~Низови~

$x:\mathbb{N}\to\mathbb{R}$ $\quad n\mapsto x(n)=x_n$

(x_n) , $(x_n)_{n\in\mathbb{N}}$

$x_1, x_2, x_3, \ldots$

Својства:

- периодичност ← $a_n=\frac{n}{4}-\left[\frac{n}{4}\right]$ $T=4$
- ограниченост

  (x_n) ограничен: $\exists M\ \forall n\in\mathbb{N}\ |x_n|<M$

  огр. одозго: $\exists M\ \forall n\in\mathbb{N}\ x_n<M$

  огр. одоздо: $\exists M\ \forall n\in\mathbb{N}\ x_n>M$

- монотоност:

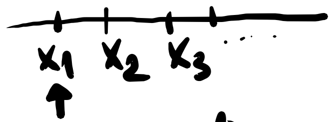
  растечки: $x_n\le x_{n+1}, \forall n\in\mathbb{N}$

  строго: $<$

  опаѓачки: $x_n\ge x_{n+1}, \forall n\in\mathbb{N}$

  строго: $>$

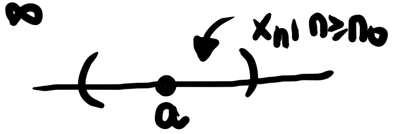
* (x_n) растечки $\Rightarrow$ (x_n) огр. одоздо



(x_n) опаѓачки $\Rightarrow$ (x_n) огр. одозго

* $a\in\mathbb{R}$

$\lim\limits_{n\to\infty} x_n=a \iff (\forall\varepsilon>0)(\exists n_0\in\mathbb{N})(\forall n\in\mathbb{N})\ n\ge n_0\Rightarrow |x_n-a|<\varepsilon$



$\lim\limits_{n\to\infty} x_n=+\infty \iff (\forall M\in\mathbb{R})(\exists n_0\in\mathbb{N})(\forall n\in\mathbb{N})\ n\ge n_0\Rightarrow x_n>M$

$\lim\limits_{n\to\infty} x_n=-\infty \iff \ldots$

$\boxed{\lim\limits_{n\to\infty} x_n=a\in\mathbb{R}}$ ← (x_n) конвергентан

$\left.\begin{array}{l}\lim\limits_{n\to\infty} x_n=+\infty/-\infty \\ (x_n) \text{ нема гр. вр.}\end{array}\right\}$ (x_n) дивергентан

* конвергира $\Rightarrow$ ограничен
  $\not\Leftarrow$

* $\lim\limits_{n\to\infty} x_n = 0 \Leftrightarrow \lim\limits_{n\to\infty} |x_n| = 0$

  $\left[\ \lim\limits_{n\to\infty} x_n = a \neq 0 \not\Leftrightarrow \lim |x_n| = |a|\ \right]$

  $\Rightarrow$

  $\not\Leftarrow$

* $$\left.\begin{array}{l}\lim\limits_{n\to\infty} x_n = 0 \\ (y_n)\text{ - ограничен}\end{array}\right\} \Rightarrow \lim\limits_{n\to\infty} (x_n \cdot y_n) = 0$$

* геометријски низ: $x_n = q^n,\ q \in \mathbb{R}$

$$\lim_{n\to\infty} q^n = \begin{cases} 0, & |q| < 1 \\ 1, & q = 1 \\ +\infty, & q > 1 \\ \text{не постоји}, & q \leq -1 \end{cases}$$

* $k > 0$ $\quad \lim\limits_{n\to\infty} \dfrac{1}{n^k} = 0$

① по дефиницији lim:

$$\lim_{n\to\infty}\frac{2n+3}{n+5}=2$$

$\underline{\varepsilon>0}$? $\exists n_0$ $\forall n\geq n_0$ $\underbrace{\left|\frac{2n+3}{n+5}-2\right|<\varepsilon}$

$\iff \left|\frac{-7}{n+5}\right|<\varepsilon$

$\iff \frac{7}{n+5}<\varepsilon$

$\iff \frac{7}{\varepsilon}-5<n$

$n_0=\left[\frac{7}{\varepsilon}\right] \to n\geq n_0 \Rightarrow |x_n-2|<\varepsilon$ ✓

$\forall\varepsilon>0 \Rightarrow \lim_{n\to\infty}\frac{2n+3}{n+5}=2$ □

II начин:

$$\lim_{n\to\infty}\frac{2n+3}{n+5}\overset{:n}{=}\lim_{n\to\infty}\frac{2+\frac{3}{n}}{1+\frac{5}{n}}=\frac{2}{1}=2$$

$\frac{3}{n}\to 0$, $2+\frac{3}{n}\to 2$; $\frac{5}{n}\to 0$, $1+\frac{5}{n}\to 1$ □

$$\lim_{n\to\infty}\frac{-2n^3-1}{n^2+50}\overset{:n^2}{=}\lim_{n\to\infty}\frac{-2n-\frac{1}{n^2}}{1+\frac{50}{n^2}}=-\infty$$

$-2n-\frac{1}{n^2}\to-\infty$, $1+\frac{50}{n^2}\to 1$

$$\lim_{n\to\infty}\frac{a_k\cdot n^k+a_{k-1}\cdot n^{k-1}+\dots+a_0}{b_m\cdot n^m+b_{m-1}\cdot n^{m-1}+\dots+b_0}\ (:n^m) \qquad a_k, b_m\neq 0$$

$$=\begin{cases}0, & k<m\\ \frac{a_k}{b_m}, & k=m\\ +\infty, & k>m \text{ и } a_k, b_m \text{ - истог знака}\\ -\infty, & k>m \text{ и } a_k, b_m \text{ - разл. знака}\end{cases}$$

2 $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} = \lim_{n\to\infty} \frac{\frac{n(n-1)\cdots(n-k+1)}{k!}}{n^k} = \lim_{n\to\infty} \frac{1}{k!} \cdot \frac{\overbrace{n(n-1)\cdots(n-k+1)}}{n^k}$$

полином deg $= k$

$n^k + \ldots$

$$= \frac{1}{k!} \cdot \frac{1}{1} = \boxed{\frac{1}{k!}}$$

3

$$\lim_{n\to\infty} \frac{1^2 + 3^2 + \cdots + (2n-1)^2}{n^3} = ?$$

$$1^2+3^2+\cdots+(2n-1)^2 = \left(\underbrace{1^2+2^2+3^2+\cdots+(2n)^2}\right) - \left(2^2+4^2+\cdots+(2n)^2\right)$$

$$= \frac{2n(2n+1)(2\cdot 2n+1)}{6} - 4\cdot(1^2+2^2+\cdots+n^2)$$

$$= \frac{2n(2n+1)(4n+1)}{6} - 4\cdot\frac{n(n+1)(2n+1)}{6}$$

пол. степ. ≤ 3

$$= \frac{16}{6}n^3 + \text{малки ст.} - \frac{8}{6}n^3 - \text{малки} \ldots$$

$$= \boxed{\frac{4}{3}n^3 + \text{малки ст.}}$$

$$= \lim_{n\to\infty} \frac{\frac{4}{3}n^3 + \text{малки ст.}}{n^3}$$

$$= \boxed{\frac{4}{3}}$$

$\boxed{4.}$ $\lim\limits_{n\to\infty} \sqrt{n^2+n+1} - \sqrt{n^2-n+1}$ $\leftarrow$ $\infty - \infty$

$$= \lim_{n\to\infty} \frac{(n^2+n+1)-(n^2-n+1)}{\sqrt{n^2+n+1}+\sqrt{n^2-n+1}}$$

$:n$

$$= \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+n+1}+\sqrt{n^2-n+1}}$$

$$= \lim_{n\to\infty} \frac{2}{\underbrace{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}}_{\to 1}+\underbrace{\sqrt{1-\frac{1}{n}+\frac{1}{n^2}}}_{\to 1}} \to 2$$

$$= \frac{2}{2} = \boxed{1}$$

$\boxed{5.}$ $\lim\limits_{n\to\infty} \Big(\underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}}_{\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cdots+\cancel{\frac{1}{n}}-\frac{1}{n+1}}\Big) =$

[illegible]
[illegible]

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$= \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = \boxed{1}$$

$\boxed{6.}$ $\lim\limits_{n\to\infty} \dfrac{2^{n+2}+5^{n+2}}{2^n-5^n}$ $\dfrac{+\infty}{-\infty}$

$:5^n$

$$= \lim_{n\to\infty} \frac{4\overbrace{\left(\frac{2}{5}\right)^n}^{\to 0}+25}{\underbrace{\left(\frac{2}{5}\right)^n}_{\to 0}-1} = \frac{25}{-1} = -25.$$

$\left|\frac{2}{5}\right| < 1$

$\boxed{\text{Т}}$ (о 2 полиц. / 3 милиса)

$(a_n), (b_n), (c_n) \qquad a_n \le b_n \le c_n, \ \forall n \ge n_0$

Ако $\exists \lim\limits_{n\to\infty} a_n = x = \lim\limits_{n\to\infty} c_n \Rightarrow \lim\limits_{n\to\infty} b_n = x$

$$\underbrace{a_n \le b_n \le c_n}_{a_n \to x,\ c_n \to x} \Rightarrow b_n \to x$$

Са изравенства:

$$\lim_{n\to\infty} \sqrt[n]{a} = 1 \quad a>0 \qquad \Leftarrow \underline{\underline{a}}^{\frac{1}{n}\to 0} \to a^0 = 1$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \qquad \Leftarrow n^{\frac{1}{n}} = e^{\frac{\ln n}{n}\to 0} \to e^0 = 1$$

$$\lim_{x\to\infty} x^{1/x} = 1$$

Пр.: $$\lim_{n\to\infty} \frac{2^{\frac{1}{n+2}} + 5^{\frac{1}{n+2}}}{2^{\frac{1}{n}} + \sqrt[n]{7}} = \lim_{n\to\infty} \frac{\overset{\to 1}{\sqrt[n+2]{2}} + \overset{\to 1}{\sqrt[n+2]{5}}}{\underset{\to 1}{\sqrt[n]{2}} + \underset{\to 1}{\sqrt[n]{7}}} = \frac{2}{2} = \boxed{1}$$

Тврдење: $a_1, \dots, a_m > 0$ $(m \in \mathbb{N})$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{a_1^n + a_2^n + \dots + a_m^n} = \max\{a_1, a_2, \dots, a_m\}$$

Д: $a = \max\{a_1, \dots, a_m\}$

$$\boxed{a} = \sqrt[n]{a^n} \leq \sqrt[n]{a_1^n + \dots + a_m^n} \underset{\substack{a_i^n \leq a^n \\ \forall i}}{\leq} \sqrt[n]{\underbrace{a^n + a^n + \dots + a^n}_{m}} = \sqrt[n]{m \cdot a^n} = a \cdot \underbrace{\sqrt[n]{m}}_{\to 1}$$

$$\boxed{a} \xrightarrow{=} a \xleftarrow{n\to\infty} a \cdot \sqrt[n]{m}$$

3 лимеса $\Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{a_1^n + \dots + a_m^n} = a$ □

Пр: $\lim\limits_{n\to\infty}\sqrt[n]{2^n+500^n+(\sqrt{3})^n} = 500$

$\lim\limits_{n\to\infty} 2^n = \lim\limits_{n\to\infty}(1^n) = \lim\limits_{n\to\infty} 1 = 1$

$2=1$ $\quad 1^n = 1$

$1^\infty \to \left(1+\underbrace{\frac{1}{n^2}}_{\to 1}\right)^{n^2}$, $\quad n^2\to\infty$

$\cancel{1^\infty = 1}$

7. $x>0$

$\lim\limits_{n\to\infty}\sqrt[n]{1+x^{2n}+x^{5n}} = \lim\limits_{n\to\infty}\sqrt[n]{1^n+(x^2)^n+(x^5)^n}$

шбр5. $\max\{1,x^2,x^5\} = \begin{cases} 1, & x\in(0,1] \\ x^5, & x\in[1,+\infty)\end{cases}$

$\leftarrow x\le 1 \Rightarrow x^5, x^2 \le 1$

$\leftarrow x>1 \quad x^5 > x^2$

за вежбу: $\lim\limits_{n\to\infty}\sqrt[n]{\frac{1}{x^{5n}}+x^{4n}+x^{5n}}$

8. $L=\lim\limits_{n\to\infty}\Bigg(\overbrace{\underbrace{\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\dots+\frac{1}{\sqrt{n^2+n}}}_{n \text{ члана}}}^{a_n}\Bigg)$

$n\cdot\underbrace{\min}_{\frac{1}{\sqrt{n^2+n}}} \le a_n \le n\cdot\underbrace{\max}_{\frac{1}{\sqrt{n^2+1}}}$

$\frac{n}{\sqrt{n^2+n}} \overset{L:n}{\le} a_n \le \frac{n}{\sqrt{n^2+1}}, \ \forall n\in\mathbb{N}$

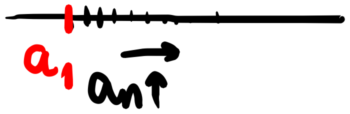
$\frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+\frac{1}{n}}} \xrightarrow[n\to\infty]{} 1$, $\quad \frac{1}{n}\to 0$

$\frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \to 1$

3 лимеса $\Rightarrow \lim\limits_{n\to\infty} a_n = 1$

$\boxed{L=1}$ □

~ Монотони низови ~



$a_n\uparrow$ $\nearrow$ $\lim\limits_{n\to\infty} a_n = +\infty$

$\searrow$ $\lim\limits_{n\to\infty} a_n = a\in\mathbb{R}$

$\boxed{T1}$ $(x_n)\uparrow$ почев од n_0

(x_n) конвергентан $\Leftrightarrow$ (x_n) огр. одозго

$\boxed{T2}$ $(x_n)\downarrow$ почев од n_0

(x_n) конвергентан $\Leftrightarrow$ (x_n) огр. одоздо

$\boxed{1}$ Испитати конвергенцију

$x_n=\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2+\sqrt{2}}}}}$ (n корена)

$x_{n+1}=\sqrt{2+x_n}\,,\ n\geq 1$

$x_1=\sqrt{2}\,,\ x_2=\sqrt{2+\sqrt{2}}$

1° $(x_n)\uparrow$ јер:

индукцијом доказујемо $\forall n\in\mathbb{N}\ x_n\leq x_{n+1}$

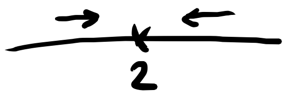
- $n=1$: $x_1\leq x_2$ $\quad \sqrt{2}\leq\sqrt{2+\sqrt{2}}$ ✓
- $n\mapsto n+1$ И.Х: $x_n\leq x_{n+1}$

за $n+1$: $x_{n+1}\overset{?}{\leq} x_{n+2}$

$\sqrt{2+x_n}\leq\sqrt{2+x_{n+1}}$ ✓

$\forall n\in\mathbb{N}$ $x_n\leq x_{n+1}$ $\boxed{(x_n)\uparrow}$

2° Доказујемо да $\forall n\in\mathbb{N}$ $x_n\leq 2$



индукција: • $n=1$ $\quad x_1=\sqrt{2}\leq 2$

- $n\mapsto n+1$ И.Х. $x_n\leq 2$

за $n+1$: $x_{n+1}=\sqrt{2+x_n}\leq\sqrt{2+2}=2$ $\quad x_{n+1}\leq 2$ ✓

$\Rightarrow x_n\leq 2,\ \forall n\in\mathbb{N}$ $\quad (x_n)$-огр. одозго ✓

1°, 2° $\overset{T1}{\Longrightarrow}$ $\boxed{\exists\lim\limits_{n\to\infty}x_n=x\in\mathbb{R}}$ $\quad x=?$

$x_{n+1}=\sqrt{2+x_n}\ \Big/\lim\limits_{n\to\infty}$

$\boxed{x=\sqrt{2+x}}$

$x^2=2+x\quad x^2-x-2=0$

$x=2\ \lor\ \cancel{x=-1}$ јер $x_n\geq 0,\ \forall n$

$\Rightarrow\boxed{x=2}\ \Big|\ \boxed{\lim x_n=2}$ ∎

$\boxed{2}$ $a>0$ дат $x_1>0$ дат

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right),\ \forall n\in\mathbb{N}$$

испитати конв.

Ако конвергира, чему?

АКО $\lim\limits_{n\to\infty} x_n = x\in\mathbb{R}$

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)\ \Big/\lim_{n\to\infty}$$

ако $x\neq 0$

$$x=\frac{1}{2}\left(x+\frac{a}{x}\right)$$

$$2x^2=x^2+a$$

$$x^2=a \quad x=\sqrt{a} \vee -\sqrt{a}$$

кандидати: $0,\ \sqrt{a},\ \cancel{-\sqrt{a}}$ јер $x_n>0, \forall n$

$\sqrt{a}$

1° Доказујемо да $x_n\geq\sqrt{a},\ \forall n\geq 2$

$$n\geq 2:\quad x_n=\frac{1}{2}\left(x_{n-1}+\frac{a}{x_{n-1}}\right)\underset{A\geq G}{\geq}\sqrt{x_{n-1}\cdot\frac{a}{x_{n-1}}}=\sqrt{a}$$

$(\forall n\geq 2)\ x_n\geq\sqrt{a}$

2° Доказујемо да $(x_n)\downarrow$ за $n\geq 2$

$$x_{n+1}\overset{?}{\leq} x_n \iff x_{n+1}-x_n\leq 0$$

$$x_{n+1}-x_n=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)-x_n=\frac{1}{2x_n}\left(-x_n^2+a\right)$$

$$=\underbrace{\frac{1}{2x_n}}_{>0}\cdot\underbrace{(\sqrt{a}-x_n)}_{\leq 0 \text{ из } 1^\circ}\underbrace{(\sqrt{a}+x_n)}_{>0}\leq 0$$

$$\Rightarrow \boxed{x_{n+1}\leq x_n,\ \forall n\geq 2}$$

$$1^\circ, 2^\circ \overset{T2}{\Longrightarrow} \exists \lim_{n\to\infty} x_n = x\in\mathbb{R}$$

$$x=?\quad \Rightarrow x\in\{\underset{x_n\geq\sqrt{a}}{\cancel{0}},\ \sqrt{a},\ \cancel{-\sqrt{a}}\}\Rightarrow x=\sqrt{a}$$

$$\Rightarrow \boxed{\lim_{n\to\infty} x_n=\sqrt{a}}\quad\square$$

3 Испитати конв. (а) $a_n = \frac{n^k}{c^n}$ $k \in \mathbb{N}, c > 1$

(б) $a_n = \frac{c^n}{n!}, c > 0$

вежба

$\frac{n^k}{c^n} \to 0$ $n^k << c^n$

$\frac{c^n}{n!} \to 0$ $c^n << n!$

$n \to \infty$

$\boxed{n^k << c^n << n!}$

$n \to \infty$

(а) I начин - из фја

II начин: $a_n = \frac{n^k}{c^n}$

$a_{n+1} = \frac{(n+1)^k}{c^{n+1}} = \underbrace{\frac{n^k}{c^n}}_{a_n} \cdot \frac{1}{n^k} \cdot \frac{1}{c} \cdot (n+1)^k$

$a_{n+1} = a_n \cdot \left(1+\frac{1}{n}\right)^k \cdot \frac{1}{c}$

☺ ако конвергира: ако $\exists \lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$

$a_{n+1} = a_n \cdot \underbrace{\left(1+\frac{1}{n}\right)^k}_{\to 1} \cdot \frac{1}{c} \quad / \lim\limits_{n\to\infty}$

$a = a \cdot 1^k \cdot \frac{1}{c}$

$a = \frac{a}{c} \overset{c>1}{\Longrightarrow} a = 0$

1° $a_n > 0, \forall n$ ✓

2° Докажимо да $(a_n) \downarrow$ почев од неког члана

$a_{n+1} \overset{?}{\leq} a_n \overset{a_n>0}{\iff} \frac{a_{n+1}}{a_n} \leq 1$

$\iff \left(1+\frac{1}{n}\right)^k \cdot \frac{1}{c} \leq 1$

$\iff 1 + \frac{1}{n} \leq c^{1/k}$

$\iff \frac{1}{n} \leq \underbrace{c^{\frac{1}{k}} - 1}_{>0} \iff \boxed{n \geq \frac{1}{c^{1/k} - 1}}$

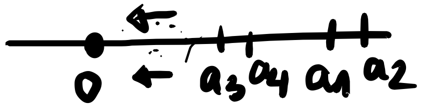
за $n \geq n_0 = \left[\frac{1}{c^{1/k}-1}\right]$ $a_{n+1} \leq a_n$ ✓

1°, 2° $\overset{T2}{\Longrightarrow}$ $\exists \lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$

☺ $\Rightarrow \boxed{a = 0}$ □

$a_n>0, \forall n$ Да ли овај низ мора опадати почев од неког члана? $\boxed{\text{НЕ}}$

$\lim\limits_{n\to\infty} a_n=0$



$a_{2n+1}=\frac{1}{n}$

$a_{2n}=2\cdot\frac{1}{n}$

$\lim\limits_{n\to\infty} a_n=0$

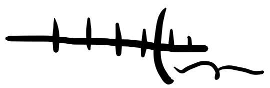
$\boxed{4}$ $x_1>0$ $x_{n+1}=x_n+\frac{1}{x_n}, \forall n\geq 1$

Испитати конвергенцију.

- $x_n>0, \forall n$ (индукција)
- $x_{n+1}=x_n+\frac{1}{x_n}>x_n$

$\boxed{x_n\uparrow}$ → ако је ограничен $\exists \lim\limits_{n\to\infty} x_n=x\in\mathbb{R}$

→ ако није ограничен $\lim\limits_{n\to\infty} x_n=+\infty$



Ако $\exists \lim\limits_{n\to\infty} x_n=x$, шта?

$\Rightarrow x>0$

$x_{n+1}=x_n+\frac{1}{x_n} \quad / \lim\limits_{n\to\infty}$

$x=x+\frac{1}{x}$

$0=\frac{1}{x}$ $\lightning$

$\Rightarrow \neg\exists \lim\limits_{n\to\infty} x_n$ у $\mathbb{R}$

$\Rightarrow \lim\limits_{n\to\infty} x_n=+\infty$

~ број e ~

$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ $\quad a_n = \left(1+\frac{1}{n}\right)^n$ растући низ

$b_n = \left(1+\frac{1}{n}\right)^{n+1}$ опадајући низ $\quad \lim_{n\to\infty} b_n = e$

$$\boxed{\left(1+\frac{1}{n}\right)^n \leq e \leq \left(1+\frac{1}{n}\right)^{n+1}, \ \forall n\in\mathbb{N}}$$

$(*)$

$\boxed{\text{ЛЕМА}}$ $(p_n), (q_n)$ $\quad \lim_{n\to\infty} p_n = +\infty, \quad \lim_{n\to\infty} q_n = -\infty$

$$\lim_{n\to\infty}\left(1+\frac{1}{p_n}\right)^{p_n} = e \qquad \lim_{n\to\infty}\left(1+\frac{1}{q_n}\right)^{q_n} = e$$

1. $\lim_{n\to\infty}\left(\frac{n^2+n+1}{n^2-n+1}\right)^{\frac{n^2-15}{n}}$ 1^∞

$$= \lim_{n\to\infty}\left(1+\frac{2n}{n^2-n+1}\right)^{\frac{n^2-15}{n}} = \lim_{n\to\infty}\left(\left(1+\frac{1}{\frac{n^2-n+1}{2n}}\right)^{\overset{p_n}{\frac{n^2-n+1}{2n}}}\right)^{\frac{2\cancel{n}}{n^2-n+1}\cdot\frac{n^2-15}{\cancel{n}}}$$

$p_n \to +\infty$, $\quad \to e$, $\quad \to 2$

$$= \boxed{e^2}$$

2 Доказати: (а) $\lim_{n\to\infty} \frac{n^n}{(n!)^2} = 0$

за вежбу → (б) $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

$\ulcorner$ $\underbrace{n^k \ll c^n \ll n!}_{c>1}$, $n\to\infty$ $\lrcorner$

$\underbrace{n! \ll n^n \ll (n!)^2}$, $n\to\infty$

$\lim_{n\to\infty} \frac{\log_a n}{n^k} = 0$: $\log_a n \ll n^k$, $n\to\infty$

(а) $a_n = \frac{n^n}{(n!)^2}$

1° $a_n > 0$, $\forall n\in\mathbb{N}$

2° да ли опада почев од неког члана?

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{(n+1)^{n+1}}{\cancel{((n+1)!)^2}} \cdot \frac{\cancel{(n!)^2}}{n^n} = \frac{(n+1)^{n+1}}{(n+1)^2} \cdot \frac{1}{n^n} =$$

$$= \underbrace{\left(1+\frac{1}{n}\right)^n}_{<e} \frac{1}{n+1} < \frac{e}{n+1} \boxed{<1} \text{ за } n\geq 2$$

$(a_n)\downarrow$ за $n\geq 2$

1° 2° $\overset{T2}{\Longrightarrow}$ $\boxed{\exists \lim_{n\to\infty} a_n = a \in\mathbb{R}}$

$a_{n+1} = a_n \cdot \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{n+1} \Big/ \lim_{n\to\infty}$

$a = a \cdot e \cdot 0$

$\Rightarrow \boxed{a=0}$ ✓ $\blacksquare$

$$\boxed{\log_a n \ll n^k \ll c^n \ll n! \ll n^n \ll (n!)^2 \ , n\to\infty}$$

$c>1,\ a>1,\ k>0$

$$2021^{2021n} = \left(2021^{2021}\right)^n \qquad \underline{c^n \ll n!}$$

Άσ.

$$\lim_{n\to\infty} \frac{\log_{15} n^2 + n^{2021} - 50n! + 2021^{2021n}}{\underbrace{\cos\left((n!)^2+500\right)}_{\text{φρ.}} - n! + \underbrace{2021^{20000\cdot n}}_{c_2^n}} =$$

L:n!

$$= \lim_{n\to\infty} \frac{\dfrac{2\cdot\log_{15} n}{n!} + \dfrac{n^{2021}}{n!} - 50 + \dfrac{c_1^n}{n!}}{\dfrac{\cos(\ldots)}{n!} - 1 + \dfrac{c_2^n}{n!}} = \frac{-50}{-1} = \boxed{50}$$

⊛: $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1} \quad / \ln$

$$n\cdot \ln\left(1+\frac{1}{n}\right) < 1 < (n+1)\cdot\ln\left(1+\frac{1}{n}\right)$$

⊛ $\boxed{\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}}$ $\forall n\in\mathbb{N}$

$\boxed{3.}$ Док. да $a_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n$ конвергира.

Г $a_n = b_n - c_n$ $\quad b_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\to+\infty$ хармониски низ

$c_n = \ln n \to +\infty$ ┘

$$a_{n+1}-a_n = \left(1+\cancel{\frac{1}{2}+\dots+\frac{1}{n}}+\frac{1}{n+1}-\ln(n+1)\right)-\left(\cancel{1+\frac{1}{2}+\dots+\frac{1}{n}}-\ln n\right)$$

$$= \frac{1}{n+1}-\ln(n+1)+\ln n$$

$$= \frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right) \overset{⊛}{<} 0$$

$\Rightarrow \boxed{a_{n+1}<a_n, \forall n\in\mathbb{N}}$ (1)

доволно да ог. одоздо:

$$\boxed{a_n} = 1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n$$

$$\overset{⊛}{>} \ln(1+1)+\ln\left(1+\frac{1}{2}\right)+\dots+\ln\left(1+\frac{1}{n}\right)-\ln n$$

$$= \ln(2)+\ln\left(\frac{3}{2}\right)+\ln\left(\frac{4}{3}\right)+\dots+\ln\frac{n+1}{n}-\ln n$$

$$= \ln\left(\cancel{2}\cdot\cancel{\frac{3}{2}}\cdot\cancel{\frac{4}{3}}\cdots\frac{n+1}{\cancel{n}}\right)-\ln n$$

$$= \ln(n+1)-\ln n$$

$\boxed{>0}$

$\Rightarrow \boxed{(\forall n\in\mathbb{N})\, a_n>0}$ (2)

(1), (2) $\overset{T2}{\Rightarrow}$ $\exists \lim\limits_{n\to\infty} a_n = \gamma \in \mathbb{R}$

Ојлерова конст. $0{,}57\dots$