②

$$A=\begin{bmatrix}14 & 15 & -15\\ -5 & -6 & 5\\ 5 & 5 & -6\end{bmatrix}$$

$$\varphi_A(\lambda)=\begin{vmatrix}14-\lambda & 15 & -15\\ -5 & -6-\lambda & 5\\ 5 & 5 & -6-\lambda\end{vmatrix}=\begin{vmatrix}14-\lambda & \lambda+1 & -\lambda-1\\ -5 & -\lambda-1 & 0\\ 5 & 0 & -\lambda-1\end{vmatrix}=(\lambda+1)^2\begin{vmatrix}14-\lambda & 1 & -1\\ -5 & -1 & 0\\ 5 & 0 & -1\end{vmatrix}=$$

$/(-1)\to +$ $\quad \uparrow +$

$$=(\lambda+1)^2\begin{vmatrix}9-\lambda & 0 & -1\\ -5 & -1 & 0\\ 5 & 0 & -1\end{vmatrix}=(\lambda+1)^2(-1)(\lambda-9+5)=-(\lambda+1)^2(\lambda-4)$$

$\lambda=-1$

$$\begin{bmatrix}14 & 15 & -15\\ -5 & -6 & 5\\ 5 & 5 & -6\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=-\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}$$

$15v_1+15v_2-15\boxed{v_3}=0$

$-5v_1\cancel{-5v_2+5v_3}=0$

$5v_1\cancel{+5v_2-5v_3}=0$

$v_1,v_2\in\mathbb{R},\quad v_3=v_1+v_2$

$$\begin{bmatrix}v_1\\ v_2\\ v_1+v_2\end{bmatrix}=v_1\begin{bmatrix}1\\0\\1\end{bmatrix}+v_2\begin{bmatrix}0\\1\\1\end{bmatrix},\ v_1,v_2\in\mathbb{R},\ v_1^2+v_2^2\neq 0$$

$\lambda=4$

$$\begin{bmatrix}14 & 15 & -15\\ -5 & -6 & 5\\ 5 & 5 & -6\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=4\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}$$

$10v_1+15v_2-15v_3=0$

$-5\boxed{v_1}-10v_2+5v_3=0\quad /\cdot(-2)$

$5v_1+5v_2-10v_3=0$

$-\boxed{v_1}-2v_2+v_3=0$

$-5\boxed{v_2}-5v_3=0$

$-5v_2-5v_3=0$

$v_3\in\mathbb{R},\quad v_2=-v_3,\quad v_1=v_3-2v_2=v_3+2v_3=3v_3$

$$\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}3v_3\\ -v_3\\ v_3\end{bmatrix}=v_3\begin{bmatrix}3\\-1\\1\end{bmatrix},\ v_3\in\mathbb{R}\setminus\{0\}$$

$$P=\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}\quad D=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 4\end{bmatrix}\quad D=P^{-1}AP\quad m_A(\lambda)=(\lambda+1)(\lambda-4)$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 3 & 1 & 0 & 0\\ 0 & 1 & -1 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 0 & 1\end{array}\right]/(-1)$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 3 & 1 & 0 & 0\\ 0 & 1 & -1 & 0 & 1 & 0\\ 0 & 1 & -2 & -1 & 0 & 1\end{array}\right]/(-1)$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 3 & 1 & 0 & 0\\ 0 & 1 & -1 & 0 & 1 & 0\\ 0 & 0 & -1 & -1 & -1 & 1\end{array}\right]/(-1)\quad /\cdot 3$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -2 & -3 & 3\\ 0 & 1 & 0 & 1 & 2 & -1\\ 0 & 0 & 1 & 1 & 1 & -1\end{array}\right]\Rightarrow P^{-1}=\begin{bmatrix}-2 & -3 & 3\\ 1 & 2 & -1\\ 1 & 1 & -1\end{bmatrix}$$

$$A^n=A\cdot\ldots\cdot A=(PD\underbrace{P^{-1})(P}_{=E}D\underbrace{P^{-1})\cdot\ldots\cdot(P}DP^{-1})=PD^nP^{-1}=$$

$$=\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}(-1)^n & 0 & 0\\ 0 & (-1)^n & 0\\ 0 & 0 & 4^n\end{bmatrix}\begin{bmatrix}-2 & -3 & 3\\ 1 & 2 & -1\\ 1 & 1 & -1\end{bmatrix}=$$

$=$

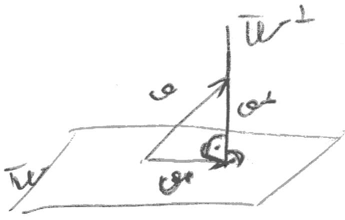
3. $W=\{(x,y,z,t)\in\mathbb{R}^4 \mid 2x-y+3z=0\}$

а) $W^{\perp}=\mathcal{L}((2,-1,3,0))$

$W=$ ___

б) $\underbrace{(5,-3,5,10)}_{v=} = \underbrace{(1,-1,-1,10)}_{\in W} + \underbrace{(4,-2,6,0)}_{\in W^{\perp}}$

орт. проj. орт. доп.



$d(v,W)=\|v^{\perp}\|=2\|(2,-1,3,0)\|=2\sqrt{14}$

$\cos\measuredangle(v,W)=\cos\measuredangle(v,v')=\dfrac{\langle v,v'\rangle}{\|v\|\,\|v'\|}=\dfrac{5+3-5+100}{\sqrt{25+9+25+100}\,\sqrt{1+1+1+100}}=$

$=\dfrac{103}{\sqrt{159}\,\sqrt{103}}=\sqrt{\dfrac{103}{159}}$

$\boxed{\measuredangle(v,W)=\arccos\sqrt{\dfrac{103}{159}}}$

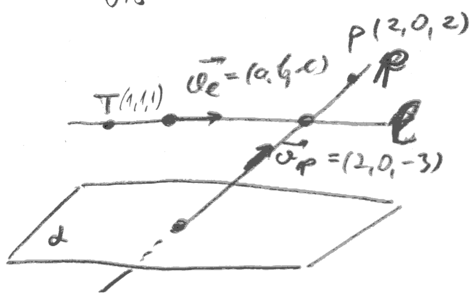
4. $\alpha: 2x-y-3z+2019=0$

$T(1,1,1)$

$p: \dfrac{x-2}{2}=\dfrac{y}{0}=\dfrac{z-2}{-3}$

l: $l\parallel\alpha$
$l\ni T$
l сече p

$v_l=(a,b,c)$ ← што још фали



• $l\parallel\alpha \iff v_l\perp\vec{n}_\alpha$

$\iff \langle(a,b,c),(2,-1,-3)\rangle=0$

$\iff \boxed{2a-b-3c=0}$

• l сече p $\iff$ v_l, v_p − лин. нез.

$[\vec{PT},\vec{v_p},\vec{v_l}]=0 \iff \begin{vmatrix}-1 & 1 & -1\\ 2 & 0 & -3\\ a & b & c\end{vmatrix}=0$

$\iff -1\begin{vmatrix}2 & -3\\ a & c\end{vmatrix}-b\begin{vmatrix}-1 & -1\\ 2 & -3\end{vmatrix}=0$

$\iff -1(2c+3a)-b(3+2)=0$

$\iff \boxed{3a+5b+2c=0}$

$2a-b-3c=0 \quad /\cdot 5$

$3a+5b+2c=0$

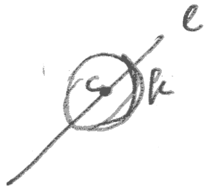
$13a \quad -13c=0 \Rightarrow a=c$

$b=-c$

$v_l=(c,-c,c)=c(1,-1,1)$

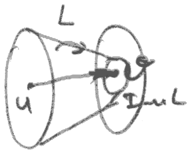
$\Rightarrow \boxed{l: \dfrac{x-1}{1}=\dfrac{y-1}{-1}=\dfrac{z-1}{1}}$

5. $\boxed{\begin{aligned} S_l:\ x'&=\tfrac{4}{5}x-\tfrac{3}{5}y+1\\ y'&=-\tfrac{3}{5}x-\tfrac{4}{5}y+3\end{aligned}}$



$S_l(k)=k$

⑥ $\operatorname{Ker} L^2 = \operatorname{Ker} L \overset{?}{\Rightarrow} \operatorname{Ker} L \cap \operatorname{Im} L = \{0\}$



(απ.α.) $\operatorname{Ker} L \cap \operatorname{Im} L \neq \{0\}$, έστω $\operatorname{Ker} L \cap \operatorname{Im} L \ni v$, $v \neq 0$

$v \in \operatorname{Im} L \Rightarrow (\exists u) : L(u) = v$

$v \in \operatorname{Ker} L \Rightarrow L(u) \in \operatorname{Ker} L \Rightarrow L^2(u) = 0 \longrightarrow u \in \operatorname{Ker} L^2$

άρα $L(u) = v \neq 0 \longrightarrow u \notin \operatorname{Ker} L$,

$\operatorname{Ker} L \neq \operatorname{Ker} L^2$ ↯