17.5.2021.

~ Испитивање функција ~

$f(x) = \dots$

$\boxed{1}$ $D_f = ?$

$\boxed{2}$ парност/непар., периодичност

$\boxed{3}$ нуле, знак

$\boxed{4}$ асимптоте, понашање на крајевима домена $(\downarrow\ \downarrow) \cup (\downarrow\ \downarrow)$

$\boxed{5}$ непр., диф.

$\boxed{6}$ $f'(x)$, интервали монотоније

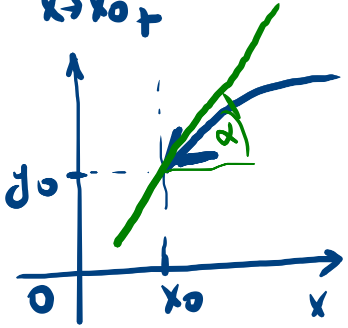
$\boxed{7}$ $f''(x)$, конвексност/конкавност

$\boxed{8}$ Γ_f

Ако

$x_0 \notin D_f$ / f није непр. у x_0

$\lim\limits_{x \to x_0+} f(x) = y_0$



? $\lim\limits_{x \to x_0+} f'(x)$

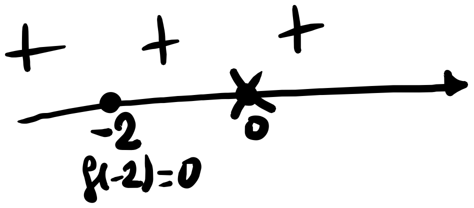
① $f(x) = |x+2| \cdot e^{-\frac{1}{x}} = \begin{cases} (x+2)\cdot e^{-1/x}, & x \geq -2, x \neq 0 \\ -(x+2)\cdot e^{-1/x}, & x < -2 \end{cases}$

1° $\boxed{D_f = (-\infty, 0) \cup (0, +\infty)}$ $\boxed{f \text{ непр. на } D_f}$

2° п/н, пер. //

3° НУЛЕ, ЗНАК: $f(x) = 0 \Leftrightarrow |x+2| = 0 \Leftrightarrow \underline{x = -2}$

$\underline{f(x) \geq 0, \forall x \in D_f}$



4° **АСИМПТОТИКА**: $(\underset{\uparrow}{-\infty}, \underset{0_-}{0}) \cup (\underset{0_+}{0}, \underset{\uparrow}{+\infty})$

$\boxed{x \to +\infty}$ $f(x) = (x+2)\cdot e^{\overset{\to 0}{-1/x}} = (x+2)\cdot\left(1 + \frac{-1}{x} + \frac{1}{2}\cdot\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right) = \underbrace{x - 1} + \frac{1}{2x} + o\left(\frac{1}{x}\right) + \underbrace{2} - \frac{2}{x} + \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)$

$|x+2| = x+2 > 0$

$= x + 1 \underbrace{- \frac{3}{2}\cdot\frac{1}{x}}_{<0} + o\left(\frac{1}{x}\right)$ $y = x+1$ КА $\underline{x \to +\infty}$ Γ_f испод АС.

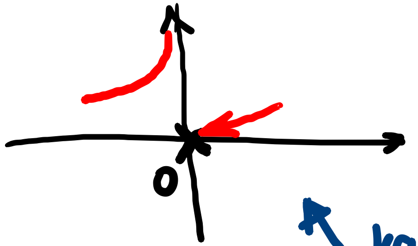
$\boxed{x \to -\infty}$ $|x+2| = -(x+2)$ $f(x) = -(x+2)\cdot e^{-1/x} = \dots = -x - 1 + \overbrace{\frac{3}{2}\cdot\frac{1}{x}}^{<0} + o\left(\frac{1}{x}\right)$ $y = -x-1$ КА. $\underline{x \to -\infty}$ Γ_f испод АС.

$\boxed{0}$ $f(x)=|x+2|\cdot e^{-1/x}$

$$\lim_{x\to0-} f(x)=\lim_{x\to0-}\underbrace{|x+2|}_{\to2}\, e^{\overset{\to+\infty}{-\frac{1}{x}}}=+\infty$$

$\boxed{\begin{matrix}x=0\ \text{B.A.}\\ x\to0-\end{matrix}}$

$$\lim_{x\to0+} f(x)=\lim_{x\to0+}\underbrace{|x+2|}_{\to2}\cdot\underbrace{e^{\overset{\to-\infty}{-\frac{1}{x}}}}_{\to0}=0$$



↑ касније:

под којим углом?

5° $f'(x)=?$

$x\in(-2,0)\cup(0,+\infty)$: $f'(x)=\left((x+2)\cdot e^{-1/x}\right)'=e^{-1/x}+(x+2)e^{-1/x}\cdot\frac{1}{x^2}=\frac{e^{-1/x}}{x^2}\cdot(x^2+x+2)$

$x\in(-\infty,-2)$: $f'(x)=\left(-(x+2)\cdot e^{-1/x}\right)'=-\frac{e^{-1/x}}{x^2}(x^2+x+2)$

$x=-2$?

$$f'_+(-2)\overset{!}{\overset{\text{ако }\exists}{=}}\lim_{x\to-2+}f'(x)=\lim_{x\to-2+}\frac{e^{-1/x}}{x^2}(x^2+x+2)=\sqrt{e}$$

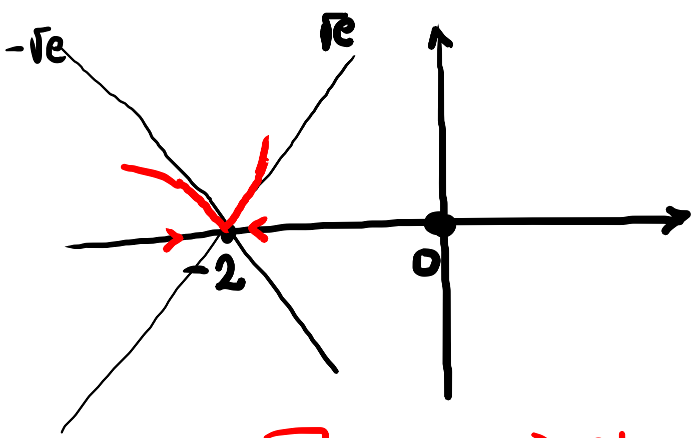
$\neq$ $\boxed{f \text{ није диф. у } -2}$

$$f'_-(-2)\overset{\text{ако }\exists}{=}\lim_{x\to-2-}f'(x)=\lim_{x\to-2-}-\frac{e^{-1/x}}{x^2}(x^2+x+2)=-\sqrt{e}$$

f диф. на $(-\infty,-2)\cup(2,0]\cup(0,+\infty)$

$$f'(x)=\begin{cases} e^{-1/x}\cdot\dfrac{x^2+x+2}{x^2}, & x\in(-2,0)\cup(0,+\infty)\\ -e^{-\frac{1}{x}}\cdot\dfrac{x^2+x+2}{x^2}, & x\in(-\infty,-2)\end{cases}$$

у -2: $f'_-(-2)=-\sqrt{e}$ $f'_+(-2)=\sqrt{e}$



$\underbrace{x^2+x+2}_{\text{увек}>0}$ $D<0$

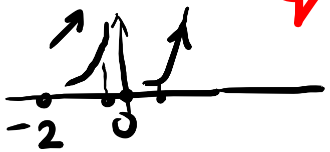
$\boxed{f'(x)>0}$ за $\forall x\in(-2,0)\cup(0,+\infty)$ ~~$\Rightarrow f\nearrow$ на $(-2,0)\cup(0,+\infty)$~~ ! не

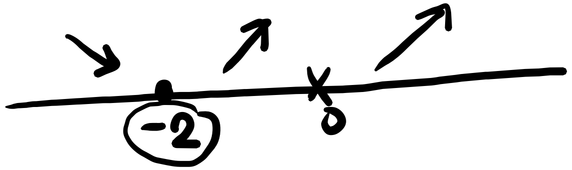
$\boxed{f'(x)<0}$ за $\forall x\in(-\infty,-2)$

$\Rightarrow \boxed{f\searrow \text{ на } (-\infty,-2)}$

$\Rightarrow$ $f\nearrow$ на $(-2,0)$
$f\nearrow$ на $(0,+\infty)$

Т $\forall x\in(a,b)$ $f'(x)\geq 0$
$\Rightarrow$ f расте на (a,b)

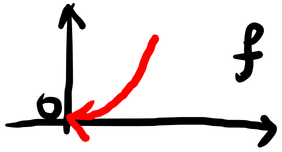




Још угао $\lim\limits_{x\to0_+}$:

$$\boxed{\lim_{x\to0_+}f'(x)}=\lim_{x\to0_+}e^{-\frac{1}{x}}\cdot\frac{\overset{\to2}{x^2+x+2}}{x^2}=2\cdot\lim_{x\to0_+}\frac{e^{-1/x\to-\infty}}{x^2}\ \frac{0/0}{t=\frac{1}{x}\to+\infty}\ 2\cdot\lim_{t\to+\infty}\frac{e^{-t}}{\frac{1}{t^2}}=2\cdot\lim_{t\to+\infty}\frac{t^2}{e^t}\overset{t^2\ll e^t}{=}\boxed{0}$$

$\overset{\shortparallel}{\operatorname{tg}\alpha}$



f се хоризонтално приљубљује
у x-осу за $x\to0_+$

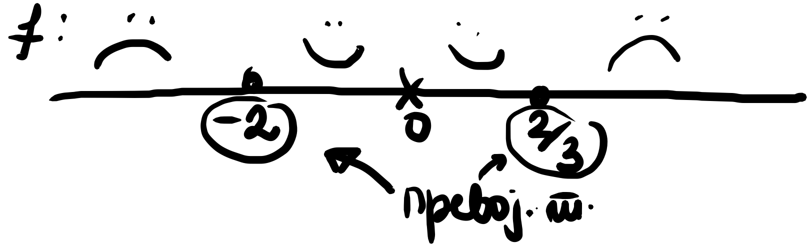
⑥ $\underline{\underline{f''(x)}}$: $\left(e^{-1/x}\cdot\frac{x^2+x+2}{x^2}\right)' \overset{\dots}{=} e^{-1/x}\cdot\frac{(2-3x)}{x^4}$

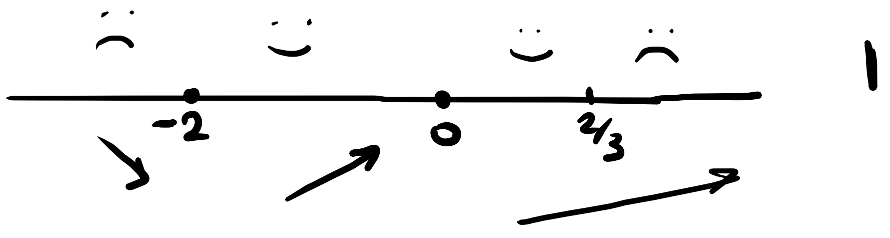
$$\rightsquigarrow f''(x)=\begin{cases} e^{-1/x}\cdot\frac{(2-3x)}{x^4}, & x\in(-2,0)\cup(0,+\infty)\\ e^{-1/x}\cdot\frac{3x-2}{x^4}, & x\in(-\infty,-2)\end{cases}$$

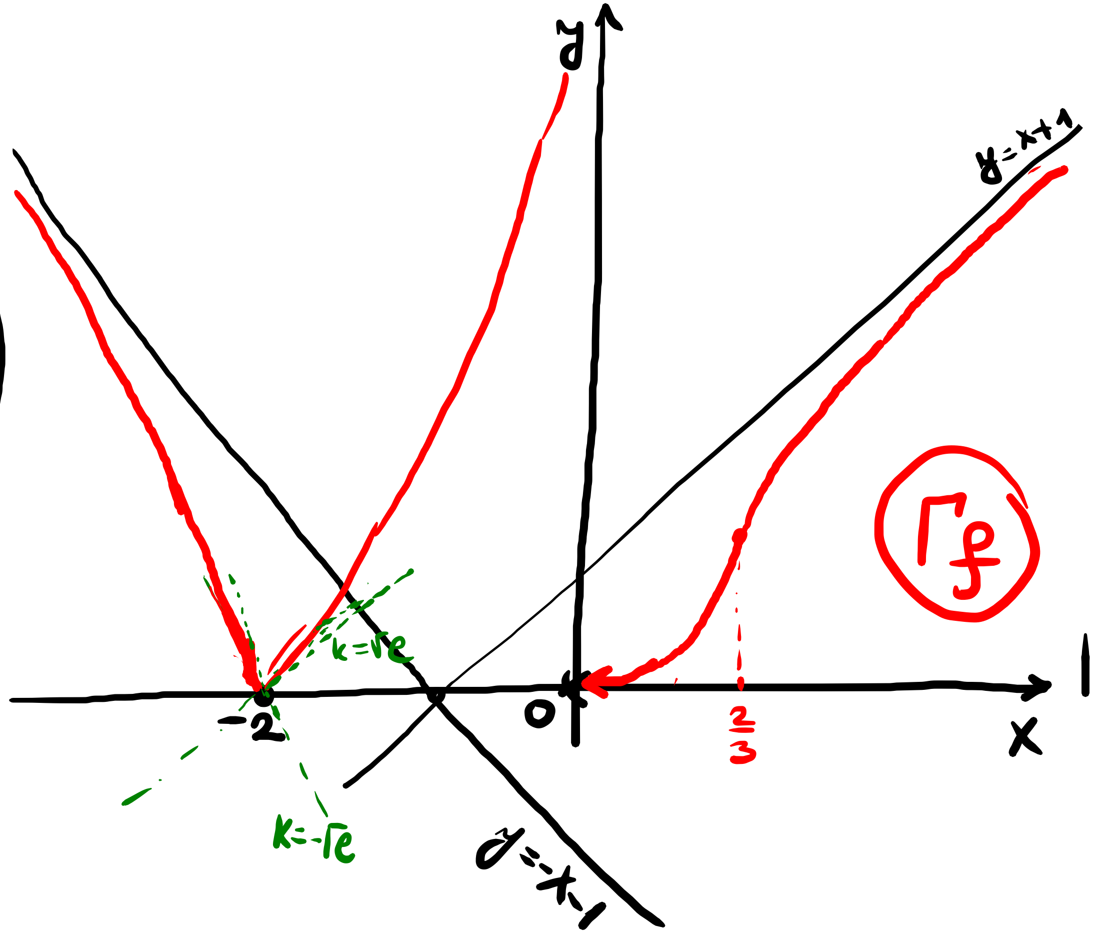
$x<-2$: $f''(x)<0$ ⟶ $\boxed{f \text{ конкавна на } (-\infty,-2)}$

$x\in(-2,0)\cup(0,\frac{2}{3})$: $f''(x)>0$ $\boxed{\begin{array}{l} f \text{ конвексна на } (-2,0)\\ f \text{ конвексна на } (0,\frac{2}{3})\end{array}}$

$x\in(\frac{2}{3},+\infty)$: $f''(x)<0$ $\boxed{f \text{ конкавна на } (\frac{2}{3},+\infty)}$







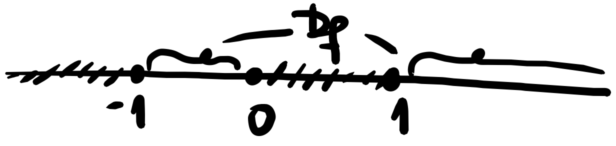
② $f(x) = -x + \ln\left(\frac{|x|-1}{x}\right)$

① Df: $x \neq 0$ $\quad \frac{|x|-1}{x} > 0$

$|x|-1 > 0 \wedge x > 0 \to x \in (1, +\infty)$

$|x|-1 < 0 \wedge x < 0 \to x \in (-1, 0)$

$\boxed{D_f = (-1, 0) \cup (1, +\infty)}$



② Df није сим $\Rightarrow$ ~~П/Н~~, није ни периодична

③ Нуле & знак: $f(x) = 0 \iff -x + \ln\left(\frac{|x|-1}{x}\right) = 0$ $\quad x = ..?$

остављамо за крај ☺

④ рубови Df:

$\lim_{x\to-1_+} f(x) = \lim_{x\to-1_+} \left(-x + \ln\left(\frac{|x|-1}{x}\right)\right) = \boxed{-\infty}$

$|x|-1 \to 0_-$, $x \to -1$, $\to 0_+$, $\to -\infty$

$\boxed{x = -1 \text{ В.А. за } x \to 1_+,\ f \to -\infty}$

$\lim_{x\to0_-} f(x) = \lim_{x\to0_-} \left(-x + \ln\left(\frac{|x|-1}{x}\right)\right) = \boxed{+\infty}$

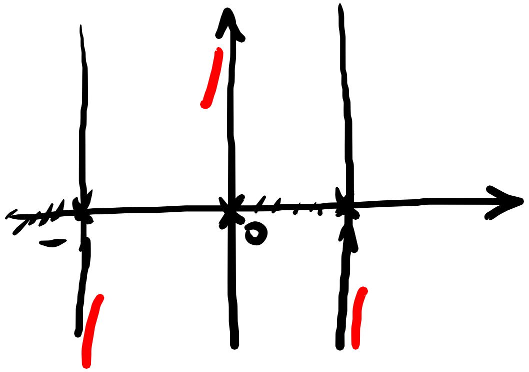
$|x|-1 \to -1$, $x \to 0_-$, $\to +\infty$

$\boxed{x = 0 \text{ В.А. за } x \to 0_-,\ f \to +\infty}$

$\lim_{x\to1_+} f(x) = \lim_{x\to1_+} \left(-x + \ln\left(\frac{|x|-1}{x}\right)\right) = \boxed{-\infty}$

$\to 0_+$

$\boxed{x = 1 \text{ В.А. за } x \to 1_+,\ f \to -\infty}$



$\boxed{x\to+\infty}$ $$a=\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\left(-1+\underbrace{\frac{\ln\left(\frac{|x|-1}{x}\right)}{x}}_{\to 0}\right)=\boxed{-1}$$ $\to 0$

$$f(x)=-x+\ln\left(\frac{|x|-1}{x}\right)$$

$$b=\lim_{x\to+\infty}(f(x)-ax)=\lim_{x\to+\infty}\ln\frac{|x|-1}{x}=\boxed{0}$$

$\boxed{y=-x \text{ је КА. } x\to+\infty}$

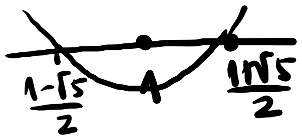
⑤° $f'(x)=?$

$\boxed{x\in(-1,0)}$: $$\boxed{f'(x)}=\left(-x+\ln\frac{-x-1}{x}\right)'=-1+\frac{1}{\frac{-x-1}{x}}\cdot\frac{1}{x^2}\doteq\boxed{-\frac{x^2+x+1}{x(x+1)}}$$

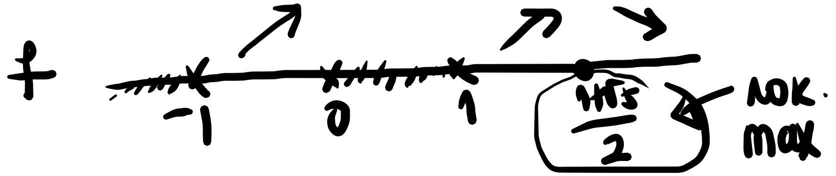
знак $f'(x)$: $x^2+x+1>0$ јер $D=-3<0$ $\leadsto$ $f'(x)>0$ за $\forall x\in(-1,0)$ $\Rightarrow$ $\boxed{f\nearrow(-1,0)}$

$\boxed{x\in(1,+\infty)}$ $$\boxed{f'(x)}=\left(-x+\ln\frac{x-1}{x}\right)'\doteq\boxed{\frac{-x^2+x+1}{x(x-1)}}=-\frac{x^2-x-1}{x(x-1)}<0$$

знак? $x^2-x-1=0$ $x_{1,2}=\frac{1\pm\sqrt5}{2}$



за $x\in\left(1,\frac{1+\sqrt5}{2}\right)$ $f'(x)>0$ $\boxed{f\nearrow\left(1,\frac{1+\sqrt5}{2}\right)}$ $x\in\left(\frac{1+\sqrt5}{2},+\infty\right)$ $f'(x)<0$ $\boxed{f\searrow}$



6° f непр. на D_f, f диф. на D_f

7° $f''(x)=(f'(x))' \overset{\cdot}{=} \dots \begin{cases} \dfrac{2x+1}{x^2(x+1)^2}, & x\in(-1,0) \\ -\dfrac{2x-1}{x^2(x-1)^2}, & x\in(1,+\infty) \end{cases}$

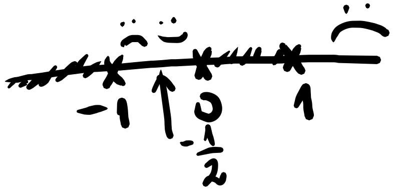
$x\in(-1,-\frac{1}{2})$ $f''(x)<0$ f конкавна

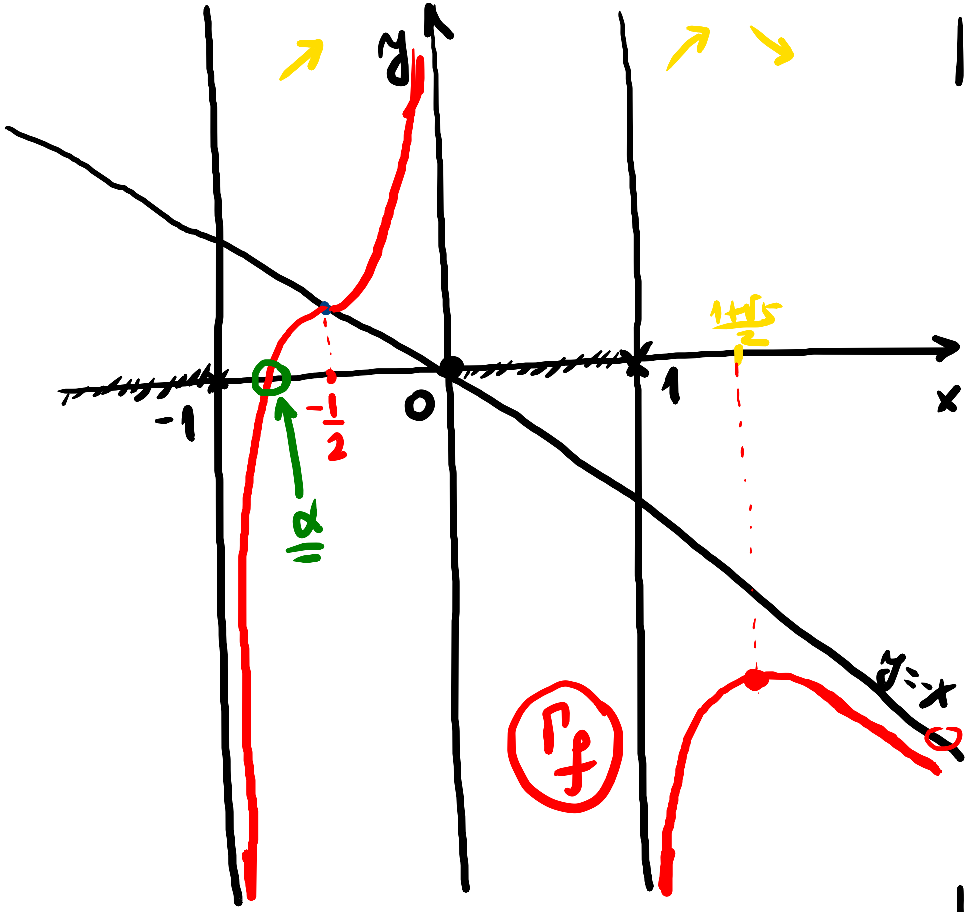
$x\in(-\frac{1}{2},0)$ $f''(x)>0$ f конвексна

$x\in(1,+\infty)$ $f''(x)<0$ f конкавна

$-\frac{1}{2}$ превојна тачка

$f(-\frac{1}{2})=+\frac{1}{2}+\ln 1=\frac{1}{2}$





Још 3° нуле и знак: $\boxed{(-1,0)}$ $\begin{cases} -\infty \quad f \text{ расте} \nearrow \quad +\infty \\ f \text{ непрекидна} \end{cases}$

БК $\Rightarrow \exists\alpha\in(-1,0)\ f(\alpha)=0$

$\forall x\in(-1,\alpha)\ f(x)<0$

$\forall x\in(\alpha,0)\ f(x)>0$

$\boxed{(1,+\infty)}$ $-\infty \nearrow \underset{x=\frac{1+\sqrt{5}}{2}}{} \searrow -\infty$

$f(\frac{1+\sqrt{5}}{2})=\dots<0$

$\Rightarrow \forall x\in(1,+\infty)\ f(x)<0$ □

③ $f(x) = -\frac{1}{|x|} + \operatorname{arcctg}\frac{2x}{x^2-1}$

1° D_f : $x \neq 0$, $x \neq \pm 1$ $D_f = (-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,+\infty)$ $\boxed{f \text{ непр. на } D_f}$

2° ~~П/Н~~, ~~пер~~

3° Нуле & знак ~> за крај

4° $f'(x) = \underbrace{+\frac{1}{|x|^2}\cdot|x|'}_{\frac{\operatorname{sgn}x}{x^2}} + \underbrace{\frac{1}{1+\left(\frac{2x}{x^2-1}\right)^2}\cdot\frac{2(x^2-1)-2x\cdot 2x}{(x^2-1)^2}}_{\cdots = -\frac{2}{x^2+1}}$

$x \in D_f$

☺ $|x| = \operatorname{sgn}x \cdot x$

$|x|' \underset{x\neq 0}{=} \operatorname{sgn}x$

$$f'(x) \underset{x\in D_f}{=} \frac{\operatorname{sgn}x}{x^2} - \frac{2}{x^2+1}$$

$$f'(x) = \begin{cases} \frac{1}{x^2} - \frac{2}{x^2+1}, & x\in(0,1)\cup(1,+\infty) \\ -\frac{1}{x^2} - \frac{2}{x^2+1}, & x\in(-\infty,-1)\cup(-1,0) \end{cases}$$

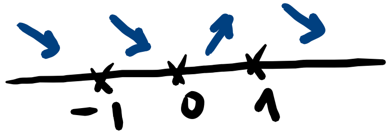
$\boxed{f \text{ диф. на } D_f}$ ✓

знак f': $\forall x<0, x\neq -1$: $f'(x)<0$

$\boxed{f\downarrow \text{ на } (-\infty,-1),\ f\downarrow \text{ на } (-1,0)}$

$x>0, x\neq 1$: $f'(x) = \frac{1-x^2}{x^2(x^2+1)}$

$\boxed{\begin{array}{l} x\in(0,1)\ f'(x)>0,\ f\nearrow \\ x\in(1,+\infty)\ f'(x)<0,\ f\searrow \end{array}}$



$$\underbrace{f(x) = -\frac{1}{|x|} + \text{arctg}\,\frac{2x}{x^2-1}}_{(-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,+\infty)}$$

(59) рубови D_f

$\boxed{-\infty}$ $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \underbrace{-\frac{1}{|x|}}_{\to 0} + \text{arctg}\underbrace{\frac{2x}{x^2-1}}_{\to 0} = \boxed{0}$

$\boxed{y=0 \text{ x.A. kag } x\to-\infty}$

$\boxed{+\infty}$ $\lim\limits_{x\to+\infty} f(x) = \boxed{0}$ $\boxed{y=0 \text{ x.A. kag } x\to+\infty}$

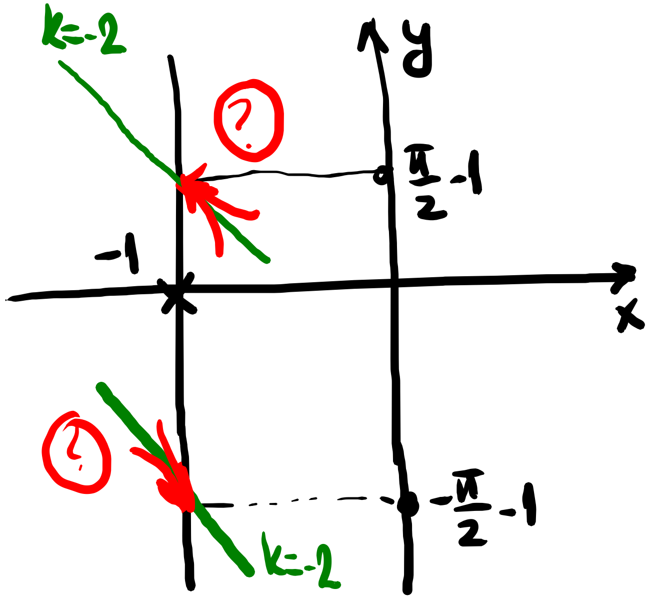
$\boxed{-1}$ $\lim\limits_{x\to-1_-} f(x) = \lim\limits_{x\to-1_-}\Big(\underbrace{-\frac{1}{|x|}}_{\to -1} + \underbrace{\text{arctg}\,\underbrace{\frac{\overset{\to -2}{2x}}{\underbrace{x^2-1}_{\to 0_+}}}_{\to -\infty}}_{\to -\frac{\pi}{2}}\Big) = -1-\frac{\pi}{2}$

$\lim\limits_{x\to-1_+} f(x) = \lim\limits_{x\to-1_+}\Big(-\frac{1}{|x|} + \underbrace{\text{arctg}\,\overset{\to+\infty}{\frac{\overset{\to -2}{2x}}{\underbrace{x^2-1}_{\to 0_-}}}}_{\to \frac{\pi}{2}}\Big) = -1+\frac{\pi}{2}$

ПОД КОЈИМ УГЛОВИМА ?

$\lim\limits_{x\to-1_-} f'(x) = \lim\limits_{x\to-1_-}\left(-\frac{1}{x^2} - \frac{2}{x^2+1}\right) = -2$

$\lim\limits_{x\to-1_+} f'(x) = \lim\limits_{x\to-1_+}\left(-\frac{1}{x^2} - \frac{2}{x^2+1}\right) = -2$



$\boxed{1}$ $\lim\limits_{x\to 1-} f(x) \overset{\dots}{=} -1-\frac{\pi}{2}$

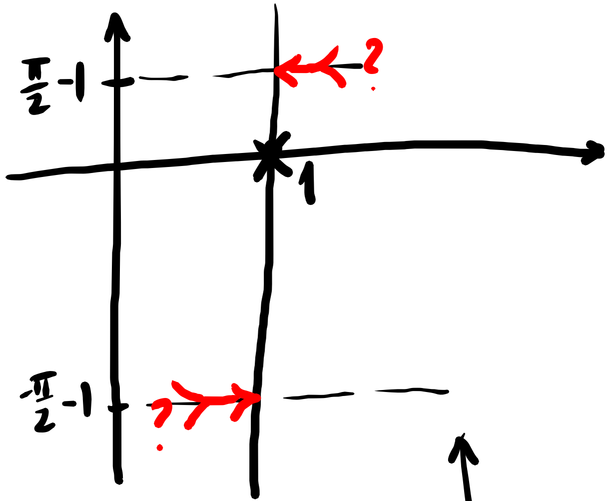
$\lim\limits_{x\to 1+} f(x) \overset{\dots}{=} -1+\frac{\pi}{2}$

(за вежбу)

услови:

$\lim\limits_{x\to 1-} f'(x) = 0$

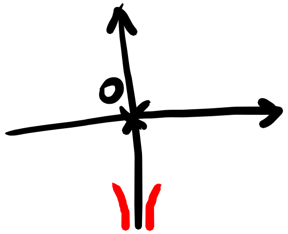
$\lim\limits_{x\to 1+} f'(x) = 0$



$\boxed{0}$ $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\Big(\underbrace{\frac{-1}{|x|}}_{-\infty} + \underbrace{\operatorname{arctg}\underbrace{\frac{2x}{x^2-1}}_{\to 0}}_{\to 0}\Big) = -\infty$

$\boxed{x=0 \text{ В.А.}}$

$x\to 0, f(x)\to -\infty$



⑥° $f''(x) = \left(\frac{\operatorname{sgn} x}{x^2} - \frac{2}{x^2+1}\right)' =$

$= \frac{-2\operatorname{sgn} x}{x^3} + \frac{2}{(x^2+1)^2}\cdot 2x,\ \forall x\in D_f$

$$f''(x) = \begin{cases} \frac{2}{x^3} + \frac{4x}{(x^2+1)^2}, & x\in(-\infty,-1)\cup(-1,0) \\ \frac{-2}{x^3} + \frac{4x}{(x^2+1)^2}, & x\in(0,1)\cup(1,+\infty) \end{cases}$$

- $x\in D_f, x<0$: $f''(x)<0$

$\boxed{\begin{array}{l} f \text{ конкавна на } (-\infty,-1) \\ f \text{ конкавна на } (-1,0) \end{array}}$

- $x\in D_f, x>0$:

$f''(x) = \frac{-2(x^2+1)^2+4x^4}{\underbrace{x^3(x^2+1)^2}_{>0}}$?

знак $f''(x)$?

|за $x>0$ $x\in D_f$|

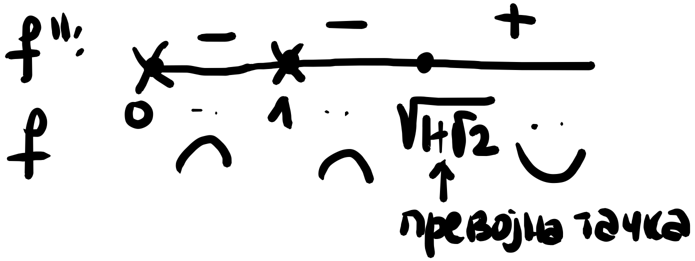
$$\boxed{f''(x)>0} \iff -2(x^2+1)^2+4x^4>0$$

$$\iff 2x^4-4x^2-2>0 \iff x^4-2x^2-1>0 \qquad t=x^2 \rightsquigarrow t^2-2t-1>0$$

$$t_{1/2}=\frac{2\pm\sqrt{8}}{2}=1\pm\sqrt{2}$$

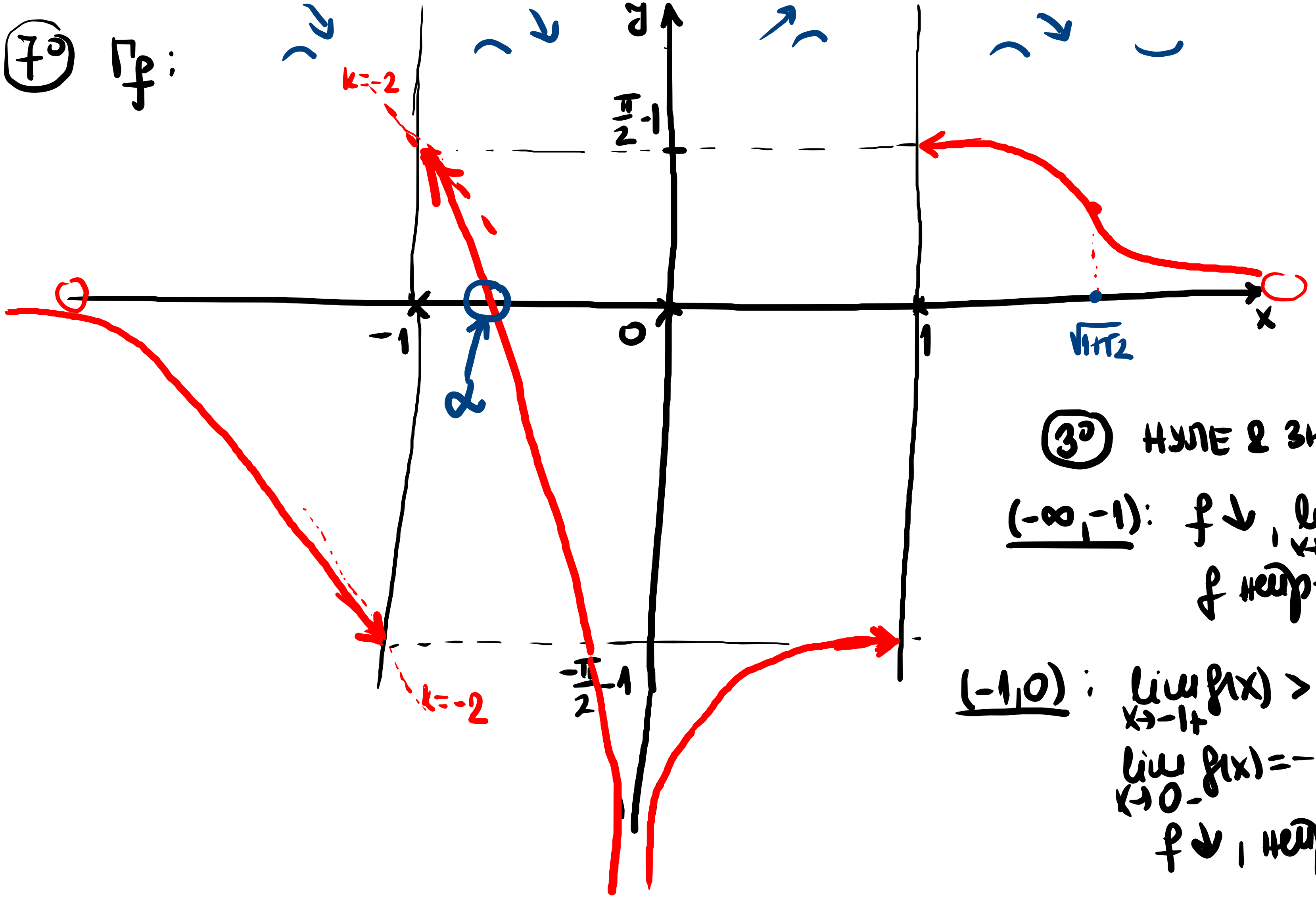
$$\iff \cancel{x^2<1-\sqrt{2}} \lor \underline{x^2>1+\sqrt{2}}$$

$$\iff \boxed{x>\sqrt{1+\sqrt{2}}}$$



f конкавна на $(0,1)$
конкавна на $(1,\sqrt{1+\sqrt{2}})$ f конвексна на $(\sqrt{1+\sqrt{2}},+\infty)$

7° Гf:



3° НУЛЕ & ЗНАК

$\underline{(-\infty,-1)}$: $f\downarrow$, $\lim\limits_{x\to-\infty} f(x)=0$, f непр. $\Big\}\Rightarrow$ $\forall x<-1$ $f(x)<0$

$\underline{(-1,0)}$: $\lim\limits_{x\to-1+} f(x)>0$, $\lim\limits_{x\to0-} f(x)=-\infty$, $f\downarrow$, непр. $\Big\}\overset{БК}{\Longrightarrow}$ $\exists\alpha\in(-1,0)$ $f(\alpha)=0$

$x\in(-1,\alpha)$: $f(x)>0$

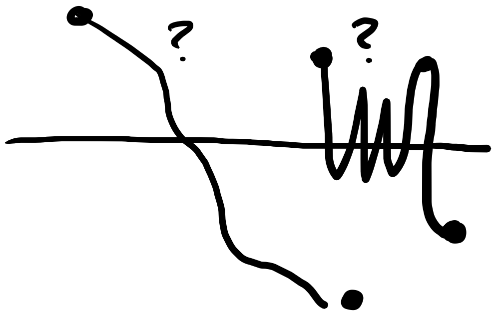
$x\in(\alpha,0)$: $f(x)<0$

$\underline{(0,1)}$: $\forall x\in(0,1)$ $f(x)<0$

$\underline{(1,+\infty)}$: $\forall x$ $f(x)>0$ ■

④ Доказати да једначина: $\ln x = \frac{1}{x^2}$ има јединствено решење.

$f(x) = \ln x - \frac{1}{x^2}$ има јединствену нулу ?



$D_f: x>0$ $\quad D_f=(0,+\infty)$

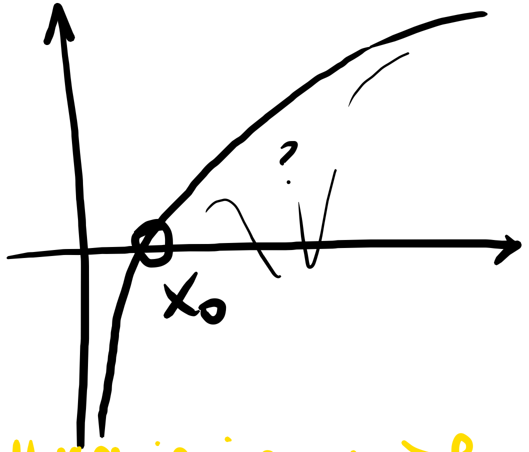
$\lim\limits_{x\to 0_+} f(x) = \lim\limits_{x\to 0+}\left(\underset{\to -\infty}{\ln x} - \frac{1}{x^2}\right) = \boxed{-\infty}$

$\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty}\left(\underset{\to +\infty}{\ln x} - \underset{\to 0}{\frac{1}{x^2}}\right) = \boxed{+\infty}$

f непр.

БК.
$\Rightarrow \boxed{\exists x_0>0 \text{ тд } f(x_0)=0}$

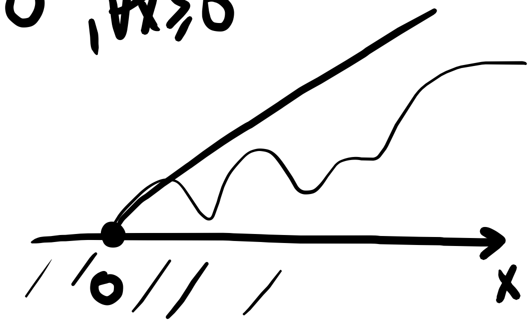
$f'(x) = \frac{1}{x} + \frac{2}{x^3} > 0, \ \forall x>0$ $\quad \boxed{f \nearrow \text{ на } (0,+\infty)}$



$\Rightarrow$ нула је јединствена ✓ □

⑤ Док. да за $\forall x \geq 0$: $\operatorname{arctg}\sqrt{x} \geq \dfrac{\sqrt{x}\cdot(5x+3)}{3(x+1)^2}$

$$f(x) = \operatorname{arctg}\sqrt{x} - \frac{\sqrt{x}\cdot(5x+3)}{3(x+1)^2} \overset{?}{\geq} 0 \ , \forall x \geq 0$$



$$f(0) = \operatorname{arctg}0 - 0 = 0$$

$$f'(x) \overset{\text{рачун...}}{=\!=} \frac{8x^2}{6\sqrt{x}(1+x)^3} \ , x \neq 0$$

$$\left.\begin{array}{l} f'(x) > 0, \forall x > 0 \Rightarrow f \text{ расте на } (0,+\infty) \\ \qquad\qquad\qquad\qquad f \text{ непр. на } [0,+\infty) \end{array}\right\} \boxed{f \nearrow \text{ на } [0,+\infty)}$$

$$\Rightarrow \forall x \geq 0 \ \ f(x) \geq f(0) = 0$$

✓

■