}, x_{18}, \ldots \to a$

$\underline{x_{6k}}$, $k\in\mathbb{N}$ - подниз низа (x_{2k}) $\Rightarrow$ $\boxed{\lim_{k\to\infty} x_{6k} = a}$

$x_{3k} \xrightarrow[k\to\infty]{} c$: $x_3, x_6, x_9, x_{12}, x_{15}, x_{18}, \ldots \to c$ $\Rightarrow$ $\boxed{\lim_{k\to\infty} x_{6k} = c}$

$\Rightarrow \boxed{a = c}$ (1)

аналогно:

x_{2k+1} : $x_1, x_3, x_5, x_7, x_9, x_{11}, x_{13}, x_{15}, x_{17}, \ldots \to b$

x_{6k+3}, $k\in\mathbb{N}_0$ - подниз низа (x_{2k+1}) $\Rightarrow$ $\boxed{\lim_{k\to\infty} x_{6k+3} = b}$

x_{3k}: $x_3, x_6, x_9, x_{12}, x_{15}, x_{18}, \ldots \to c$ $\Rightarrow$ $\boxed{\lim_{k\to\infty} x_{6k+3} = c}$

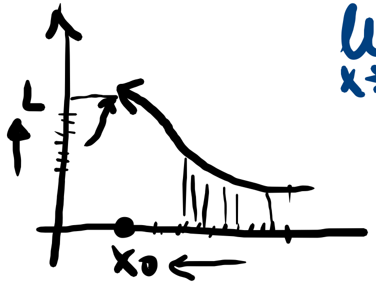
$\Rightarrow \boxed{b = c}$ (2

(1), (2) $\Rightarrow \boxed{a = b}$

$\underline{\text{зад. } \boxed{4}} \Rightarrow \boxed{\lim x_n = a} = b = c$

~ Веза лимеса низа и лимеса функције ~

ХАЈНЕ $f: X \to \mathbb{R}$, $x_0 \in \overline{\mathbb{R}}$ тачка нагом. скупа X. Тада:



$$\lim_{x\to x_0} f(x) = L \in \overline{\mathbb{R}} \iff \text{за } \forall \text{ низ } x_n \in X \setminus x_0 \text{ важи импликација:}$$

$$\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = L$$

Последица $f: X \to \mathbb{R}$ непрекидна у $x_0 \in X \iff$ за $\forall$ низ (x_n) у X важи:

$$\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)$$

$\boxed{1.}$ Да ли постоји $a \in \mathbb{R}$ тако да функција $f(x) = \begin{cases} \cos^2 \frac{1}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$ буде непр. на $\mathbb{R}$?

$x \neq 0$: $\cos^2 \frac{1}{x}$ непр. у X ✓

$x = 0$: Тражимо да $\exists a \in \mathbb{R}$ тд. f непр. у $x=0$

Хајне → $\boxed{\lim_{n\to\infty} x_n = 0 \Rightarrow \lim_{n\to\infty} \underbrace{\cos^2 \frac{1}{x_n}}_{f(x_n)} = a}$

• $\boxed{x_n = \frac{1}{2n\pi}} \to 0, n \to \infty$ $\lim_{n\to\infty} \cos^2 \frac{1}{x_n} = \lim_{n\to\infty} \cos^2 2n\pi = \boxed{1} \Rightarrow a = 1$ (1)

$$\boxed{x_n'} = \frac{1}{2n\pi+\frac{\pi}{2}} \to 0,\ n\to\infty$$

$$\lim_{n\to\infty} f(x_n') = \lim_{n\to\infty} \cos^2\left(2n\pi+\frac{\pi}{2}\right) = 0 \Rightarrow \boxed{a=0} \qquad (2)$$

(1), (2) ↯ $\Rightarrow$ $\boxed{\text{граница } a \text{ не постоји}}$

$$\left[\; f(0) \overset{\text{ако } \exists}{=} \lim_{x\to 0}\left(\cos^2 \underbrace{\frac{1}{x}}_{= t}\right) \quad \xleftarrow{x\to 0+} \lim_{t\to+\infty}\cos^2 t \qquad \xleftarrow{x\to 0-} \lim_{t\to-\infty}\cos^2 t \;\right]$$

$2n\pi \to +\infty \quad \cos 2n\pi = 1 \to 1$

$2n\pi+\frac{\pi}{2} \to +\infty \quad \cos\left(2n\pi+\frac{\pi}{2}\right) = 0 \to 0$

$\boxed{2}$ $L=\lim\limits_{n\to\infty} n\cdot\left(\dfrac{\sqrt[n]{e}+1}{\sqrt[n]{e}-1}-2n\right)$ „$\infty\cdot(\infty-\infty)$"

$\sqrt[n]{e}=\underbrace{e^{\frac{1}{n}}}_{f\left(\frac{1}{n}\right)}$, $\frac{1}{n}\to 0$, $n\to\infty$ Маклоренов развој $\underbrace{e^x}_{f(x)}$, $x\to 0$

$0 \leftarrow \frac{1}{n}$

$$e^{\frac{1}{n}}=1+\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{6\cdot n^3}+o\left(\frac{1}{n^3}\right),\ n\to\infty$$

$$L=\lim_{n\to\infty} n\cdot\left(\frac{e^{\frac{1}{n}}+1}{e^{\frac{1}{n}}-1}-2n\right)=\lim_{n\to\infty} n\cdot\left(\frac{2+\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right)}{\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right)}-2n\right)$$

$$=\boxed{\frac{1}{n}}\left(1+\frac{1}{2n}+\frac{1}{6n^2}+o\left(\frac{1}{n^2}\right)\right)$$

$$=\lim_{n\to\infty} n\cdot\left(n\cdot\left(2+\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right)\right)\cdot\Big(1+\underbrace{\frac{1}{2n}+\frac{1}{6n^2}+o\left(\frac{1}{n^2}\right)}_{t}\Big)^{-1}-2n\right)$$

$$(1+t)^{-1}=1-t+t^2+o(t^2)$$

$$=\lim_{n\to\infty} n^2\left(\Big(2+\frac{1}{n}+\frac{1}{2n^2}+\underbrace{\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right)}_{o\left(\frac{1}{n^2}\right)}\Big)\cdot\left(1-\left(\frac{1}{2n}+\frac{1}{6n^2}+o\left(\frac{1}{n^2}\right)\right)+\left(\frac{1}{2n}+\frac{1}{6n^2}+o\left(\frac{1}{n^2}\right)\right)^2+o\left(\frac{1}{n^2}\right)\right)-2\right)$$

$$=\lim_{n\to\infty} n^2\cdot\left(\left(2+\frac{1}{n}+\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)\cdot\left(1-\frac{1}{2n}-\frac{1}{6n^2}+\frac{1}{4n^2}+o\left(\frac{1}{n^2}\right)\right)-2\right)$$

$$= \dots$$

$$= \lim_{n\to\infty} n^2 \cdot \left(\frac{1}{6} \cdot \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{6} + \underbrace{o(1)}_{\substack{\text{niz koji} \to 0 \\ n\to\infty}} \right)$$

$$= \boxed{\frac{1}{6}}$$

За крај ћемо размотрити једну специјалну класу низова:

~Низови задати линеарним диференцним једначинама реда 2 ~

Нека знамо прва два члана низа: x_1, x_2 (или x_0, x_1)

а сваки следећи је дат са: $\boxed{x_{n+2} = a\cdot x_{n+1} + b\cdot x_n}$, $n\in\mathbb{N}$ (или $x_n = a x_{n-1} + b x_{n-2}$ ...)

Поставља се питање како експлицитно израчунати конкретан члан низа,

нпр. $x_{2020} = ?$.

Поступак је следећи: од $\boxed{x_{n+2} = a\cdot x_{n+1} + b\cdot x_n}$

формирамо карактеристичну једначину:

$$\boxed{t^2 = a\cdot t + b}$$

њеним решавањем добијамо или два различита решења $t_1 \neq t_2$,

или једно двоструко $t_1 = t_2 = t_0$.

1) Ако $t_1 \neq t_2$ → низ x_n је облика $x_n = c_1\cdot t_1^n + c_2 t_2^n$

c_1, c_2 – константе које одређујемо из x_1 и x_2

назива се
ОПШТЕ РЕШЕЊЕ

2) Ако $t_1 = t_2 = t_0$ → низ x_n је облика $x_n = c_1\cdot t_0^n + c_2\cdot n\cdot t_0^n$

c_1, c_2 – из x_1 и x_2, такође

14. Одредити формулу за општи члан низа (x_n), $x_1 = \frac{13}{6}$, $x_0 = 5$

$$6x_{n+2} = 5x_{n+1} - x_n, \quad n \geq 0$$

карактеристична једначина: $6t^2 = 5t - 1$

решавамо: $6t^2 - 5t + 1 = 0$ $\quad t_{1,2} = \frac{5\pm\sqrt{25-4\cdot 6}}{2\cdot 6} = \frac{5\pm 1}{12}$ → $\boxed{1/2 = t_1}$, $\boxed{1/3 = t_2}$

$$\Rightarrow \boxed{x_n = c_1\cdot\left(\tfrac{1}{2}\right)^n + c_2\cdot\left(\tfrac{1}{3}\right)^n}$$

$c_1, c_2 = ?$

$x_0 = 5 = c_1\cdot\left(\frac{1}{2}\right)^0 + c_2\cdot\left(\frac{1}{3}\right)^0$ тј. $5 = c_1 + c_2$

$x_1 = \frac{13}{6} = c_1\cdot\left(\frac{1}{2}\right)^1 + c_2\cdot\left(\frac{1}{3}\right)^1$ тј. $\frac{13}{6} = \frac{c_1}{2} + \frac{c_2}{3}$ $\Big/\cdot 6$

систем

решавамо систем: $\left.\begin{array}{l} 5 = c_1 + c_2 \\ 13 = 3c_1 + 2c_2 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} c_1 = 3 \\ c_2 = 2 \end{array}}$

дакле, $\boxed{x_n = 3\cdot\left(\tfrac{1}{2}\right)^n + 2\cdot\left(\tfrac{1}{3}\right)^n}$ ∎

15. (а) Одредити низ (y_n): $y_1 = \frac{5}{6}$, $y_2 = \frac{1}{3}$, $y_{n+1} = \frac{2}{3}y_n - \frac{1}{9}y_{n-1}$, $n \geq 2$

(б) За низ (x_n) из претходног задатка одредити: $\lim\limits_{n\to\infty} \frac{x_n}{y_n}$.

(в) — || — — || — : $\lim\limits_{n\to\infty} \sqrt[n]{\frac{x_n}{y_n}}$.

(а) $y_{n+1} = \frac{2}{3}y_n - \frac{1}{9}y_{n-1}$

$t^2 = \frac{2}{3}t - \frac{1}{9}$

$t^2 - \frac{2}{3}t + \frac{1}{9} = 0 \iff \left(t - \frac{1}{3}\right)^2 = 0 \to$ двоструко решење $t_{1,2} = \frac{1}{3}$, па:

$$\underline{y_n = C_1\left(\frac{1}{3}\right)^n + C_2 \cdot n \cdot \left(\frac{1}{3}\right)^n}$$

$C_1, C_2 = ?$

$$\left.\begin{array}{l} \frac{5}{6} = y_1 = C_1 \cdot \frac{1}{3} + C_2 \cdot 1 \cdot \frac{1}{3} \Rightarrow 5 = 2C_1 + 2C_2 \\ \frac{1}{3} = y_2 = C_1\left(\frac{1}{3}\right)^2 + C_2 \cdot 2\left(\frac{1}{3}\right)^2 \overset{/\cdot 9}{\Rightarrow} 3 = C_1 + 2C_2 \end{array}\right\} \Rightarrow C_1 = 2,\ C_2 = 1/2$$

низ: $y_n = 2\left(\frac{1}{3}\right)^n + \frac{n}{2} \cdot \left(\frac{1}{3}\right)^n$

$$\boxed{y_n = \left(2 + \frac{n}{2}\right) \cdot \left(\frac{1}{3}\right)^n} \quad \forall n \in \mathbb{N}$$

(б) сада користимо и $x_n = 3 \cdot \left(\frac{1}{2}\right)^n + 2 \cdot \left(\frac{1}{3}\right)^n$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{3\left(\frac{1}{2}\right)^n + 2 \cdot \left(\frac{1}{3}\right)^n}{\left(2 + \frac{n}{2}\right) \cdot \left(\frac{1}{3}\right)^n} \quad /\cdot 3^n$$

$$= \lim_{n\to\infty} \frac{3 \cdot \left(\frac{1}{2}\right)^n \cdot 3^n + 2}{2 + n/2} = \lim_{n\to\infty} \frac{3 \cdot \left(3/2\right)^n + 2}{2 + \frac{n}{2}}$$

експоненцијална q^n, $q > 1$ је јача од полинома тј. $2 + \frac{n}{2} \ll 3\left(\frac{3}{2}\right)^n + 2$, $n \to \infty$

$$= \boxed{+\infty}$$

можете и доследно, скратити са $\left(\frac{3}{2}\right)^n$: ...

(в) $$\lim_{n\to\infty} \sqrt[n]{\frac{x_n}{y_n}} = \lim_{n\to\infty} \sqrt[n]{\frac{3 \cdot \left(\frac{1}{2}\right)^n + 2 \cdot \left(\frac{1}{3}\right)^n}{\left(2 + \frac{n}{2}\right) \cdot \left(\frac{1}{3}\right)^n}} \quad /\cdot 3^n$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{3 \cdot \left(\frac{3}{2}\right)^n + 2}{2 + n/2}} = \lim_{n\to\infty} \frac{\sqrt[n]{\left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^n + 1^n + 1^n}}{\sqrt[n]{2 + n/2}}$$

$\sqrt[n]{\left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^n + \left(\frac{3}{2}\right)^n + 1^n + 1^n} \xrightarrow[n\to\infty]{} \max\left\{\frac{3}{2}, 1\right\} = 3/2$

$\sqrt[n]{2 + n/2} \xrightarrow[n\to\infty]{} 1$ из ⊛

$$\frac{3/2}{1} = \boxed{\frac{3}{2}}$$

□

⊛ $$\lim_{n\to\infty} \sqrt[n]{2 + \frac{n}{2}} = \lim_{n\to\infty} \sqrt[n]{n} \cdot \sqrt[n]{\frac{1}{2} + \frac{2}{n}}$$

$$= \lim_{n\to\infty} \underbrace{\sqrt[n]{n}}_{\to 1} \cdot \underbrace{\left(\frac{1}{2} + \frac{2}{n}\right)}_{\to 1/2}^{\overbrace{1/n}^{\to 0}} = 1 \cdot \left(\frac{1}{2}\right)^0 = \underline{\underline{1}}$$

~ Крај 🙂 ~

Срећно даље!