②
$$\begin{array}{l}
2x + 5y + \boxed{z} + 3t = 2 \quad /\cdot(-3),\ /\cdot(-1) \\
4x + 6y + 3z + 5t = 4 \\
4x + 14y + z + 7t = 4 \\
2x - 3y + 3z + \lambda t = 7 \\
\hline
-2\boxed{x} - 9y \qquad -4t = -2 \quad /\cdot(-2) \\
2x + 9y \qquad +4t = 2 \\
-4x - 18y \qquad (\lambda - 9)t = 1 \\
\hline
(\lambda - 1)t = 5
\end{array}$$

$\underline{1^\circ\ \lambda = 1} \rightsquigarrow 0 = 5 \Rightarrow$ НЕМА РЕШЕЊА

$\underline{2^\circ\ \lambda \neq 1} \rightsquigarrow (\lambda-1)\boxed{t} = 5$

$\boxed{y \in \mathbb{R}}$

$\boxed{t = \frac{5}{\lambda-1}}$

$x = 1 - 2t - \frac{9}{2}y = \boxed{1 - \frac{10}{\lambda-1} - \frac{9}{2}y}$

$z = 2 - 2x - 5y - 3t$

$= \cancel{2} - (\cancel{2} - \frac{20}{\lambda-1} - 9y) - 5y - \frac{15}{\lambda-1} = \frac{5}{\lambda-1} - 14y$

$(x,y,z,t) = (1 - \frac{10}{\lambda-1} - \frac{9}{2}y, y, \frac{5}{\lambda-1} - 14y, \frac{5}{\lambda-1}),\ y \in \mathbb{R}$

$= \boxed{(1 - \frac{10}{\lambda-1}, 0, \frac{5}{\lambda-1}, \frac{5}{\lambda-1}) + y(-\frac{9}{2}, 1, -14, 0),\ y \in \mathbb{R}}$

∞ РЕШЕЊА

③ $\mathbb{U} = \{ p \in \mathbb{R}^3[x] \mid p(0) + p(1) = 0 \}$

$\mathbb{W} = \{ p \in \mathbb{R}^3[x] \mid p(1) + p'(1) = 0 \}$

а) $\mathbb{U} = \{ \underbrace{a_2x^2 + a_1x + a_0}_{p \in \mathbb{R}^3[x]} \mid \underbrace{\underbrace{a_0}_{p(0)} + \underbrace{(a_2 + a_1 + a_0)}_{p(1)} = 0}_{\boxed{a_2} + a_1 + 2a_0 = 0,\ a_1, a_0 \in \mathbb{R}},\ a_0, a_1, a_2 \in \mathbb{R} \} = \{ (-a_1 - 2a_0)x^2 + a_1x + a_0 \mid a_1, a_0 \in \mathbb{R} \}$

$= \{ a_1\underbrace{(-x^2+x)}_{= e_1} + a_0\underbrace{(-2x^2+1)}_{= e_2} \mid a_1, a_0 \in \mathbb{R} \}$

$= \boxed{\underbrace{\mathcal{L}(e_1, e_2)}_{\in \mathbb{R}^3[x]}} \leq \mathbb{R}^3[x]$ ЛЕМА

$\begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ лин. нез. $\{e_1, e_2\}$ - база, $\dim \mathbb{U} = 2$

$\mathbb{W} = \{ \underbrace{a_2x^2 + a_1x + a_0}_{p =} \mid \underbrace{(a_2 + a_1 + a_0) + (2a_2 + a_1) = 0}_{3a_2 + 2a_1 + \boxed{a_0} = 0,\ a_1, a_2 \in \mathbb{R}},\ a_0, a_1, a_2 \in \mathbb{R} \} = \{ a_2x^2 + a_1x + (-3a_2 - 2a_1) \mid a_1, a_2 \in \mathbb{R} \}$

$p' = 2a_2x + a_1$

$= \{ a_2\underbrace{(x^2 - 3)}_{= e_3} + a_1\underbrace{(x - 2)}_{= e_4} \mid a_2, a_1 \in \mathbb{R} \}$

$= \boxed{\underbrace{\mathcal{L}(e_3, e_4)}_{\in \mathbb{R}^3[x]}} \leq \mathbb{R}^3[x]$

$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}$ лин. нез. $\{e_3, e_4\}$ - база, $\dim \mathbb{W} = 2$

б) • $\underbrace{\dim \mathbb{U}}_{= 2} + \underbrace{\dim \mathbb{W}}_{= 2} = \underbrace{\dim(\mathbb{U} + \mathbb{W})}_{\leq 3\ (\dim \mathbb{R}^3[x])} + \dim(\mathbb{U} \cap \mathbb{W})$ чија је база $\{x^2, x, 1\}$

$\Rightarrow \dim \mathbb{U} \cap \mathbb{W} = 4 - \underbrace{\dim(\mathbb{U} + \mathbb{W})}_{\leq 3} \geq 1$

$\Rightarrow \mathbb{U} \cap \mathbb{W} \neq \{0_V\}$

$\Rightarrow$ сума $\mathbb{U} + \mathbb{W}$ није директна.

• Да ли је $\mathbb{U} + \mathbb{W} = \mathbb{R}^3[x]$? $\Leftrightarrow \underbrace{\mathbb{U} + \mathbb{W} \subseteq \mathbb{R}^3[x]}_{\text{ТРИВ}} \wedge \underbrace{\mathbb{U} + \mathbb{W} \supseteq \mathbb{R}^3[x]}_{?}$

$\underset{\text{дато}}{p} = a_2x^2 + a_1x + a_0 = \underbrace{\alpha_1(-x^2+x) + \alpha_2(-2x^2+1)}_{\in \mathbb{U}} + \underbrace{\alpha_3(x^2-3) + \alpha_4(x-2)}_{\in \mathbb{W}}\ (\forall x \in \mathbb{R})$

ТРАЖИМО

$a_2x^2 + a_1x + a_0 = (-\alpha_1 - 2\alpha_2 + 3\alpha_3)x^2 + (\alpha_1 + \alpha_4)x + (\alpha_2 - 3\alpha_3 - 2\alpha_4)\ (\forall x \in \mathbb{R})$

$$\begin{aligned} a_2 &= -d_1 - 2d_2 + 3d_3 + d_4 \\ a_1 &= \boxed{d_1} \\ a_0 &= d_2 - 3d_3 - 2d_4 \end{aligned}$$

дато $\quad$ тражимо

$$\begin{aligned} a_2 + a_1 &= -2d_2 + 3d_3 + d_4 \\ a_0 &= \boxed{d_2} - 3d_3 - 2d_4 \ /\cdot 2 \\ \hline a_2 + a_1 + 2a_0 &= -3\boxed{d_2} - 3d_4 \qquad d_4 \in \mathbb{R} \end{aligned}$$

$\Rightarrow$ има ∞ решења тј. има ∞ начина да се вектор из $\mathbb{R}^3[x]$ разбије на суму једног из U и једног из W $\Rightarrow \mathbb{R}^3[x] \subseteq U+W$.

4. $L(x,y,z,t) = (x-y+z-t,\ 2x-y-z+2t,\ 3x-z-t)$, $L: \mathbb{R}^4 \to \mathbb{R}^3$

$e = \{e_1=(1,0,0,0),\ e_2=(0,1,0,0),\ e_3=(0,0,1,0),\ e_4=(0,0,0,1)\}$

$f = \{f_1=(1,0,0),\ f_2=(0,1,0),\ f_3=(0,0,1)\}$

$L(e_1) = L(1,0,0,0) = (1,2,3) = 1\cdot f_1 + 2\cdot f_2 + 3f_3$

$L(e_2) = (-1,-1,0) = -1\cdot f_1 - f_2 + 0\cdot f_3$

$L(e_3) = (1,-1,-1) = 1\cdot f_1 - f_2 - f_3$

$L(e_4) = (-1,2,-1) = -1f_1 + 2f_2 - f_3$

$$[L]_{ef} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & -1 & -1 & 2 \\ 3 & 0 & -1 & -1 \end{bmatrix}$$

$\operatorname{Ker} L = \{v \in \mathbb{R}^4 \mid L(v) = (0,0,0)\} = \{(a,b,c,d) \in \mathbb{R}^4 \mid L(a,b,c,d) = (0,0,0)\} = \{(d,2d,2d,d) \mid d \in \mathbb{R}\}$

$$\begin{aligned} a - b + c - d &= 0 \ /(-1) \\ 2a - b - c + 2d &= 0 \\ 3a - c - d &= 0 \\ \hline a - 2c + 3d &= 0 \ /(-3) \\ 3a - c - d &= 0 \\ \hline 5c - 10d &= 0,\ d \in \mathbb{R} \end{aligned}$$

$c = 2d$

$a = 2c - 3d = d$

$b = a + c - d = d + 2d - d = 2d$

$= \{d\underbrace{(1,2,2,1)}_{v} \mid d \in \mathbb{R}\}$

$= \mathcal{L}(v)$ $\quad \{v\}$ - база за $\operatorname{Ker} L$

$\dim \operatorname{Ker} L = 1$

$\operatorname{Im} L = \mathcal{L}(L(e_1), L(e_2), L(e_3), L(e_4))$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 0 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & -3 & -4 \\ 0 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & -10 \end{bmatrix}$$

$\{L(e_1), L(e_2), L(e_3)\}$ - база за $\operatorname{Im} L$

$\rho(L) = \dim \operatorname{Im} L = 3$

5.
$$A=\begin{bmatrix}0 & 1 & \alpha+1 & -1\\ 0 & \alpha & 6 & -2\\ 0 & -1 & -3 & \alpha-1\\ 2019 & 12 & 14 & 2020\end{bmatrix}$$

а) $$\det A=\begin{vmatrix}0 & 1 & \alpha+1 & -1\\ 0 & \alpha & 6 & -2\\ 0 & -1 & -3 & \alpha-1\\ 2019 & 12 & 14 & 2020\end{vmatrix}=2019\,(-1)^{4+1}\begin{vmatrix}1 & \alpha+1 & -1\\ \alpha & 6 & -2\\ -1 & -3 & \alpha-1\end{vmatrix}=-2019\begin{vmatrix}0 & \alpha+1 & -1\\ \alpha-2 & 6 & -2\\ \alpha-2 & -3 & \alpha-1\end{vmatrix}=$$

$$=-2019(\alpha-2)\begin{vmatrix}0 & \alpha+1 & -1\\ 1 & 6 & -2\\ 1 & -3 & \alpha-1\end{vmatrix}/\cdot(-1)=-2019(\alpha-2)\begin{vmatrix}0 & \alpha+1 & -1\\ 1 & 6 & -2\\ 0 & -9 & \alpha+1\end{vmatrix}=-2019(\alpha-2)(-1)^{2+1}\begin{vmatrix}\alpha+1 & -1\\ -9 & \alpha+1\end{vmatrix}$$

$$=2019(\alpha-2)\left((\alpha+1)^2-9\right)=2019(\alpha-2)(\alpha-2)(\alpha+4)=\boxed{2019(\alpha-2)^2(\alpha+4)}$$

б) $\boxed{A^{-1}\text{ постоји}} \Leftrightarrow \det A\neq 0 \Leftrightarrow \boxed{\alpha\neq 2 \wedge \alpha\neq -4}$

в) $\boxed{\alpha\neq 2 \wedge \alpha\neq -4} \Rightarrow \det A\neq 0 \Rightarrow$ у врстама (колонама) су лин. нез. вектори $\Rightarrow \boxed{\rho(A)=4}$

$\underline{\alpha=2}$

$$A=\begin{bmatrix}0 & 1 & 3 & -1\\ 0 & 2 & 6 & -2\\ 0 & -1 & -3 & 1\\ 2019 & 12 & 14 & 2020\end{bmatrix}/\cdot 2 \sim \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -1 & -3 & 1\\ 2019 & 12 & 14 & 2020\end{bmatrix}\qquad \rho(A)=2$$

$\underline{\alpha=-4}$

$$A=\begin{bmatrix}0 & 1 & -3 & -1\\ 0 & -4 & 6 & -2\\ 0 & -1 & -3 & -5\\ 2019 & 12 & 14 & 2020\end{bmatrix}/\cdot(-4) \sim \begin{bmatrix}0 & 0 & -6 & -6\\ 0 & 0 & 18 & 18\\ 0 & -1 & -3 & 3\\ 2019 & 12 & 14 & 2020\end{bmatrix}/\cdot\tfrac{1}{3}\qquad \rho(A)=3$$

6. $\mathbb{U},\mathbb{W}\leq\mathbb{V}$, $\boxed{\mathbb{U}\neq\mathbb{W}}$ #

$\dim\mathbb{U},\ \dim\mathbb{W}=4$

$\dim\mathbb{V}=5$

$\dim(\mathbb{U}+\mathbb{W}),\ \dim\mathbb{U}\cap\mathbb{W}=?$

• $\underbrace{\dim(\mathbb{U}+\mathbb{W})}_{\leq 5}+\dim(\mathbb{U}\cap\mathbb{W})=\underbrace{\dim\mathbb{U}}_{=4}+\underbrace{\dim\mathbb{W}}_{=4}$

(јер $\mathbb{U}+\mathbb{W}\leq\mathbb{V}$ и $\dim\mathbb{V}=5$)

$\Rightarrow \dim(\mathbb{U}\cap\mathbb{W})=8-\underbrace{\dim(\mathbb{U}+\mathbb{W})}_{\leq 5}\geq 3$

$\left.\begin{array}{l}\mathbb{U}\cap\mathbb{W}\leq\mathbb{U},\mathbb{W}\\ \dim\mathbb{U}=\dim\mathbb{W}=4\end{array}\right\}\Rightarrow \dim\mathbb{U}\cap\mathbb{W}\leq 4$

$\Rightarrow \boxed{3\leq\dim(\mathbb{U}\cap\mathbb{W})\leq 4}$

• Ако је $\left.\begin{array}{l}\dim\mathbb{U}\cap\mathbb{W}=4\\ \mathbb{U}\cap\mathbb{W}\leq\mathbb{U},\mathbb{W}\\ \dim\mathbb{U}=\dim\mathbb{W}=4\end{array}\right\}\Rightarrow \mathbb{U}\cap\mathbb{W}=\mathbb{U}=\mathbb{W} \Rightarrow \mathbb{U}=\mathbb{W}$ ↯ са #

$\Rightarrow \boxed{\dim\mathbb{U}\cap\mathbb{W}=3}$

$\boxed{\dim\mathbb{U}+\mathbb{W}=5}$

ПРИМЕР

$e_1=(1,0,0,0,0)$ $\quad\mathbb{U}=\mathcal{L}(e_1,e_2,e_3,e_4)$

$e_2=(0,1,0,0,0)$ $\quad\mathbb{W}=\mathcal{L}(e_1,e_2,e_3,e_5)$

$e_3=(0,0,1,0,0)$

$e_4=(0,0,0,1,0)$ $\quad\mathbb{U}\cap\mathbb{W}=\mathcal{L}(e_1,e_2,e_3)$

$e_5=(0,0,0,0,1)$ $\quad\mathbb{U}+\mathbb{W}=\mathbb{R}^5$