I група

① $W \subseteq \mathbb{R}^3$

1° $W = \{(x,y,z) \in \mathbb{R}^3 \mid x+2y+2z=0\} =$

$v = (x,y,z) \in \mathbb{R}^3$

$e = (1,2,2)$

$= \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{(1,2,2)}_{e} \circ (x,y,z) = 0\}$

$= \{v \in \mathbb{R}^3 \mid e \circ v = 0\} = (\mathcal{L}(e))^\perp$

$W = \{(x,y,z) \in \mathbb{R}^3 \mid \boxed{x} + 2y + 2z = 0\}$

$\boxed{x} + 2y + 2z = 0$

$y = a,\ a \in \mathbb{R}$

$z = b,\ b \in \mathbb{R}$

$x = -2y - 2z = -2a - 2b$

$W = \{(-2a-2b, a, b) \in \mathbb{R}^3 \mid a, b \in \mathbb{R}\} =$

$= \{a\underbrace{(-2,1,0)}_{e_1} + b\underbrace{(-2,0,1)}_{e_2} \mid a,b \in \mathbb{R}\}$

$= \mathcal{L}(e_1, e_2)$ $(*)$

↑

генератриса за W

$$\begin{bmatrix} -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} /\cdot(-1) \sim \begin{bmatrix} \boxed{-2} & 1 & 0 \\ 0 & \boxed{1} & 1 \end{bmatrix}$$ — линеарно су независни $(**)$

$\overset{(*)(**)}{\Rightarrow} \{e_1, e_2\}$ база за W

$\Downarrow$

$\dim W = 2$

Из 1°: $W = (\mathcal{L}(e))^\perp$ $\qquad \underline{(W^\perp)^\perp = W}$

$W^\perp = ((\mathcal{L}(e))^\perp)^\perp = \mathcal{L}(e)$ ← генератриса за $W^\perp$

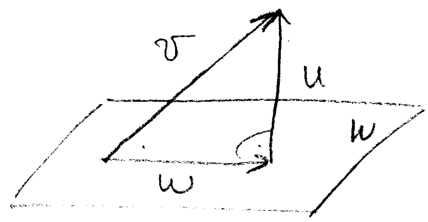
↑ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Downarrow$

линеарно независан $\Rightarrow$ база за $W^\perp \Rightarrow$

$\dim W^{\perp} = 1$

б) $v = (1, 0, 4)$

$$v = \underbrace{w}_{\in W} + \underbrace{u}_{\in W^{\perp}}$$



$$u = \alpha e$$

$$v = w + \alpha e \ / \circ e$$

$$v \circ e = \underbrace{w \circ e}_{=0} + \alpha e \circ e$$

$= 0$ јер $w \in W$ и $e \in W^{\perp}$ и $W \perp W^{\perp}$

$$v \circ e = \alpha e \circ e$$

$$v \circ e = (1, 0, 4) \circ (1, 2, 2) = 1 + 0 + 8 = 9$$

$$e \circ e = (1, 2, 2) \circ (1, 2, 2) = 1 + 4 + 4 = 9$$

$$9 = \alpha 9 \Rightarrow \alpha = 1$$

$u = \alpha e = 1 \cdot e = e \rightarrow$ ортогонална пројекција за $W^{\perp}$
тј. ортогонална допуна за W

$$w = v - u = (1, 0, 4) - (1, 2, 2) = (0, -2, 2)$$

$\uparrow$
ортогонална пројекција за W
тј. ортогонална допуна за $W^{\perp}$

$$d(v, w) = \|u\| = \|(1, 2, 2)\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = 3$$

$$d(v, u) = \|w\| = \|(0, -2, 2)\| = \sqrt{0^2 + (-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2} \quad ?$$

$w \in W$, $u \in W^{\perp}$

база за $W \Rightarrow \{e_1, e_2\} = \{(-2, 1, 0), (-2, 0, 1)\}$

$u \perp \{e_1, e_2\} \overset{?}{\Leftrightarrow} u \perp \mathcal{L}(e_1, e_2)$

$\boxed{\Leftarrow}$ $u \perp \mathcal{L}(e_1, e_2) \overset{?}{\Rightarrow} u \perp \{e_1, e_2\}$

Ово је тачно јер ако је u нормалан на генератрису векторског простора онда је нормалан и на све векторе тог простора.

$\boxed{\Rightarrow}$ $u \perp \{e_1, e_2\} \overset{?}{\Rightarrow} u \perp \mathcal{L}(e_1, e_2)$

$u \perp \{e_1, e_2\} \Rightarrow \langle u, e_1 \rangle = 0$

$\langle u, e_2 \rangle = 0$

$u \perp \mathcal{L}(e_1, e_2) \Rightarrow u \perp \alpha e_1 + \beta e_2 \Rightarrow$

$\Rightarrow \langle u, \alpha e_1 + \beta e_2 \rangle = 0$

$\langle u, \alpha e_1 \rangle + \langle u, \beta e_2 \rangle = 0$

$\alpha \underbrace{\langle u, e_1 \rangle}_{0} + \beta \underbrace{\langle u, e_2 \rangle}_{0} = 0$

$\alpha \cdot 0 + \beta \cdot 0 = 0$ ✓

2. $L: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$$L(x,y,z,t) = (x-y+z-t,\ 2x-y-z+2t,\ 3x-z-t)$$

$$E = \{\underbrace{(1,0,0,0)}^{e_1}, \underbrace{(0,1,0,0)}^{e_2}, \underbrace{(0,0,1,0)}^{e_3}, \underbrace{(0,0,0,1)}^{e_4}\}$$

↳ канонска база за $\mathbb{R}^4$

$$F = \{\underbrace{(1,0,0)}_{f_1}, \underbrace{(0,1,0)}_{f_2}, \underbrace{(0,0,1)}_{f_3}\}$$

↳ канонска база за $\mathbb{R}^3$

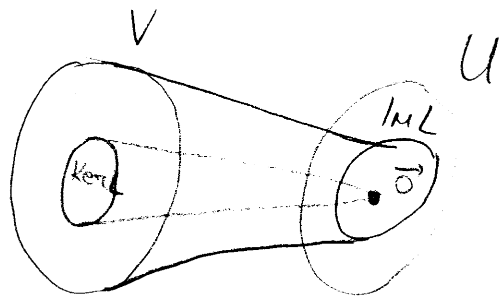
$$L(e_1) = L(1,0,0,0) = 1 \cdot f_1 + 2 \cdot f_2 + 3 \cdot f_3 = (1,2,3)$$

$$L(e_2) = L(0,1,0,0) = -1 \cdot f_1 - 1 \cdot f_2 + 0 \cdot f_3 = (-1,-1,0)$$

$$L(e_3) = L(0,0,1,0) = 1 \cdot f_1 - 1 \cdot f_2 - 1 \cdot f_3 = (1,-1,-1)$$

$$L(e_4) = L(0,0,0,1) = -1 \cdot f_1 + 2 \cdot f_2 - 1 \cdot f_3 = (-1,2,-1)$$

$$[L]_E^F = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & -1 & -1 & 2 \\ 3 & 0 & -1 & -1 \end{bmatrix}$$



$$\operatorname{Ker} L = \{v \in V \mid L(v) = \vec{0}\} =$$

$$= \{(x,y,z,t) \in \mathbb{R}^4 \mid L(x,y,z,t) = (0,0,0)\}$$

$$= \left\{(x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x - y + z - t = 0 \\ 2x - y - z + 2t = 0 \\ 3x \quad\ \ - z - t = 0 \end{array}\right\}$$

$$\begin{array}{ll} x - y + z - t = 0 & / \cdot (-2) \ / \cdot (-3) \\ 2x - y - z + 2t = 0 & \leftarrow + \\ 3x \quad\ \ - z - t = 0 & \leftarrow + \end{array}$$

$$\begin{aligned}
\boxed{x} - y + z - t &= 0 \\
\boxed{y} - 3z + 4t &= 0 \quad / \cdot (-3) \\
3y - 4z + 2t &= 0
\end{aligned}$$

$$\begin{aligned}
\boxed{x} - y + z - t &= 0 \\
\boxed{y} - 3z + 4t &= 0 \\
5z - 10t &= 0 \quad /:5
\end{aligned}$$

$$\begin{aligned}
\boxed{x} - y + z - t &= 0 \\
\boxed{y} - 3z + 4t &= 0 \\
\boxed{z} - 2t &= 0
\end{aligned}$$

$t = a,\ a \in \mathbb{R}$

$z = 2t = 2a$

$y = 3z - 4t = 3 \cdot 2a - 4a = 6a - 4a = 2a$

$x = y - z + t = \cancel{2a} - \cancel{2a} + a = a$

$\operatorname{Ker} L = \{(a, 2a, 2a, a) \in \mathbb{R}^4 \mid a \in \mathbb{R}\} =$

$= \{a(1,2,2,1) \mid a \in \mathbb{R}\} = \mathcal{L}((\underbrace{1,2,2,1}_{e})) = \mathcal{L}(e)$

Генератриса за Ker L, линеарно је независан, па је онда и база за Ker L ∇

$\dim(\operatorname{Ker} L) = \delta(L) = 1 \Leftarrow$

$\operatorname{Im} L = \mathcal{L}(L(e_1), L(e_2), L(e_3), L(e_4)\} =$

$= \mathcal{L}((1,2,3), (-1,-1,0), (1,-1,-1), (-1,2,-1))$

$\uparrow$

(*) Генератриса за Im L

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 0 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{matrix} /\cdot 1 \quad /\cdot(-1) \\ + \\ + \\ + \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & -3 & -4 \\ 0 & 4 & 2 \end{bmatrix} \begin{matrix} \\ /\cdot 3 \quad /\cdot(-4) \\ + \\ + \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & -10 \end{bmatrix} \begin{matrix} \\ \\ /\cdot 2 \\ + \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ (**) линеарно су независни $L(e_1), L(e_2), L(e_3)$

Из (*) и (**) $\{L(e_1), L(e_2), L(e_3)\}$ је база за $\mathrm{Im}\, L$

$$\rho(L) = \dim(\mathrm{Im}\, L) = 3$$

(3) $L : \mathbb{R}^3 \to \mathbb{R}^3$

$L(x,y,z) = (x-2y+z,\ -x+3y+2z,\ x-y+5z)$

a) $u = (u_1, u_2, u_3)$

$w = (w_1, w_2, w_3)$

1° адитивност

$L(u+w) \stackrel{?}{=} L(u) + L(w)$

1) $L(u+w) = L(u_1+w_1, u_2+w_2, u_3+w_3) =$

$= (u_1 - 2u_2 + u_3 + w_1 - 2w_2 + w_3,$

$-u_1 + 3u_2 + 2u_3 - w_1 + 3w_2 + 2w_3,$

$u_1 - u_2 + 5u_3 + w_1 - w_2 + 5w_3)$

2) $L(u) + L(w) = L(u_1, u_2, u_3) + L(w_1, w_2, w_3) =$

$= (u_1 - 2u_2 + u_3,\ -u_1 + 3u_2 + 2u_3,\ u_1 - u_2 + 5u_3) +$

$+ (w_1 - 2w_2 + w_3,\ -w_1 + 3w_2 + 2w_3,\ w_1 - w_2 + 5w_3) =$

$= (u_1 - 2u_1 + u_3 + w_1 - 2w_2 + w_3,\ -u_1 + 3u_2 + 2u_3 - w_1 + 3w_2 + 2w_3,$

$u_1 - u_2 + 5u_3 + w_1 - w_2 + 5w_3)$

$\stackrel{1),2)}{\Rightarrow}$ $L(u+w) = L(u) + L(w)$

2° хомогеност:

$L(\alpha u) \stackrel{?}{=} \alpha \cdot L(u)$

1) $L(\alpha u) = L(\alpha(u_1, u_2, u_3)) = L(\alpha u_1, \alpha u_2, \alpha u_3) =$

$= (\alpha(u_1 - 2u_2 + u_3),\ \alpha(-u_1 + 3u_2 + 2u_3),\ \alpha(u_1 - u_2 + 5u_3)) =$

$= (\alpha u_1 - 2\alpha u_2 + \alpha u_3,\ -\alpha u_1 + 3\alpha u_2 + 2\alpha u_3,\ \alpha u_1 - \alpha u_2 + 5\alpha u_3)$

2) $\alpha \cdot L(u) = \alpha \cdot L(u_1, u_2, u_3) = \alpha \cdot (u_1 - 2u_2 + u_3,\ -u_1 + 3u_2 + 2u_3,$

$u_1 - u_2 + 5u_3) =$

$\longrightarrow$

$$= (\alpha(u_1-2u_2+u_3), \alpha(-u_1+3u_2+2u_3), \alpha(u_1-u_2+5u_3)) =$$
$$= (\alpha u_1 - 2\alpha u_2 + \alpha u_3, -\alpha u_1 + 3\alpha u_2 + 2\alpha u_3, \alpha u_1 - \alpha u_2 + 5\alpha u_3)$$

$$\overset{1), 2)}{\Longrightarrow} \quad L(\alpha u) = \alpha \cdot L(u)$$

б) L инвертибилан $\Leftrightarrow \operatorname{Ker} L = \{\vec{0}\}$

$$v = (x, y, z) \in \operatorname{Ker} L$$

$$\operatorname{Ker} L = \{(x,y,z) \in \mathbb{R}^3 \mid L(x,y,z) = \vec{0}\} =$$

$$= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x - 2y + z = 0 \\ -x + 3y + 2z = 0 \\ x - y + 5z = 0 \end{array}\right\}$$

$$\begin{array}{l} \boxed{x} - 2y + z = 0 \quad / \cdot(-1) \\ -x + 3y + 2z = 0 \\ x - y + 5z = 0 \end{array}$$

$$\begin{array}{l} \boxed{x} - 2y + z = 0 \\ 5y + z = 0 \\ \boxed{y} + 4z = 0 \quad / \cdot(-5) \end{array}$$

$$\begin{array}{l} \boxed{x} - 2y + z = 0 \\ -9\boxed{z} = 0 \\ \boxed{y} + 4z = 0 \end{array}$$

$$\left.\begin{array}{l} z = 0 \\ y = 0 \\ x = 0 \end{array}\right\} \Rightarrow (x,y,z) = (0,0,0) \Rightarrow \operatorname{Ker} L = \{\vec{0}\}$$

$$\Downarrow$$

L je инвертибилан

$$E = \{\overbrace{(1,0,0)}^{e_1}, \overbrace{(0,1,0)}^{e_2}, \overbrace{(0,0,1)}^{e_3}\}$$

↳ канонска база за $\mathbb{R}^3$

$$L(e_1) = L(1,0,0) = (1,-1,1)$$

$$L(e_2) = L(0,1,0) = (-2,3,-1)$$

$$L(e_3) = L(0,0,1) = (1,2,5)$$

$$[L]_E = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -1 & 3 & 2 & 0 & 1 & 0 \\ 1 & -1 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} /+ \quad /\cdot(-1) \\ \\ + \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 4 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ /\cdot(-1) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \begin{array}{l} \quad + \\ + \\ /\cdot(-3) \; /\cdot(-1) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 3 & 1 & -1 \\ 0 & 1 & 0 & 7 & 4 & -3 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \begin{array}{l} + \\ /\cdot 2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 17 & 9 & -7 \\ 0 & 1 & 0 & 7 & 4 & -3 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \qquad [L^{-1}]_E = \begin{bmatrix} 17 & 9 & -7 \\ 7 & 4 & -3 \\ -2 & -1 & 1 \end{bmatrix}$$

$$\|$$

$$[L^{-1}]_E$$

4.

$$\begin{vmatrix} -2 & 6 & 6 & 8 \\ \boxed{-1} & 4 & 2 & 2 \\ 0 & 4 & 3 & 3 \\ -7 & 3 & 3 & 9 \end{vmatrix} \begin{matrix} + \\ /\cdot(-2) \ /\cdot(-7) \\ + \\ \end{matrix} \sim$$

$$\sim \begin{vmatrix} 0 & -2 & 2 & 4 \\ \boxed{-1} & 4 & 2 & 2 \\ 0 & 4 & 3 & 3 \\ 0 & -25 & -11 & -5 \end{vmatrix} =$$

$$= (-1)\cdot(-1)^{1+2}\cdot \left|\begin{matrix} -2 & 2 & 4 \\ 4 & 3 & 3 \\ -25 & -11 & -5 \end{matrix}\right| \begin{matrix} -2 & 2 \\ 4 & 3 \\ -25 & -11 \end{matrix} =$$

$= (-2)\cdot 3\cdot(-5) + 2\cdot 3\cdot(-25) + 4\cdot 4\cdot(-11) - (-25)\cdot 3\cdot 4 - (-2)\cdot 3\cdot(-11)$

$- 2\cdot 4\cdot(-5) = 30 - 150 - 176 + 300 - 66 + 40 = 370 - 392 =$

$= \underline{\underline{-22}}$

5.

$$A = \begin{bmatrix} -2 & 4 & -3 \\ -2 & 7 & -6 \\ -2 & 8 & -7 \end{bmatrix}$$

$$\varphi_A(t) = \det(A - tE) = \left|\begin{matrix} -2-t & 4 & -3 \\ -2 & 7-t & -6 \\ -2 & 8 & -7-t \end{matrix}\right| \begin{matrix} -2-t & 4 \\ -2 & 7-t \\ -2 & 8 \end{matrix} =$$

$= +(t+2)(7-t)(t+7) + 48 + 48 - 6(7-t) - 48(t+2)$

$- 8(t+7) =$

$$= (t+2)(49-t^2) + \cancel{98} - 42 + 6t - 48t - \cancel{98} - 8t - 56 =$$

$$= 49t + \cancel{98} - t^3 - 2t^2 - 50t - \cancel{98} =$$

$$= -t^3 - 2t^2 - t = -t(t^2 + 2t + 1) = -t(t+1)^2$$

$$\varphi_A(t) = -t(t+1)^2$$

кандидати за $m_A(t)$:

- $t(t+1)$
- $t(t+1)^2$

$$t(t+1)\Big|_{t=A} = A \cdot (A + 1E) =$$

$$= \begin{bmatrix} -2 & 4 & -3 \\ -2 & 7 & -6 \\ -2 & 8 & -7 \end{bmatrix} \begin{bmatrix} -1 & 4 & -3 \\ -2 & 8 & -6 \\ -2 & 8 & -6 \end{bmatrix} =$$

$$= \begin{bmatrix} 2-8+6 & -8+32-24 & 6-24+18 \\ 2-14+12 & -8+56-48 & 6-42+36 \\ 2-16+14 & -8+64-56 & 6-42+42 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow m_A(t) = t(t+1)$$

$$\varphi_A(t) = -t(t+1)^2$$

$$-t(t+1)^2 = 0$$

$$t = 0 \lor t = -1$$

$$\underbrace{\lambda_1 = 0 \qquad \lambda_2 = -1}_{\text{сопствене вредности}}$$

$1^\circ \ \lambda_1 = 0$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \qquad Av = \lambda_1 v$$

$$Av = 0 \cdot v$$

$$Av = 0$$

$$\begin{bmatrix} -2 & 4 & -3 \\ -2 & 7 & -6 \\ -2 & 8 & -7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -2x + 4y - 3z &= 0 \quad / \cdot (-1) \\ -2x + 7y - 6z &= 0 \quad + \\ -2x + 8y - 7z &= 0 \quad + \end{aligned}$$

$$\begin{aligned} -2x + 4y - 3z &= 0 \\ 3y - 3z &= 0 \quad / \cdot \tfrac{1}{3} \\ 4y - 4z &= 0 \quad / \cdot \tfrac{1}{4} \end{aligned}$$

$$\begin{aligned} -2x + 4y - 3z &= 0 \\ \boxed{y} - z &= 0 \\ \cancel{y - z = 0} & \end{aligned}$$

$$z = a, \ a \in \mathbb{R}$$

$$y = z = a$$

$$x = -\tfrac{1}{2} \cdot (-4y + 3z) = -\tfrac{1}{2}(-4a + 3a) = -\tfrac{1}{2}(-a) = \tfrac{1}{2} a$$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{a}{2} \\ a \\ a \end{bmatrix} = a \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}, \quad a \in \mathbb{R} \setminus \{0\}$$

сви сопствени вектори за сопствену вредност $\lambda_1 = 0$

$2^\circ \quad \lambda_2 = -1$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \qquad Av = \lambda_2 v$$

$$Av = -1 \cdot v$$

$$\begin{bmatrix} -2 & 4 & -3 \\ -2 & 7 & -6 \\ -2 & 8 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -1 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$-2x + 4y - 3z = -x$$
$$-2x + 7y - 6z = -y$$
$$-2x + 8y - 7z = -z$$

$$-x + 4y - 3z = 0 \quad / \cdot (-2)$$
$$-2x + 8y - 6z = 0 \quad +$$
$$-2x + 8y - 6z = 0 \quad +$$

$$-x + 4y - 3z = 0$$
$$0 = 0$$
$$0 = 0$$

$y = a, \; a \in \mathbb{R}$

$z = b, \; b \in \mathbb{R}$

$x = 4y - 3z = 4a - 3b$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4a-3b \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad a, b \in \mathbb{R}, \quad a^2 + b^2 > 0$$

сви сопствени вектори за сопствени вредности $\lambda_2 = -1$

5. Наставак:

$\overset{1^\circ, 2^\circ}{\Longrightarrow}$ $\left(\frac{1}{2}, 1, 1\right)$ ← припадају различитим сопственим вредностима па су линеарно независни $\overset{*}{\Longrightarrow}$

$(4, 1, 0), (-3, 0, 1)$

$$\begin{bmatrix} 4 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 1 \end{bmatrix} /\cdot 3 \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \end{bmatrix}$$ – линеарно независни

$\overset{*}{\Longrightarrow}$ Има их $3 =$ ред матрице $A \Rightarrow$ Матрица A је слична дијагоналној матрици D

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{2} & 4 & -3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc|ccc l} \frac{1}{2} & 4 & -3 & 1 & 0 & 0 & /\cdot(-2) \\ 1 & 1 & 0 & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 1 & \\ \hline -1 & -8 & 6 & -2 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 1 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 1 & \\ \hline -1 & -8 & 6 & -2 & 0 & 0 & \\ 0 & -7 & 6 & -2 & 1 & 0 & \\ 0 & -8 & 7 & -2 & 0 & 1 & /\cdot(-1) \\ \hline -1 & -8 & 6 & -2 & 0 & 0 & \\ 0 & 1 & -1 & 0 & 1 & -1 & /\cdot 8 \\ 0 & -8 & 7 & -2 & 0 & 1 & \\ \hline -1 & -8 & 6 & -2 & 0 & 0 & \\ 0 & 1 & -1 & 0 & 1 & -1 & \\ 0 & 0 & -1 & -2 & 8 & -7 & /\cdot 6 \; /\cdot(-1) \\ \hline -1 & -8 & 0 & -14 & 48 & -42 & \\ 0 & 1 & 0 & 2 & -7 & 6 & /\cdot 8 \\ 0 & 0 & -1 & -2 & 8 & -7 & \end{array}$$

$$\left.\begin{array}{ccc|ccc} -1 & 0 & 0 & 2 & -8 & 6 \\ 0 & 1 & 0 & 2 & -7 & 6 \\ 0 & 0 & -1 & -2 & 8 & -7 \end{array}\right. \begin{array}{l} /\cdot(-1) \\ \\ /\cdot(-1) \end{array}$$

$$\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 8 & -6 \\ 0 & 1 & 0 & 2 & -7 & 6 \\ 0 & 0 & 1 & 2 & -8 & 7 \end{array}$$

$$P^{-1} = \begin{bmatrix} -2 & 8 & -6 \\ 2 & -7 & 6 \\ 2 & -8 & 7 \end{bmatrix}$$

$$P \backslash \; D = P^{-1} A P \; / P^{-1}$$

$$PDP^{-1} = \underbrace{PP^{-1}}_{E} A \underbrace{PP^{-1}}_{E}$$

$$\underline{A = PDP^{-1}}$$

$$A^n = \underbrace{(PDP^{-1})(PDP^{-1})(PDP^{-1})\ldots(PDP^{-1})}_{n} =$$

$$A^n = PD^nP^{-1}$$

$$D^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{bmatrix}$$

$$A^n = \begin{bmatrix} \frac{1}{2} & 4 & -3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{bmatrix} \begin{bmatrix} -2 & 8 & -6 \\ 2 & -7 & 6 \\ 2 & -8 & 7 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 4\cdot(-1)^n & (-3)\cdot(-1)^n \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{bmatrix} \begin{bmatrix} -2 & 8 & -6 \\ 2 & -7 & 6 \\ 2 & -8 & 7 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 8\cdot(-1)^n - 6(-1)^n & -28\cdot(-1)^n + 24\cdot(-1)^n & 24\cdot(-1)^n - 21\cdot(-1)^n \\ 2\cdot(-1)^n & -7\cdot(-1)^n & 6\cdot(-1)^n \\ 2\cdot(-1)^n & -8\cdot(-1)^n & 7\cdot(-1)^n \end{bmatrix} =$$

$$= \begin{bmatrix} 2\cdot(-1)^n & -4\cdot(-1)^n & 3\cdot(-1)^n \\ 2\cdot(-1)^n & -7\cdot(-1)^n & 6\cdot(-1)^n \\ 2\cdot(-1)^n & -8\cdot(-1)^n & 7\cdot(-1)^n \end{bmatrix}$$

(1) $W \subseteq \mathbb{R}^3$ $\quad 2x-y-2z=0$

II ГРУПА

a) $W=\{w\in\mathbb{R}^3 \mid 2x-y-2z=0\}$

$$W=\left\{w\in\mathbb{R}^3 \;\middle|\; \begin{array}{r} 2x-2z=y \\ x=a,\, a\in\mathbb{R} \\ z=b,\, b\in\mathbb{R} \\ 2a-2b=y \end{array}\right\}$$

$$W=\{(a,2a-2b,b) \mid a,b\in\mathbb{R}\}$$

$$W=\{\underbrace{a(1,2,0)}_{f_1}+\underbrace{b(0,-2,1)}_{f_2} \mid a,b\in\mathbb{R}\}$$

baza za $W=[\{f_1,f_2\}]$

$$\begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{-2} & 1 \end{bmatrix} \leftarrow \text{lin. nezavisni} \quad \dim W = 2$$

$$W^\perp=\{x\in\mathbb{R} \mid x\perp\mathcal{L}(f_1,f_2)\}$$

$$=\left\{x\in\mathbb{R} \;\middle|\; \begin{array}{l} x\circ f_1=0 \\ x\circ f_2=0 \end{array}\right\}$$

$$=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \begin{array}{l} (x,y,z)\circ(1,2,0)=0 \\ (x,y,z)\circ(0,-2,1)=0 \end{array}\right.$$

$$\begin{array}{l} \boxed{x}+2y=0 \\ -2y+z=0 \end{array}$$

$z=a,\, a\in\mathbb{R}$

$+2y=+a$

$\boxed{y=\frac{a}{2}}$

$x+\cancel{2}\frac{a}{\cancel{2}}=0$

$\boxed{x=-a}$

$$=\{(-a,\tfrac{a}{2},a) \mid a\in\mathbb{R}\}$$

$$=\{\underbrace{a(-1,\tfrac{1}{2},1)}_{e} \mid a\in\mathbb{R}\}$$

Baza za $W^\perp=[\{e\}]$ $\quad \dim W^\perp = 1$

b) $\vartheta = (0,1,4)$

$\vartheta = \vartheta' + \vartheta^{\perp}$ $\qquad \vartheta' \perp \vartheta^{\perp}$

$\vartheta' = \alpha_1 f_1 + \alpha_2 f_2$

$\vartheta = \alpha_1 f_1 + \alpha_2 f_2 + \vartheta^{\perp} \quad / \circ f_1$

$\vartheta \circ f_1 = \alpha_1 f_1 \circ f_1 + \alpha_2 f_2 \circ f_1 + \underbrace{\vartheta^{\perp} \circ f_1}_{0}$

$(0,1,4)\circ(1,2,0) = \alpha_1 (1,2,0)\circ(1,2,0) + \alpha_2 (0,-2,1)\circ(1,2,0)$

$2 = \alpha_1 (1+4) + \alpha_2 (-4)$

$2 = 5\alpha_1 - 4\alpha_2$

$\vartheta = \alpha_1 f_1 + \alpha_2 f_2 + \vartheta^{\perp} \quad / \circ f_2$

$\vartheta \circ f_2 = \alpha_1 f_1 \circ f_2 + \alpha_2 f_2 \circ f_2 + \underbrace{\vartheta^{\perp} \circ f_2}_{0}$

$(0,1,4)\circ(0,-2,1) = \alpha_1 (1,2,0)\circ(0,-2,1) + \alpha_2 (0,-2,1)\circ(0,-2,1)$

$-2+4 = \alpha_1(-4) + \alpha_2(4+1)$

$2 = -4\alpha_1 + 5\alpha_2$

$$\begin{aligned} 5\alpha_1 - 4\alpha_2 &= 2 \quad \Big| \cdot \tfrac{4}{5} \\ -4\alpha_1 + 5\alpha_2 &= 2 \end{aligned}$$

$\frac{\cancel{9}}{\cancel{5}}\alpha_2 = \frac{\cancel{18}^{\,2}}{\cancel{5}}$

$\boxed{\alpha_2 = 2}$

$5\alpha_1 - 4\alpha_2 = 2$

$5\alpha_1 = 2 + 4\alpha_2$

$5\alpha_1 = 2 + 4\cdot 2$

$5\alpha_1 = 2 + 8$

$5\alpha_1 = 10$

$\boxed{\alpha_1 = 2}$

$$\vartheta' = \alpha_1 f_1 + \alpha_2 f_2$$
$$= 2(1,2,0) + 2(0,-2,1)$$
$$= (2,4,0) + (0,-4,2)$$
$$= (2,0,2)$$

$$\vartheta^{\perp} = \vartheta - \vartheta'$$
$$\vartheta^{\perp} = (0,1,4) - (2,0,2)$$
$$= (-2,1,2)$$

$$d(\vartheta, W) = \|\vartheta^{\perp}\| = \sqrt{(-2)^2 + 1^2 + 2^2}$$
$$= \sqrt{4+1+4}$$
$$= \sqrt{9} = 3$$

$$d(\vartheta, W^{\perp}) = \|\vartheta'\| = \sqrt{2^2 + 0^2 + 2^2}$$
$$= \sqrt{4+4}$$
$$= \sqrt{8}$$

Vektor ϑ je bliži prostoru $W^{\perp}$.

2. $\mathbb{R}^4 \to \mathbb{R}^3$ $\quad L(x,y,z,t) = (-x+y-z+t,\ x-2y-2z+t,\ x+2y-2t)$

Kanonske baze:
$$E = \{ e_1 = (1,0,0,0),\ e_2 = (0,1,0,0),\ e_3 = (0,0,1,0),\ e_4 = (0,0,0,1) \}$$
$$E' = \{ e_1' = (1,0,0),\ e_2' = (0,1,0),\ e_3' = (0,0,1) \}$$

$$L(e_1) = L(1,0,0,0) = (-1,1,1) = (-1)e_1' + 1 \cdot e_2' + e_3'$$
$$L(e_2) = L(0,1,0,0) = (1,-2,2) = e_1' - 2e_2' + 2e_3'$$
$$L(e_3) = L(0,0,1,0) = (-1,-2,0) = -1e_1' - 2e_2'$$
$$L(e_4) = L(0,0,0,1) = (1,1,-2) = e_1' + e_2' - 2e_3'$$

$$[L]_E^{E'} = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -2 & -2 & 1 \\ 1 & 2 & 0 & -2 \end{bmatrix}$$

$[L(e_1)]_{E'}$ $\quad [L(e_2)]_{E'}$ $\quad [L(e_3)]_{E'}$ $\quad [L(e_4)]_{E'}$

$\operatorname{Im} L = \mathcal{L}(L(e_1), L(e_2), L(e_3), L(e_4))$

$= \mathcal{L}((-1,1,1), (1,-2,2), (-1,-2,0), (1,1,-2))$

$$\begin{array}{c} L(e_1) \\ L(e_2) \\ L(e_3) \\ L(e_4) \end{array} \begin{bmatrix} -\boxed{1} & 1 & 1 \\ 1 & -2 & 2 \\ -1 & -2 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{array}{c} +\ (-1)\ + \\ \\ \\ \\ \end{array} \sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \\ 0 & 2 & -1 \end{bmatrix} \begin{array}{c} \\ (-3)\ 2 \\ \\ \\ \end{array}$$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & -10 \\ 0 & 0 & 5 \end{bmatrix} \begin{array}{c} \\ \\ /:-10 \\ /:5 \end{array} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{c} \\ \\ -1 \\ \\ \end{array}$$

$$\sim \begin{bmatrix} -\boxed{1} & 1 & 0 \\ 0 & -\boxed{1} & 3 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

Baza za $\operatorname{Im} L = [\{e_1, e_2, e_3\}]$

$\rho(L) = 3$

$\operatorname{Ker} L \stackrel{?}{=} \{\vec{0}\}$

$\operatorname{Ker} L = \{v \in \mathbb{R}^4 \mid v = \vec{0}\}$

$\{(x,y,z,t) \in \mathbb{R}^4 \mid$

$$\begin{aligned} -\boxed{x} + y - z + t &= 0 \\ x - 2y - 2z + t &= 0 \\ x + 2y \quad - 2t &= 0 \end{aligned}$$

$$\begin{aligned} -y - 3z + 2t &= 0 \quad 3 \\ 3y - z - t &= 0 \end{aligned}$$

$-10z + 5t = 0 \quad /:5$

$-2z + t = 0$

$\boxed{t = 2z}, \ z \in \mathbb{R}$

$3y - z - t = 0$

$3y = t + z$

$3y = 2z + z$

$3y = 3z$

$\boxed{y = z}$

$x + 2y - 2t = 0$

$x = 2t - 2y$

$x = 2 \cdot 2z - 2z$

$x = 4z - 2z$

$\boxed{x = 2z}$

$\ker L = \{(2z, z, z, 2z) \mid z \in \mathbb{R}\}$

$= \{ z\underbrace{(2,1,1,2)} \mid z \in \mathbb{R}\}$

baza za $\ker L = [\{2,1,1,2\}]$ $\quad \mathcal{J}(L) = 1$

(5) $L : \mathbb{R}^3 \to \mathbb{R}^3$

$L(x,y,z) = (-x+y+2z, -x+2y-z, -x+2y-2z)$

1) ADITIVNOST

$L(v_1+v_2) \stackrel{?}{=} L(v_1) + L(v_2)$

$L(v_1+v_2) = L((x_1,y_1,z_1)+(x_2,y_2,z_2))$

$= L(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (-(x_1+x_2)+y_1+y_2+2(z_1+z_2), -(x_1+x_2)+2(y_1+y_2)-(z_1+z_2),$
$-(x_1+x_2)+2(y_1+y_2)-2(z_1+z_2))$

$L(v_1)+L(v_2) = L(x_1,y_1,z_1) + L(x_2,y_2,z_2)$

$= (-x_1+y_1+2z_1, -x_1+2y_1-z_1, -x_1+2y_1-2z_1)$
$+ (-x_2+y_2+2z_2, -x_2+2y_2-z_2, -x_2+2y_2-2z_2)$

$= (-x_1+y_1+2z_1+(-x_2+y_2+2z_2), -x_1+2y_1-z_1$
$+(-x_2+2y_2-z_2), -x_1+2y_1-2z_1+(-x_2+2y_2-2z_2))$

$= (-(x_1+x_2)+(y_1+y_2)+2(z_1+z_2), -(x_1+x_2)+2(y_1+y_2)$
$-(z_1+z_2), -(x_1+x_2)+2(y_1+y_2)-2(z_1+z_2))$

$=$

Jednaki su zahvaljujući asocijativnosti i komutativnosti sabiranja

2) Homogenost

$kL(v) = L(kv)$

$kL(v) = kL(x,y,z) = k(-x+y+2z, -x+2y-z, -x+2y-2z)$

$L(kv) = L(k(x,y,z)) = L(kx,ky,kz) = (-kx+ky+2kz,$
$-kx+2ky-kz,$
$-kx+2ky-2kz)$

$=$

zbog distributivnosti množenja

$= (k(-x+y+2z), k(-x+2y-z),$
$k(-x+2y-2z))$

$= k(-x+y+2z, -x+2y-z, -x+2y-2z)$

b) $\operatorname{Ker} L \overset{?}{=} \{\vec{0}\}$

$\operatorname{Ker} L = \{v \in \mathbb{R}^3 \mid L(v) = \vec{0}\}$

$= \{(x,y,z) \in \mathbb{R}^3 \mid (-x+y+2z, -x+2y-z, -x+2y-2z) = (0,0,0)\}$

$= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} -x+y+2z=0 \\ -x+2y-z=0 \\ -x+2y-2z=0 \end{array}\right\}$

$$\begin{array}{l} \boxed{-x}+y+2z=0 \\ -x+2y-z=0 \\ -x+2y-2z=0 \\ \hline y-3z=0 \\ y-4z=0 \\ \hline -z=0 \end{array}$$

$\left.\begin{array}{l} \Rightarrow z=0 \\ \Rightarrow y=0 \\ \Rightarrow x=0 \end{array}\right\} \Rightarrow \exists L^{-1}$

$= \{(x,y,z) \in \mathbb{R}^3 \mid (x,y,z) = 0\}$

Jeste invertibilan ✓

kanonske baze

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 \\ 0 & 1 & -4 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right] /\cdot(-1) \sim \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & -1 & 4 & -3 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 2 & -6 & 5 \\ 0 & 1 & 0 & -1 & 4 & -3 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 6 & -5 \\ 0 & 1 & 0 & -1 & 4 & -3 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$[L^{-1}]_E = \begin{bmatrix} -2 & 6 & -5 \\ -1 & 4 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

(4)

$$\begin{vmatrix} -2 & 6 & 6 & 8 \\ -1 & 0 & 2 & 2 \\ 0 & 4 & 3 & 3 \\ -6 & 3 & 3 & 9 \end{vmatrix} = \begin{vmatrix} -2 & 6 & 2 & 4 \\ -1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 3 \\ -6 & 3 & -9 & -3 \end{vmatrix} = \underbrace{(-1)(-1)^{1+2}}_{=1} \begin{vmatrix} 6 & 2 & 4 \\ 4 & 3 & 3 \\ 3 & -9 & -3 \end{vmatrix}$$

2

2

$$= \begin{vmatrix} 6 & 2 & 4 \\ 4 & 3 & 3 \\ 3 & -9 & -3 \end{vmatrix} \begin{matrix} 6 & 2 \\ 4 & 3 \\ 3 & -9 \end{matrix} = 6\cdot3\cdot(-3) + 2\cdot3\cdot3 + 4\cdot4\cdot(-9) - 4\cdot3\cdot3 - 6\cdot3\cdot(-9) - 2\cdot4\cdot(-3)$$

$$= -54 + 18 - 144 - 36 + 162 + 24$$

$$= -30$$

(5)

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ 10 & 5 & -4 \end{bmatrix}$$

$$\varphi_{A}(t) = \det(A - tE) \Rightarrow \begin{vmatrix} 3-t & 1 & -1 \\ 4 & 3-t & -2 \\ 10 & 5 & -4-t \end{vmatrix} \begin{matrix} 3-t & 1 \\ 4 & 3-t \\ 10 & 5 \end{matrix}$$

$$= (3-t)^2(-4-t) - 20 - 20 + 10(3-t) + 10(3-t) - (-4-t)4$$

$$= (9-6t+t^2)(-4-t) - 40 + 30 - 10t + 30 - 10t - (-16-4t)$$

$$= (-36 - 9t + 24t + 6t^2 - 4t^2 - t^3) - 10 - 10t + 30 - 10t + 16 + 4t$$

$$= -36 - 9t + 24t + 6t^2 - 4t^2 - t^3 - 10 - 10t + 30 - 10t + 16 + 4t$$

$$= -t^3 + 2t^2 - t = -t(t^2 - 2t + 1) = \underline{-t(t-1)^2}$$

Karakteristični polinom

Kandidati za minimalni polinom $m_{A}(t)$:

$t(t-1)$;

$t(t-1)^2$

$$t(t-1)\Big|_{t=A} \Rightarrow A(A-E) = \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ 10 & 5 & -4 \end{bmatrix}\begin{bmatrix} 3-1 & 1 & -1 \\ 4 & 3-1 & -2 \\ 10 & 5 & -4-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ 10 & 5 & -4 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 4 & 2 & -2 \\ 10 & 5 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\overset{6}{3\cdot 2} + \overset{4}{4\cdot 1} - 10$

$\overset{8}{4\cdot 2} + \overset{12}{3\cdot 4} - 2\cdot 10$

$10\cdot 2 + 5\cdot 4 - 4\cdot 10$

$20 \quad 20 \quad 40$

$3\cdot 1 + 2\cdot 1 - 5$

$\overset{4}{4\cdot 1} + \overset{6}{3\cdot 2} - 2\cdot 5$

$\overset{10}{10\cdot 1} + \overset{10}{5\cdot 2} - \overset{20}{4\cdot 5}$

$\overset{-3}{3\cdot(-1)} + \overset{-2}{1\cdot(-2)} + \overset{5}{(-1)(-5)}$

$\overset{-4}{4\cdot(-1)} + \overset{-6}{3(-2)} + \overset{10}{(-2)(-5)}$

$10\cdot(-1) + 5(-2) + (-4)(-5)$

$-10 \quad -10 \quad 20$

$m_A(t) = t(t-1)$

Sopstvene vrednosti: $\lambda_1 = 1$

$\lambda_2 = 0$

$v_1 A = v_1 \lambda_1$

$v_1 A = v_1 1$

$v_1 A - v_1 = 0$

$v_1(A-1) = 0$

$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} 3-1 & 1 & -1 \\ 4 & 3-1 & -2 \\ 10 & 5 & -4-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ 4 & 2 & -2 \\ 10 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x + y - z \\ 4x + 2y - 2z \\ 10x + 5y - 5z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x+y-z=0$$
$$4x+2y-2z=0$$
$$10x+5y-5z=0$$
(-2, -5)

$$0=0$$
$$0=0$$

$$2x+y-z=0$$

$$2x+y=z \quad , x,y\in\mathbb{R}$$

$$\vartheta_1=\begin{bmatrix}x\\y\\2x+y\end{bmatrix}=x\begin{bmatrix}1\\0\\2\end{bmatrix}+y\begin{bmatrix}0\\1\\1\end{bmatrix}$$

sopstveni prostori za sopstvenu vrednost $\lambda_1=1$, sopstveni vektori $\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$

$$\lambda_2=0$$
$$\vartheta_2 A=\vartheta_2\lambda_2$$
$$\vartheta_2 A=0$$

$$\begin{bmatrix}3&1&-1\\4&3&-2\\10&5&-4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}3x+y-z\\4x+3y-2z\\10x+5y-4z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow 3x+y-z=0$$
$$4x+3y-2z=0$$
$$10x+5y-4z=0$$
$(-\frac{4}{3}, -\frac{10}{3})$

$$\frac{5}{3}y-\frac{2}{3}z=0$$

$$\cancel{\frac{5}{3}y-\frac{2}{3}z=0}$$

$$\frac{5}{\cancel{3}}y=\frac{2}{\cancel{3}}z$$

$$2z=5y$$

$$z=\frac{5}{2}y, \ y\in\mathbb{R}$$

$$3x+y-z=0$$
$$3x=z-y$$
$$3x=\frac{5}{2}y-\frac{2y}{2}$$
$$\cancel{3}x=\frac{\cancel{3}}{2}y$$
$$x=\frac{1}{2}y \quad ,y\in\mathbb{R}$$

$$\Rightarrow \vartheta_2=\begin{bmatrix}\frac{1}{2}y\\y\\\frac{5}{2}y\end{bmatrix}, y\in\mathbb{R}$$

$$=\vartheta_2=y\begin{bmatrix}\frac{1}{2}\\1\\\frac{5}{2}\end{bmatrix} \cdot 2 \quad ,y\in\mathbb{R}$$

$\Rightarrow$ Imamo 3 sopstvena vektora, dakle slična je dijagonalnoj

$$\vartheta_2=y\begin{bmatrix}1\\2\\5\end{bmatrix}, y\in\mathbb{R}$$

$$D = P^{-1} A P \qquad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 5 & 0 & 0 & 1 \end{array}\right] \cdot (-2) \quad \cdot(-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 & 1 \end{array}\right] \cdot (-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \cdot (-2) \quad \cdot(-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & -1 \\ 0 & 1 & 0 & 4 & 3 & -2 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ -2 & -1 & 1 \end{bmatrix}$$

$$D = P^{-1} A P = \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ 10 & 5 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^n = P D^n P^{-1}$$

$$A^n = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ -2 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ -2 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ -2 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ 10 & 5 & -4 \end{bmatrix}$$ ista kao i matrica A