ГРАМ-ШМИТОВ ПОСТУПАК ОРТОГОНАЛИЗАЦИЈЕ

- Нека је $\{f_1, f_2, \dots, f_n\}$ база унитарног простора V. Грам-Шмитовим поступком налазимо ортогоналну базу $\{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\}$ такву да је:

$$\left.\begin{aligned}
\hat{e}_1 &= f_1\\
\hat{e}_2 &= f_2 - \frac{f_2 \circ \hat{e}_1}{\hat{e}_1 \circ \hat{e}_1}\hat{e}_1\\
\hat{e}_3 &= f_3 - \left(\frac{f_3 \circ \hat{e}_1}{\hat{e}_1 \circ \hat{e}_1}\hat{e}_1 + \frac{f_3 \circ \hat{e}_2}{\hat{e}_2 \circ \hat{e}_2}\hat{e}_2\right)\\
&\dots\\
\hat{e}_n &= f_n - \left(\frac{f_n \circ \hat{e}_1}{\hat{e}_1 \circ \hat{e}_1}\hat{e}_1 + \dots + \frac{f_n \circ \hat{e}_{n-1}}{\hat{e}_{n-1} \circ \hat{e}_{n-1}}\hat{e}_{n-1}\right)
\end{aligned}\right\} \text{ортогонална база}$$

$$\left.\begin{aligned}
e_1 &= \frac{\hat{e}_1}{\|\hat{e}_1\|}\\
e_2 &= \frac{\hat{e}_2}{\|\hat{e}_2\|}\\
&\vdots\\
e_n &= \frac{\hat{e}_n}{\|\hat{e}_n\|}
\end{aligned}\right\} \text{ОНБ}$$

1. Грам-Шмитовим поступком одредити ОНБ потпростора W простора $\mathbb{R}^4$ генерисаног векторима $f_1 = (1,1,1,1)$, $f_2 = (1,0,1,0)$, $f_3 = (-1,2,0,1)$. $W = \mathcal{L}(f_1, f_2, f_3)$

Затим одредити базу ортогоналног комплемента W ($W^\perp$).

$$\hat{e}_1 = f_1 = \boxed{(1,1,1,1)}$$

$$\hat{e}_2 = f_2 - \frac{f_2 \circ \hat{e}_1}{\hat{e}_1 \circ \hat{e}_1}\hat{e}_1 = (1,0,1,0) - \frac{(1,0,1,0)\circ(1,1,1,1)}{(1,1,1,1)\circ(1,1,1,1)}(1,1,1,1) = (1,0,1,0) - \frac{2}{4}(1,1,1,1) = \boxed{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)}$$

$$\hat{e}_3 = f_3 - \left(\frac{f_3 \circ \hat{e}_1}{\hat{e}_1 \circ \hat{e}_1}\hat{e}_1 + \frac{f_3 \circ \hat{e}_2}{\hat{e}_2 \circ \hat{e}_2}\hat{e}_2\right) = (-1,2,0,1) - \frac{(-1,2,0,1)\circ(1,1,1,1)}{(1,1,1,1)\circ(1,1,1,1)}(1,1,1,1) - \frac{(-1,2,0,1)\circ\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)}{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\circ\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)}\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$= (-1,2,0,1) - \frac{2}{4}(1,1,1,1) - \frac{-2}{1}\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) =$$

$$= (-1,2,0,1) - \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + (1,-1,1,-1) = \boxed{\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)}$$

НОРМИРАМО ДОБИЈЕНУ ОРТОГОНАЛНУ БАЗУ:

$$e_1 = \frac{\hat{e}_1}{|\hat{e}_1|} = \frac{1}{\sqrt{1^2+1^2+1^2+1^2}}(1,1,1,1) = \boxed{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}$$

$$e_2 = \frac{\hat{e}_2}{|\hat{e}_2|} = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}}\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) = \boxed{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)}$$

(e_1, e_2, e_3) – ОРТОНОРМИРАНА БАЗА простора W.

$$e_3 = \frac{\hat{e}_3}{|\hat{e}_3|} = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}}\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) = \boxed{\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)}$$

$$W^\perp = \{v \in \mathbb{R}^4 \mid v \perp \overset{\mathcal{L}(f_1,f_2,f_3)}{\overset{\parallel}{W}}\} = \{v \in \mathbb{R}^4 \mid v \perp f_1, f_2, f_3\} = \{v \in \mathbb{R}^4 \mid v \circ f_i = 0,\ i = 1,2,3\}$$

$v = (x,y,z,t)$

$$\begin{aligned}
v \circ f_1 &= x + y + z + t = 0 \quad /\cdot(-1)\\
v \circ f_2 &= x + z = 0\\
v \circ f_3 &= -x + 2y + t = 0\\
&-y - t = 0 \quad /\cdot 3\\
&3y + z + 2t = 0\\
&z - t = 0
\end{aligned}$$

$\Rightarrow$ $\boxed{z = t,\ t \in \mathbb{R}}$, $\boxed{y = -t}$, $\boxed{x = -t}$

$x = -y - z - t$
$= t - t - t$

$\Rightarrow v = (x,y,z,t) = (-t,-t,t,t) = t\underbrace{(-1,-1,1,1)}_{\text{база}}$

$\Rightarrow W^\perp = \{t(-1,-1,1,1) \mid t \in \mathbb{R}\} = \boxed{\mathcal{L}((-1,-1,1,1))}$

$W \oplus W^\perp = \mathbb{R}^4$

$\dim \mathbb{R}^4 = \dim W + \dim W^\perp - \dim \underbrace{W \cap W^\perp}_{\{0\}}$

$4 = 3 + \dim W^\perp$

$\Rightarrow \dim W^\perp = 1$

2. Нека је $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 - x_4 = 0 \, , \; 3x_1 - x_2 - x_3 = 0\}$. Користећи Грам-Шмитов поступак наћи ОНБ за V

$$\left.\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 0 \\ 3x_1 - x_2 - x_3 &= 0 \end{aligned}\right\} \Rightarrow x_2 = 3x_1 - x_3$$

$$x_1 = a, \quad x_3 = b \;, \; a, b \in \mathbb{R}, \quad x_2 = 3a - b$$

$$\mathrm{I} \Rightarrow x_4 = x_1 + x_2 + x_3 = a + 3a - \cancel{b} + \cancel{b} = 4a$$

$$V = \{(a, 3a-b, b, 4a) \mid a, b \in \mathbb{R}\} = \{a\underbrace{(1,3,0,4)}_{f_1} + b\underbrace{(0,-1,1,0)}_{f_2} \mid a, b \in \mathbb{R}\} = \mathcal{L}(f_1, f_2)$$

$$\begin{bmatrix} 1 & 3 & 0 & 4 \\ 0 & -1 & 1 & 0 \end{bmatrix} \Rightarrow f_1, f_2 \text{ - лин. нез.}$$

$\Rightarrow f_1, f_2$ база за V

Нама треба ОНБ:

- ортогонална:

$$\hat{e}_1 = f_1 = (1,3,0,4)$$

$$\hat{e}_2 = f_2 - \frac{f_2 \circ \hat{e}_1}{\hat{e}_1 \circ \hat{e}_1}\hat{e}_1 = (0,-1,1,0) - \frac{(0,-1,1,0)\circ(1,3,0,4)}{(1,3,0,4)\circ(1,3,0,4)}(1,3,0,4) = (0,-1,1,0) - \frac{-3}{26}(1,3,0,4)$$

$$= \left(\frac{3}{26}, \frac{-17}{26}, 1, \frac{6}{13}\right)$$

- нормирамо:

$$e_1 = \frac{\hat{e}_1}{\|\hat{e}_1\|} = \frac{1}{\sqrt{26}}(1,3,0,4)$$

$$e_2 = \frac{\hat{e}_2}{\|\hat{e}_2\|} = \frac{1}{\sqrt{\frac{3}{26^2} + \frac{17^2}{26^2} + 1^2 + \frac{6^2}{13^2}}}\left(\frac{3}{26}, \frac{-17}{26}, 1, \frac{6}{13}\right)$$

$$\boxed{\{e_1, e_2\} \text{ је ОНБ за } V}$$

РАСТОЈАЊЕ И УГАО

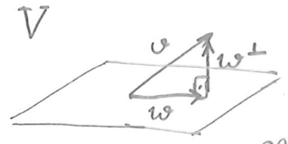
(Т) Ако је $W \leq V$, тада је $W^{\perp} \leq V$ и $V = W \oplus W^{\perp}$

$\forall v \in V$ може да се представи на јединствен начин као

$$v = \underbrace{w}_{\in W} + \underbrace{w^{\perp}}_{\in W^{\perp}}$$

ортогонална пројекција

ортогонална допуна



$\left[(W^{\perp})^{\perp} = W\right]$ $\qquad \dim W + \dim W^{\perp} = \dim V$

ортогонална допуна

Деф. Растојање вектора $v \in V$ од потпростора $W \leq V$ је $d(v, W) \overset{деф.}{=} \|w^{\perp}\|$

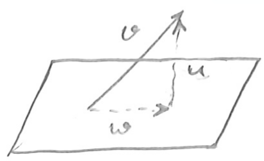
Угао између вектора $v \in V$ и потпростора $W \leq V$ је дат са $\quad \measuredangle(v, W) = \measuredangle(v, w)$

ортогонална пројекција

(1.) Одредити угао који вектор $v = (1,0,-1)$ заклапа са скупом решења W ј-не $-2x+y+z=0$ у односу на стандардни скаларни производ у $\mathbb{R}^3$. Затим одредити растојање вектора v од датог потпростора W.

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid -2x+y+z=0\} = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{(-2,1,1)}_{e} \circ (x,y,z) = 0\} = \{v \in \mathbb{R}^3 \mid v \circ e = 0\} = (\mathcal{L}(e))^{\perp}$$

$$W^{\perp} = \left((\mathcal{L}(e))^{\perp}\right)^{\perp} = \mathcal{L}(e)$$



$$v = \underbrace{w}_{\in W} + \underbrace{u}_{\in W^{\perp}}$$

$$v = w + \alpha e \quad / \circ e$$

$$v \circ e = \underbrace{w \circ e}_{=0 \text{ јер } w \perp e} + \alpha e \circ e$$

$$v \circ e = (1,0,-1) \circ (-2,1,1) = -3$$

$$e \circ e = (-2,1,1) \circ (-2,1,1) = 6$$

$$\Rightarrow -3 = \alpha \cdot 6$$

$$\Rightarrow \alpha = -\frac{1}{2}$$

$$\Rightarrow u = -\frac{1}{2} e = -\frac{1}{2}(-2,1,1)$$

$$\boxed{u = \left(1, -\frac{1}{2}, -\frac{1}{2}\right)} \leftarrow \text{ортогонална допуна}$$

$$\Rightarrow v = w - \frac{1}{2} e$$

$$w = v + \frac{1}{2} e = (1,0,-1) - \left(1, -\frac{1}{2}, -\frac{1}{2}\right) = \boxed{\left(0, -\frac{1}{2}, -\frac{1}{2}\right)} \leftarrow \text{ортогонална пројекција}$$

$$\cos \measuredangle(v, W) = \cos \measuredangle(v, \underset{\substack{\uparrow \\ \text{ортогонална} \\ \text{проj. } v \text{ на } W}}{w}) = \frac{v \circ w}{\|v\| \cdot \|w\|} = \frac{(1,0,-1) \circ \left(0, -\frac{1}{2}, -\frac{1}{2}\right)}{\sqrt{1^2+0^2+(-1)^2}\sqrt{0^2+\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2}} = \frac{\frac{1}{2}}{\sqrt{2}\sqrt{\frac{1}{2}}} = \frac{1}{2}$$

$$\Rightarrow \boxed{\measuredangle(v, W) = \frac{\pi}{3}}$$

$$d(v, W) = \|\underset{\substack{\uparrow \\ \text{ортогонална} \\ \text{допуна за } v}}{u}\| = \left\|\left(1, -\frac{1}{2}, -\frac{1}{2}\right)\right\| = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

II НАЧИН

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid -2x+y+z=0\} = \{(x,y,z) \in \mathbb{R}^3 \mid z = 2x-y\} = \{(x,y,2x-y) \mid x,y \in \mathbb{R}\} = \{x\underbrace{(1,0,2)}_{e_1} + y\underbrace{(0,1,-1)}_{e_2} \mid x,y \in \mathbb{R}\}$$
$$= \mathcal{L}(e_1, e_2)$$

$$v = \underbrace{w}_{\in W} + \underbrace{v'}_{\in W}$$

ОРТОГОНАЛНА ДОПУНА

$$v = \alpha e_1 + \beta e_2 + v' \quad / \circ e_1$$

$$v \circ e_1 = \alpha e_1 \circ e_1 + \beta e_2 \circ e_1 + 0$$

$$v \circ e_2 = \alpha e_1 \circ e_2 + \beta e_2 \circ e_2 + 0$$

$$\curvearrowright \quad (1,0,-1) \circ (1,0,2) = \alpha \cdot 5 + \beta \cdot (-2)$$

$$(1,0,-1) \circ (0,1,-1) = \alpha \cdot (-2) + \beta \cdot 2$$

$$\curvearrowright \quad \left.\begin{array}{l} -1 = 5\alpha - 2\beta \\ 1 = -2\alpha + 2\beta \end{array}\right\} + \Rightarrow \alpha = 0, \quad \beta = \frac{1}{2}$$

$$w = \frac{1}{2} e_2 = \boxed{\left(0, \frac{1}{2}, -\frac{1}{2}\right)}$$

$$v' = v - w =$$
$$= (1,0,-1) - \left(0, \frac{1}{2}, \ldots\right)$$

② Дат је векторски простор W решења система ј-на

$$4x+2y-4z+2t=0$$
$$5x+y-3z+3t=0$$

а) Наћи базу и dim за W

б) Наћи базу и dim за $W^{\perp}$

в) Растојање вектора $v=(2,6,-2,6)$ од W

г) угао између $v=(2,6,-2,6)$ и W.

$$W=\{(x,y,z,t)\in\mathbb{R}^4 \mid \underbrace{(x,y,z,t)\circ(4,2,-4,2)=0}_{f_1},\ \underbrace{(x,y,z,t)\circ(5,1,-3,3)=0}_{f_2}\} = \{v\in\mathbb{R}^4 \mid v\perp f_1,\ v\perp f_2\} = (\mathcal{L}(f_1,f_2))^{\perp}$$

$$\begin{bmatrix} 4 & 2 & -4 & 2 \\ 5 & 1 & -3 & 3 \end{bmatrix} \cdot\left(-\tfrac{5}{4}\right) \sim \begin{bmatrix} 4 & 2 & -4 & 2 \\ \boxed{0} & -\tfrac{3}{2} & & \end{bmatrix}$$

$\Rightarrow f_1,f_2$ - лин. нез.

б) $W^{\perp}=\mathcal{L}(f_1,f_2)$, f_1,f_2 - лин. нез. $\Rightarrow$ $\boxed{f_1,f_2 \text{ база за } W^{\perp},\ \dim W^{\perp}=2}$

$$W=\{(x,y,z,t)\in\mathbb{R}^4 \mid 4x+2y-4z+2t=0 \ /:2,\ 5x+y-3z+3t=0\}$$

$$\boxed{2x}+y-2z+t=0 \quad /\cdot(-5)$$
$$5x+y-3z+3t=0 \quad /\cdot 2$$
$$\overline{-3y+6z+t=0}$$

$\boxed{y=a,\ z=b}$ $a,b\in\mathbb{R}$

$\Rightarrow t=3y-6z$

$\boxed{t=3a-6b}$

$2x+a-2b+3a-6b=0$

$2x=-4a+8b$

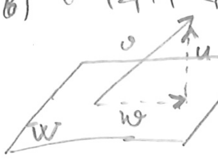
$\boxed{x=-2a+4b}$

а) $W=\{(-2a+4b,\ a,\ b,\ 3a-6b) \mid a,b\in\mathbb{R}\} = \{a\underbrace{(-2,1,0,3)}_{e_1}+b\underbrace{(4,0,1,-6)}_{e_2} \mid a,b\in\mathbb{R}\} = \mathcal{L}(e_1,e_2)$

$$\begin{bmatrix} -2 & 1 & 0 & 3 \\ 4 & 0 & 1 & -6 \end{bmatrix} \cdot 2 \sim \begin{bmatrix} -2 & 1 & 0 & 3 \\ \boxed{0} & 2 & 1 & 0 \end{bmatrix}$$

$\Rightarrow e_1,e_2$ - лин. нез. $\Rightarrow$ $\boxed{e_1,e_2 \text{ база за } W,\ \dim W=2}$

в) $v=(2,6,-2,6)$



$\underset{\in W}{v=w}+\underset{\in W^{\perp}}{u}$ $\quad u\in W^{\perp}=\mathcal{L}(f_1,f_2)$

$v=w+\alpha f_1+\beta f_2 \quad /\circ f_1 \quad /\circ f_2$

$v\circ f_1=\alpha f_1\circ f_1+\beta f_2\circ f_1$

$v\circ f_2=\alpha f_1\circ f_2+\beta f_2\circ f_2$

$v\circ f_1=(2,6,-2,6)\circ(4,2,-4,2)=8+12+8+12=40$

$v\circ f_2=(2,6,-2,6)\circ(5,1,-3,3)=10+6+6+18=40$

$f_1\circ f_1=(4,2,-4,2)\circ(4,2,-4,2)=16+4+16+4=40$

$f_1\circ f_2=(4,2,-4,2)\circ(5,1,-3,3)=20+2+12+6=40$

$f_2\circ f_2=(5,1,-3,3)\circ(5,1,-3,3)=25+1+9+9=44$

$$40=40\alpha+40\beta \quad /\cdot(-1)$$
$$40=40\alpha+44\beta$$
$$\overline{4\beta=0} \Rightarrow \boxed{\beta=0},\ \boxed{\alpha=1}$$

$u=\alpha f_1+\beta f_2=f_1=(4,2,-4,2)$

$d(v,W)=\underset{=\|v-w\|}{d(v,w)}=\|u\|=\|(4,2,-4,2)\|=\sqrt{40}=\boxed{2\sqrt{10}}$

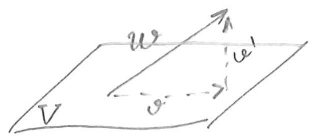
г) $\measuredangle(v,W)=\measuredangle(v,w)$

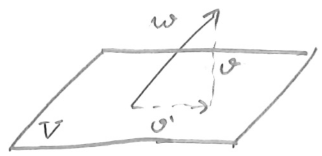
$w=v-u=(2,6,-2,6)-(4,2,-4,2)=(-2,4,2,4)$

$$\cos\measuredangle(v,w)=\frac{v\circ w}{\|v\|\|w\|}=\frac{(2,6,-2,6)\circ(-2,4,2,4)}{\sqrt{4+36+4+36}\ \sqrt{4+16+4+16}}=\frac{-4+24-4+24}{\sqrt{80}\ \sqrt{40}}=\frac{40}{40\sqrt{2}}=\frac{\sqrt{2}}{2} \Rightarrow \measuredangle(v,w)=\frac{\pi}{4}$$

$\Rightarrow \boxed{\measuredangle(v,W)=\frac{\pi}{4}}$

3. Нека је V векторски потпростор простора $\mathbb{R}^4$ генерисан векторима $e_1=(1,2,1,2)$ и $e_2=(4,3,1,2)$
а) Одредити ортогоналну пројекцију вектора $w=(1,2,-1,-2)$ на V и $V^\perp$.
б) Којем од потпростора V и $V^\perp$ је ближи вектор w?





а) $\mathbb{R}^4 = V \oplus V^\perp$ $\qquad V = \mathcal{L}(e_1, e_2)$

$w = \underset{\in V}{v} + \underset{\in V^\perp}{v'}$

$w = \alpha e_1 + \beta e_2 + v' \quad / \circ e_1$

$w\circ e_1 = \alpha e_1\circ e_1 + \beta e_2 \circ e_1$

$w\circ e_2 = \alpha e_1\circ e_2 + \beta e_2 \circ e_2$

$0 = 10\alpha + 15\beta \quad /:5$

$5 = 15\alpha + 30\beta \quad /:5$

$2\alpha + 3\beta = 0 \quad /\cdot(-3)$

$3\alpha + 6\beta = 1 \quad /\cdot 2$

$3\beta = 2 \Rightarrow \boxed{\beta = \frac{2}{3}}$, $\boxed{\alpha = -1}$

$e_1 = (1,2,1,2)$

$e_2 = (4,3,1,2)$

$w = (1,2,-1,-2)$

$w\circ e_1 = 1+4-1-4 = 0$

$w\circ e_2 = 4+6-1-4 = 5$

$e_1\circ e_1 = 1+4+1+4 = 10$

$e_1\circ e_2 = 4+6+1+4 = 15$

$e_2\circ e_2 = 16+9+1+4 = 30$

$\Rightarrow$ Ортогонална пројекција w на V је $v = \alpha e_1 + \beta e_2 = -(1,2,1,2) + \frac{2}{3}(4,3,1,2) = \boxed{\left(\frac{5}{3}, 0, -\frac{1}{3}, -\frac{2}{3}\right)}$

Ортогонална пројекција вектора w на $V^\perp$ је $v' = w - v = (1,2,-1,-2) - \left(\frac{5}{3}, 0, -\frac{1}{3}, -\frac{2}{3}\right) = \boxed{\left(-\frac{2}{3}, 2, -\frac{2}{3}, -\frac{4}{3}\right)}$

б) $d(w,V) = d(w,v) = \|w-v\| = \|v'\| = \sqrt{\frac{4}{9} + 4 + \frac{4}{9} + \frac{16}{9}} = \sqrt{\frac{24+36}{9}} = \sqrt{\frac{60}{9}} = \frac{2\sqrt{15}}{3}$

$d(w,V^\perp) = d(w,v') = \|w-v'\| = \|v\| = \sqrt{\frac{25}{9} + 0 + \frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{30}{9}} = \frac{\sqrt{30}}{3}$

$d(w,V) = \frac{\sqrt{60}}{3} > \frac{\sqrt{30}}{3} = d(w,V^\perp)$

w је ближи потпростору $V^\perp$.

4. Дат је векторски потпростор $W = \{M \in M_2(\mathbb{R}) \mid M = M^T\}$
а) Наћи базу и dim за W
б) Наћи базу и dim за $W^\perp$ ако је скаларни производ задат са $A\circ B = tr(AB^T)$
в) Одредити угао који матрица $A = \begin{bmatrix}2&0\\2&0\end{bmatrix}$ заклапа са простором W у односу на скаларни производ $A\circ B = Tr(AB^T)$.

а) $W = \{M\in M_2(\mathbb{R}) \mid M = M^T\} = \left\{\begin{bmatrix}a&b\\c&d\end{bmatrix} \in M_2(\mathbb{R}) \;\middle|\; \begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}a&b\\c&d\end{bmatrix}^T\right\} = \left\{\begin{bmatrix}a&b\\c&d\end{bmatrix}\in M_2(\mathbb{R}) \;\middle|\; \begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}a&c\\b&d\end{bmatrix}\right\}$

$\Leftrightarrow a=a$, $b=c$, $d=d$

$= \left\{\begin{bmatrix}a&b\\b&d\end{bmatrix} \;\middle|\; a,b,d\in\mathbb{R}\right\} = \left\{a\underbrace{\begin{bmatrix}1&0\\0&0\end{bmatrix}}_{E_1} + b\underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{E_2} + d\underbrace{\begin{bmatrix}0&0\\0&1\end{bmatrix}}_{E_3} \;\middle|\; a,b,d\in\mathbb{R}\right\} = \mathcal{L}(E_1,E_2,E_3)$

$\alpha E_1 + \beta E_2 + \gamma E_3 = O_M$

$\Rightarrow \begin{bmatrix}\alpha&0\\0&0\end{bmatrix} + \begin{bmatrix}0&\beta\\\beta&0\end{bmatrix} + \begin{bmatrix}0&0\\0&\gamma\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}\alpha&\beta\\\beta&\gamma\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} \Rightarrow \alpha=\beta=\gamma=0$

$\Rightarrow E_1, E_2, E_3$ - лин. нез.

$\Rightarrow E_1, E_2, E_3$ су база за W.

$\Rightarrow \dim W = 3$

б) $W \oplus W^{\perp} = V = M_2(\mathbb{R})$

ГРАСМАН:

$$\underbrace{\dim W}_{=3} + \dim W^{\perp} = \dim M_2(\mathbb{R}) = 4 \quad \Rightarrow \boxed{\dim W^{\perp} = 1}$$

$$W^{\perp} = \{ M \in M_2(\mathbb{R}) \mid M \perp \underset{\mathcal{L}(E_1, E_2, E_3)}{W} \} = \{ M \in M_2(\mathbb{R}) \mid M \circ E_1 = 0,\ M \circ E_2 = 0,\ M \circ E_3 = 0 \}$$

$$= \left\{ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \;\middle|\; M \circ E_1 = 0,\ M \circ E_2 = 0,\ M \circ E_3 = 0 \right\} = (*)$$

$$M \circ E_1 = Tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T \right) = Tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = Tr \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a$$

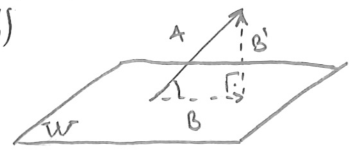
$$M \circ E_2 = Tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T \right) = Tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = Tr \begin{bmatrix} b & a \\ d & c \end{bmatrix} = b + c$$

$$M \circ E_3 = Tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T \right) = Tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = Tr \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = d$$

$$(*) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \;\middle|\; a = 0,\ b + c = 0,\ d = 0 \right\} =$$

$$= \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \;\middle|\; b \in \mathbb{R} \right\} = \left\{ b \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{E_4} \;\middle|\; b \in \mathbb{R} \right\} = \mathcal{L}(E_4) \quad \Rightarrow \dim W^{\perp} = 1$$

в)



$$\measuredangle(A, W) = \measuredangle(A, B)$$

$$A = \underset{\in W = \mathcal{L}(E_1, E_2, E_3)}{B} + \underset{\in W^{\perp} = \mathcal{L}(E_4)}{B'}$$

$$A = \underbrace{\alpha E_1 + \beta E_2 + \gamma E_3}_{=B} + \underbrace{\delta E_4}_{=B'}$$

$$A = B + \delta E_4 \quad / \circ E_4$$

$$A \circ E_4 = \delta E_4 \circ E_4$$

$$-2 = \delta \cdot 2 \quad \Rightarrow \delta = -1$$

$$A \circ E_4 = Tr\left(\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T \right) = Tr\left(\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = Tr \begin{bmatrix} 0 & -2 \\ 0 & -2 \end{bmatrix} = -2$$

$$E_4 \circ E_4 = Tr\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T \right) = Tr\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = Tr \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

$$\Rightarrow A = B - E_4$$

$$B = A + E_4 = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$ ← ОРТОГОНАЛНА ПРОЈЕКЦИЈА A на W

$$\cos\measuredangle(A, B) = \frac{A \circ B}{\|A\| \cdot \|B\|}$$

$$A \circ B = Tr(A \cdot B^T) = Tr\left(\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \right) = Tr \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} = 6$$

$$A \circ A = Tr(A \cdot A^T) = Tr\left(\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \right) = Tr \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = 8$$

$$B \circ B = Tr(B \cdot B^T) = Tr\left(\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \right) = Tr \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = 6$$

$$\Rightarrow \cos\measuredangle(A, B) = \frac{6}{\sqrt{8}\sqrt{6}} = \sqrt{\frac{6}{8}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \quad \Rightarrow \boxed{\measuredangle(A, B) = \frac{\pi}{6}}$$

5. Ако је на простору $V=\mathbb{R}^4[x]$ скаларни производ дефинисан са

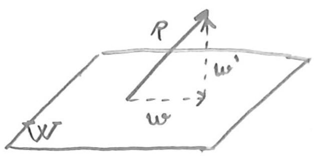
$$p\circ q = p(0)\cdot q(0) + p'(0)\cdot q'(0) + p''(0)\cdot q''(0) + p'''(0)\cdot q'''(0),$$ и

$W=\{p\in V \mid p(1)+p(-1)=0\}$, одредити угао који полином $p(x)=-x^2-x+2$ заклапа са простором W.

$$W=\{p(x)=\alpha x^3+\beta x^2+\gamma x+\delta \mid p(1)+p(-1)=0\} = \{p(x)=\alpha x^3+\beta x^2+\gamma x+\delta \mid \delta=-\beta,\ \alpha,\beta,\gamma,\delta\in\mathbb{R}\}$$

$$p(1)=\alpha+\beta+\gamma+\delta$$
$$+\ p(-1)=-\alpha+\beta-\gamma+\delta$$
$$p(1)+p(-1)=2\beta+2\delta=0$$
$$\beta+\delta=0 \Rightarrow \delta=-\beta$$

$$= \{p(x)=\alpha x^3+\beta x^2+\gamma x-\beta \mid \alpha,\beta,\gamma\in\mathbb{R}\} = \{p(x)=\alpha x^3+\beta(x^2-1)+\gamma x \mid \alpha,\beta,\gamma\in\mathbb{R}\} = \mathcal{L}(\underbrace{x^3}_{p_1},\underbrace{x^2-1}_{p_2},\underbrace{x}_{p_3}) = \mathcal{L}(p_1,p_2,p_3)$$



$$p = \underset{\in W}{w} + \underset{\in W^{\perp}}{w'}$$

$$p=\underbrace{\alpha p_1+\beta p_2+\gamma p_3}_{=w}+w' \quad /\circ p_1 \quad /\circ p_2 \quad /\circ p_3$$

$$\left.\begin{aligned} p\circ p_1 &= \alpha\, p_1\circ p_1 + \beta\, p_2\circ p_1 + \gamma\, p_3\circ p_1 \\ p\circ p_2 &= \alpha\, p_1\circ p_2 + \beta\, p_2\circ p_2 + \gamma\, p_3\circ p_2 \\ p\circ p_3 &= \alpha\, p_1\circ p_3 + \beta\, p_2\circ p_3 + \gamma\, p_3\circ p_3 \end{aligned}\right\}$$ систем по α,β,γ

$p\circ p_1 = (-x^2-x+2)\circ(x^3) = 2\cdot 0+(-1)\cdot 0+(-2)\cdot 0+0\cdot 6=0$

$p'=-2x-1 \qquad p_1'=3x^2$

$p''=-2 \qquad p_1''=6x$

$p'''=0 \qquad p_1'''=6$

$p\circ p_2 = (-x^2-x+2)\circ(x^2-1) = 2\cdot(-1)+(-1)\cdot 0+(-2)\cdot 2+0\cdot 0 = -2-4=-6$

$p'=-2x-1 \qquad p_2'=2x$

$p''=-2 \qquad p_2''=2$

$p'''=0 \qquad p_2'''=0$

$p\circ p_3 = (-x^2-x+2)\circ(x) = 2\cdot 0+(-1)\cdot 1+(-2)\cdot 0+0\cdot 0=-1$

$p'=-2x-1 \qquad p_3'=1$

$p''=-2 \qquad p_3''=0$

$p'''=0 \qquad p_3'''=0$

$p_1\circ p_1 = (x^3)\circ(x^3) = 0\cdot 0+0\cdot 0+0\cdot 0+6\cdot 6=36$

$p_1'=3x^2$

$p_1''=6x$

$p_1'''=6$

$p_1\circ p_2 = (x^3)\circ(x^2-1) = 0\cdot(-1)+0\cdot 0+0\cdot 2+6\cdot 0=0$

$p_2'=2x$

$p_2''=2$

$p_2'''=0$

$p_1\circ p_3 = (x^3)\circ(x) = 0\cdot 0+0\cdot 1+0\cdot 0+6\cdot 0=0$

$p_1'=3x^2 \qquad p_2'=1$

$p_1''=6x \qquad p_2''=0$

$p_1'''=6 \qquad p_3'''=0$

$p_2\circ p_2 = (x^2-1)\circ(x^2-1) = (-1)(-1)+0\cdot 0+2\cdot 2+0\cdot 0=5$

$p_2'=2x$

$p_2''=2$

$p_2'''=0$

$p_2\circ p_3 = (x^2-1)\circ(x) = (-1)\cdot 0+0\cdot 1+2\cdot 0+0\cdot 0=0$

$p_2'=2x \qquad p_3'=1$

$p_2''=2 \qquad p_3''=0$

$p_2'''=0 \qquad p_3'''=0$

$p_3\circ p_3 = (x)\circ(x) = 0\cdot 0+1\cdot 1+0\cdot 0+0\cdot 0=1$

$$\left.\begin{aligned} 0 &= 36\alpha+0+0 \\ -6 &= 0+5\beta+0 \\ -1 &= 0\quad 0+\gamma \end{aligned}\right| \Rightarrow \begin{aligned} \alpha&=0 \\ \beta&=-\tfrac{6}{5} \\ \gamma&=-1 \end{aligned}$$

$$w=-\frac{6}{5}(x^2-1)-x=-\frac{6}{5}x^2-x+\frac{6}{5}$$

$$\cos\sphericalangle(p,W)=\cos\sphericalangle(p,w)=\frac{p\circ w}{\|p\|\cdot\|w\|}=\frac{\frac{41}{5}}{3\cdot\sqrt{\frac{41}{5}}}=\frac{1}{3}\sqrt{\frac{41}{5}}=\boxed{\frac{\sqrt{205}}{15}}$$

$p\circ p = (-x^2-x+2)\circ(-x^2-x+2) = 2\cdot 2+(-1)(-1)+(-2)(-2)+0\cdot 0 = 9$

$p'=-2x-1$

$p''=-2 \qquad \Rightarrow \boxed{\|p\|=3}$

$p'''=0$

$w\circ w = \left(-\frac{6}{5}x^2-x+\frac{6}{5}\right)\circ\left(-\frac{6}{5}x^2-x+\frac{6}{5}\right) = \frac{6}{5}\cdot\frac{6}{5}+(-1)(-1)+\left(-\frac{12}{5}\right)\left(-\frac{12}{5}\right)+0\cdot 0 = \frac{36+25+144}{25} = \frac{205}{25}=\frac{41}{5}$

$w'=-\frac{12}{5}x-1$

$w''=-\frac{12}{5}$

$w'''=0$

$p\circ w = (-x^2-x+2)\circ\left(-\frac{6}{5}x^2-x+\frac{6}{5}\right) = 2\cdot\frac{6}{5}+(-1)(-1)+(-2)\left(-\frac{12}{5}\right)+0\cdot 0 = \frac{12}{5}+1+\frac{24}{5}=\frac{41}{5}$

$$\Rightarrow \boxed{\sphericalangle(p,W)=\arccos\frac{\sqrt{205}}{15}}$$

⑥ Нека је V векторски простор квадратних матрица реда 2 са скаларним производом $A\circ B = Tr\left(A^T\cdot\begin{bmatrix}1&0\\0&4\end{bmatrix}\cdot B\right)$ и нека је дат потпростор $W=\{A\in M_2(\mathbb{R}) \mid Tr A=0\}$ простора V.

а) Одредити ортогоналну пројекцију матрице $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ на W и $W^{\perp}$.

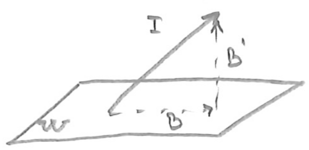
б) Ком простору је вектор I ближи?

$$W=\left\{A=\begin{bmatrix}a&b\\c&d\end{bmatrix} \;\middle|\; Tr A=0\right\}=\left\{\begin{bmatrix}a&b\\c&-a\end{bmatrix} \;\middle|\; a,b,c\in\mathbb{R}\right\}=\left\{a\underbrace{\begin{bmatrix}1&0\\0&-1\end{bmatrix}}_{E_1}+b\underbrace{\begin{bmatrix}0&1\\0&0\end{bmatrix}}_{E_2}+c\underbrace{\begin{bmatrix}0&0\\1&0\end{bmatrix}}_{E_3} \;\middle|\; a,b,c\in\mathbb{R}\right\}$$

$a+d=0$

$d=-a$

$$=\mathcal{L}(E_1,E_2,E_3)$$



$I=\underset{\in W}{B}+\underset{\in W^{\perp}}{B'}$

B - ортогонална пројекција матрице I на W

B' - ортогонална пројекција матрице I на $W^{\perp}$

$$I=\underbrace{aE_1+bE_2+cE_3}_{=B}+B' \qquad /\circ E_1$$

$$\left.\begin{aligned}I\circ E_1&=aE_1\circ E_1+bE_2\circ E_1+cE_3\circ E_1\\ I\circ E_2&=aE_1\circ E_2+bE_2\circ E_2+cE_3\circ E_2\\ I\circ E_3&=aE_1\circ E_3+bE_2\circ E_3+cE_3\circ E_3\end{aligned}\right\}$$ систем линеарних ј-на по a,b,c

$I\circ E_1=Tr\left(I^T\begin{bmatrix}1&0\\0&4\end{bmatrix}E_1\right)=Tr\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right)=Tr\left(\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right)=Tr\begin{bmatrix}1&0\\0&-4\end{bmatrix}=-3$

$I\circ E_2= \quad\quad =Tr\left(\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=Tr\begin{bmatrix}0&\\&0\end{bmatrix}=0$

$I\circ E_3= \quad\quad =Tr\left(\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=Tr\begin{bmatrix}0&\\&0\end{bmatrix}=0$

$E_1\circ E_1=Tr\left(E_1^T\begin{bmatrix}1&0\\0&4\end{bmatrix}E_1\right)=Tr\left(\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right)=Tr\left(\begin{bmatrix}1&0\\0&-4\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right)=Tr\begin{bmatrix}1&\\&4\end{bmatrix}=5$

$E_1\circ E_2= \quad\quad =Tr\left(\begin{bmatrix}1&0\\0&-4\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=Tr\begin{bmatrix}0&1\\0&0\end{bmatrix}=0$

$E_1\circ E_3= \quad\quad =Tr\left(\begin{bmatrix}1&0\\0&-4\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=Tr\begin{bmatrix}0&0\\-4&0\end{bmatrix}=0$

$E_2\circ E_2=Tr\left(E_2^T\begin{bmatrix}1&0\\0&4\end{bmatrix}E_2\right)=Tr\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=Tr\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=Tr\begin{bmatrix}0&0\\0&1\end{bmatrix}=1$

$E_2\circ E_3= \quad\quad =Tr\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=Tr\begin{bmatrix}0&0\\0&0\end{bmatrix}=0$

$E_3\circ E_3=Tr\left(E_3^T\begin{bmatrix}1&0\\0&4\end{bmatrix}E_3\right)=Tr\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=Tr\left(\begin{bmatrix}0&4\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=Tr\begin{bmatrix}4&0\\0&0\end{bmatrix}=4$

$$\left.\begin{aligned}-3&=5a+0+0\\0&=0+b+0\\0&=0+0+4c\end{aligned}\right\}\Rightarrow \begin{aligned}&b=c=0\\&a=-\tfrac{3}{5}\end{aligned}$$

$$\Rightarrow I=-\frac{3}{5}E_1+B'$$

ортогонална пројекција I на W је $B=\frac{3}{5}E_1=-\frac{3}{5}\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}-\frac{3}{5}&0\\0&\frac{3}{5}\end{bmatrix}$

ортогонална пројекција I на $W^{\perp}$ је $B'=I-\frac{3}{5}E_1=\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}-3/5&0\\0&3/5\end{bmatrix}=\begin{bmatrix}8/5&0\\0&2/5\end{bmatrix}$

$d(I,W)=d(I,B)=\|I-B\|=\|B'\|=\sqrt{B'\circ B'}=\frac{8\sqrt{2}}{5}$

$B'\circ B'=Tr\left(B'^T\begin{bmatrix}1&0\\0&4\end{bmatrix}B'\right)=Tr\left(\begin{bmatrix}8/5&0\\0&2/5\end{bmatrix}\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}8/5&0\\0&2/5\end{bmatrix}\right)=Tr\left(\begin{bmatrix}8/5&0\\0&8/5\end{bmatrix}\begin{bmatrix}8/5&0\\0&8/5\end{bmatrix}\right)=Tr\begin{bmatrix}64/25&0\\0&64/25\end{bmatrix}=\frac{128}{25}$

$d(I,W^{\perp})=d(I,B')=\|I-B'\|=\|B\|=\sqrt{B\circ B}=\frac{3\sqrt{5}}{5}$

$B\circ B=Tr\left(B^T\begin{bmatrix}1&0\\0&4\end{bmatrix}B\right)=Tr\left(\begin{bmatrix}-3/5&0\\0&3/5\end{bmatrix}\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}-3/5&0\\0&3/5\end{bmatrix}\right)=Tr\left(\begin{bmatrix}-3/5&0\\0&12/5\end{bmatrix}\begin{bmatrix}-3/5&0\\0&3/5\end{bmatrix}\right)=$

$=Tr\begin{bmatrix}9/25&0\\0&36/25\end{bmatrix}=\frac{45}{25}=\frac{9}{5}$

$\underline{d(I,W)=\sqrt{\frac{128}{25}}}\ \underline{>}\ \underline{\sqrt{\frac{45}{25}}=d(I,W^{\perp})} \Rightarrow I$ је ближи простору $W^{\perp}$

7. Нека је $V = M_2(\mathbb{R})$ векторски простор матрица над пољем $\mathbb{R}$ са скаларним производом $A \circ B = Tr(B^T \circ A)$. Ако је $W = \{A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R}\} \leq V$ векторски простор дијагоналних матрица наћи ортогоналну базу за $W^\perp$.

$$W = \{A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R}\} = \{a \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1} + b \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_2} \mid a, b \in \mathbb{R}\} = \mathcal{L}(E_1, E_2)$$

$\Rightarrow \{E_1, E_2\}$ база за W

$\dim W = 2$

$$a E_1 + b E_2 = 0_V$$
$$\Rightarrow \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow E_1 \text{ и } E_2 \text{ су лин. нез.}$$
$$\Rightarrow a = b = 0$$

$$W \oplus W^\perp = V = M_2(\mathbb{R})$$

$$\dim W + \dim W^\perp = \dim(W \oplus W^\perp) - \underbrace{\dim(\underbrace{W \cap W^\perp}_{=\{0\}})}_{=0} \quad \Leftarrow \text{ГРАСМАН}$$

$$\dim W + \dim W^\perp = \dim V$$

$$\underset{2}{\dim W} + \dim W^\perp = 4 \Rightarrow \boxed{\dim W^\perp = 2}$$

$$W^\perp = \{B \in V \mid B \perp \underset{\mathcal{L}(E_1, E_2)}{W}\} = \{B \in V \mid (\forall A \in W)\ A \circ B = 0\} = \{B \in V \mid (aE_1 + bE_2) \circ B = 0;\ a, b \in \mathbb{R}\}$$

$$= \{B \in V \mid E_1 \circ B = 0,\ E_2 \circ B = 0\} = \{B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid E_1 \circ B = 0,\ E_2 \circ B = 0\} =$$

$$E_1 \circ B = Tr(B^T E_1) = Tr\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = Tr\begin{bmatrix} a & \\ & 0 \end{bmatrix} = a$$

$$E_2 \circ B = Tr(B^T E_2) = Tr\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = Tr\begin{bmatrix} 0 & \\ & d \end{bmatrix} = d$$

$$= \{B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a = 0,\ d = 0\}$$
$$= \{\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \mid b, c \in \mathbb{R}\}$$
$$= \{b \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_3} + c \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_4} \mid b, c \in \mathbb{R}\} =$$
$$= \mathcal{L}(E_3, E_4) \Rightarrow \{E_3, E_4\} \text{ база за } W^\perp$$

E_3, E_4 - лин. нез.

Да ли су ортогонални?

$$E_3 \circ E_4 = Tr(E_4^T E_3) = Tr\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = Tr\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \Rightarrow E_3 \perp E_4$$

$\{E_3, E_4\}$ ортогонална база за $W^\perp$.