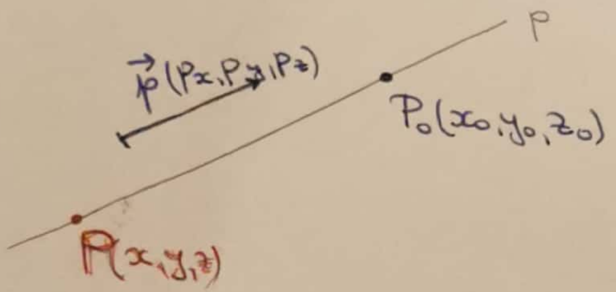
ПРАВА И РАВАН У ПРОСТОРУ

ПРАВА У ПРОСТОРУ



$P_0(x_0, y_0, z_0)$ – ТАЧКА СА ПРАВЕ p

$\vec{p}(p_x, p_y, p_z)$ – ВЕКТОР ПРАВЦА ПРАВЕ p

$P(x, y, z)$ – ПРОИЗВОЉНА ТАЧКА ПРАВЕ p

* ПАРАМЕТАРСКА Ј-НА ПРАВЕ У $\mathbb{R}^3$

$$\boxed{\begin{aligned} x &= p_x t + x_0 \\ y &= p_y t + y_0 \\ z &= p_z t + z_0 \end{aligned} \quad , t \in \mathbb{R}}$$

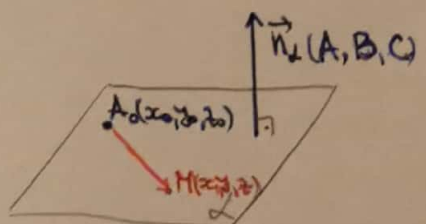
$$\uparrow \vec{p} \qquad \uparrow P_0$$

* КАНОНСКА Ј-НА ПРАВЕ

$$\boxed{\frac{x - x_0}{p_x} = \frac{y - y_0}{p_y} = \frac{z - z_0}{p_z} \left(= t\right)}$$

РАВАН У ПРОСТОРУ

ИМПЛИЦИТНА Ј-НА РАВНИ



$A_0(x_0, y_0, z_0)$ – ТАЧКА ИЗ РАВНИ α

$\vec{n}_\alpha(A, B, C)$ – ВЕКТОР НОРМАЛЕ НА РАВАН α

$M(x, y, z)$ – ПРОИЗВОЉНА ТАЧКА РАВНИ α

$\vec{n}_\alpha \perp \alpha \Rightarrow \vec{n}_\alpha \perp \overrightarrow{A_0M} \Leftrightarrow \vec{n}_\alpha \circ \overrightarrow{A_0M} = 0$

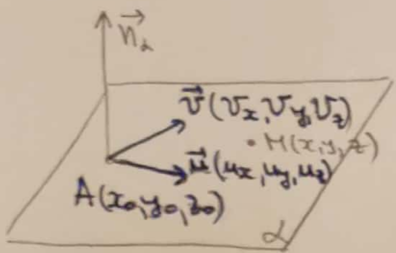
$$\boxed{\alpha: Ax + By + Cz + D = 0}$$

$0 = \vec{n}_\alpha \circ \overrightarrow{A_0M} = (A, B, C) \circ (x - x_0, y - y_0, z - z_0) =$
$= A(x - x_0) + B(y - y_0) + C(z - z_0)$

$$\Rightarrow \boxed{\alpha: A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$$

Scanned with CamScanner

ПАРАМЕТАРСКА Ј-НА РАВНИ



$A(x_0,y_0,z_0)$ – ТАЧКА ИЗ РАВНИ α

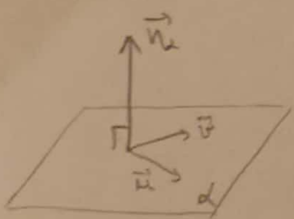
$\left.\begin{array}{l}\vec{u}(u_x,u_y,u_z)\\ \vec{v}(v_x,v_y,v_z)\end{array}\right\}$ ВЕКТОРИ ПАРАЛЕЛНИ РАВНИ α

$M(x,y,z)$ – ПРОИЗВОЉНА ТАЧКА РАВНИ α

$$\boxed{\begin{array}{l} x = \underbrace{x_0}_{A} + \underbrace{u_x}_{\vec{u}}\, t + \underbrace{v_x}_{\vec{v}}\, s \\ y = y_0 + u_y\, t + v_y\, s \\ z = z_0 + u_z\, t + v_z\, s \end{array} \quad , \ t, s \in \mathbb{R}}$$

(10) Равни $x-y+3z-2=0$ одредити параметарску једначину.

$\alpha: x-y+3z-2=0 \Rightarrow \underline{\vec{n}_\alpha(1,-1,3)}$



Потребно је одредити координате вектора $\vec{u}$ и $\vec{v}$ који су паралелни равни α.

$\left.\begin{array}{l}\vec{u},\vec{v} \parallel \alpha \\ \vec{n}_\alpha \perp \alpha\end{array}\right\} \Rightarrow \vec{u},\vec{v} \perp \vec{n}_\alpha \Leftrightarrow \vec{u}\circ\vec{n}_\alpha = 0,\ \vec{v}\circ\vec{n}_\alpha = 0$

$\Leftrightarrow \underbrace{(x,y,z)}_{\text{координате вектора паралелних равни } \alpha} \circ \underbrace{(1,-1,3)}_{\vec{n}_\alpha} = 0$

$\Leftrightarrow \underline{x-y+3z=0} \quad (*)$

Потребно је да одредимо векторе $\vec{u}$ и $\vec{v}$:

нпр. • $z=0 \leadsto (*) \Rightarrow x-y=0 \Rightarrow x=y \Rightarrow \boxed{\vec{u}(1,1,0)}$

• $x=0 \leadsto (*) \Rightarrow -y+3z=0 \Rightarrow y=3z \Rightarrow \boxed{\vec{v}(0,3,1)}$

провера: $\vec{u}\circ\vec{n}_\alpha = (1,1,0)\circ(1,-1,3) = 1-1=0 \Rightarrow \vec{u}\perp\vec{n}_\alpha$

$\vec{v}\circ\vec{n}_\alpha = (0,3,1)\circ(1,-1,3) = -3+3=0 \Rightarrow \vec{v}\perp\vec{n}_\alpha$

Тачка са равни α

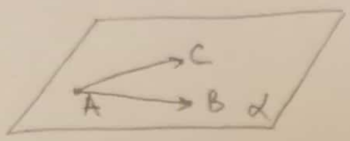
нпр. $\underline{y=z=0} \leadsto \alpha \Rightarrow x-2=0 \Rightarrow \underline{x=2} \Rightarrow \boxed{A(2,0,0)\in\alpha}$

$$\Downarrow$$

$$\boxed{\alpha: \begin{array}{l} x = 2+t \\ y = t+3s \\ z = s \end{array} \quad t,s\in\mathbb{R}}$$

(11) Одредити ј-ну равни која садржи тачке $A(1,2,3)$, $B(-1,2,-1)$ и $C(0,0,1)$.

ПАРАМЕТАРСКА:



$\vec{u} \sim \vec{AB} = (-2,0,-4) \Rightarrow \vec{u}(1,0,2)$

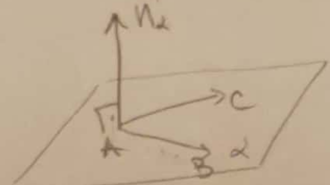
$\vec{v} \sim \vec{AC} = (-1,-2,-2) \Rightarrow \vec{v}(1,2,2)$

$C(0,0,1) \in \alpha$

$$\alpha: \begin{array}{l} x = + 1\cdot t + 1 s \\ y =   + 2 s \\ z = 1 + 2t + 2s \end{array}, \quad t, s \in \mathbb{R}$$

$\uparrow C \quad \uparrow \vec{u} \quad \uparrow \vec{v}$

ИМПЛИЦИТНА



$$\vec{n}_\alpha \sim \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & -4 \\ -1 & -2 & -2 \end{vmatrix} = (-8,0,4) \Rightarrow \underline{\underline{\vec{n}_\alpha(-2,0,1)}}$$

$\underline{C(0,0,1) \in \alpha}$

$\alpha: -2(x-0) + 0(y-0) + 1\cdot(z-1) = 0$ тј.

$\boxed{\alpha: -2x + z - 1 = 0}$

ПРАВА КАО ПРЕСЕК ДВЕ РАВНИ

$\alpha: Ax + By + Cz + D = 0$

$\beta: A_1x + B_1y + C_1z + D_1 = 0$ (*)

- Ако су $\vec{n}_\alpha, \vec{n}_\beta$ колинеарни вектори (тј. $\|\vec{n}_\alpha \times \vec{n}_\beta\| = 0$) равни су ПАРАЛЕЛНЕ или СЕ ПОКЛАПАЈУ.
- Ако су $\vec{n}_\alpha, \vec{n}_\beta$ линеарно независни, $\alpha \cap \beta = \{p\}$, p – ЈЕДИНСТВЕНА ПРЕСЕЧНА ПРАВА

$$\left.\begin{array}{l} p \in \alpha \Rightarrow \vec{p} \perp \vec{n}_\alpha \\ p \in \beta \Rightarrow \vec{p} \perp \vec{n}_\beta \end{array}\right\} \Rightarrow \vec{p} \sim \vec{n}_\alpha \times \vec{n}_\beta$$

Тачка $P \in p$ је једно решење система (*)

Ако је права p дата каноничком ј-ном $p: \frac{x-x_0}{p_x} = \frac{y-y_0}{p_y} = \frac{z-z_0}{p_z} = t$, ($\alpha$:, β:) две равни које се секу по правој p су нпр: $\alpha: \frac{x-x_0}{p_x} = \frac{y-y_0}{p_y}$, $\beta: \frac{y-y_0}{p_y} = \frac{z-z_0}{p_z}$

13) Праву $p: x=t+4, y=-2t+1, z=3t-2, t\in\mathbb{R}$ записати као пресек две равни.

$$p: \begin{cases} x=t+4 & \Rightarrow t=x-4 \\ y=-2t+1 & \Rightarrow t=\frac{-y+1}{2}=\frac{y-1}{-2} \\ z=3t-2 & \Rightarrow t=\frac{z+2}{3} \end{cases} \Rightarrow \boxed{p: \frac{x-4}{1}=\frac{y-1}{-2}=\frac{z+2}{3}\,(=t)}$$

β

α

$\alpha: \frac{x-4}{1}=\frac{y-1}{-2}$

$-2(x-4)=y-1$

$-2x-y+9=0$ /·(-1)

$\boxed{\alpha: 2x+y-9=0}$

$\beta: \frac{y-1}{-2}=\frac{z+2}{3}$

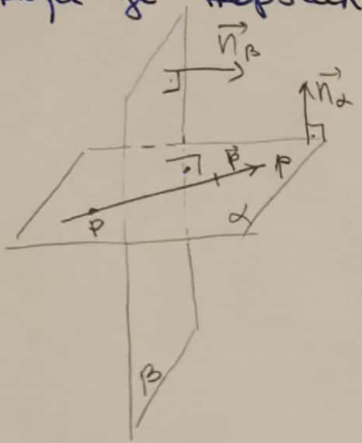
$3(y-1)=-2(z+2)$ тј.

$\boxed{\beta: 3y+2z+1=0}$

14) Одредити једну раван α која садржи праву $p: \frac{x-2}{1}=\frac{y-11}{5}=\frac{z-2}{1}$ и која је нормална на раван $\beta: y=0$.

P

$\vec{p}$



$$\left.\begin{array}{l} p\in\alpha: \underline{P(2,11,2)\in\alpha} \\ p\in\alpha: \vec{p}(1,5,1) \quad \vec{n}_\alpha \perp \vec{p} \\ \beta\perp\alpha \Rightarrow \vec{n}_\beta \perp \vec{n}_\alpha, \ \vec{n}_\beta(0,1,0) \end{array}\right\} \Rightarrow \vec{n}_\alpha \sim \vec{p}\times\vec{n}_\beta$$

$$\vec{p}\times\vec{n}_\beta = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 5 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1\cdot\vec{e}_1 - 0\vec{e}_2 + 1\,\vec{e}_3 = \underline{(-1,0,1)}$$

$$\Rightarrow \vec{n}_\alpha \sim \vec{p}\times\vec{n}_\beta \Rightarrow \underline{\vec{n}_\alpha=(-1,0,1)}$$

$\alpha: -1\cdot(x-2)+0(y-11)+1\cdot(z-2)=0$

$-x+2+z-2=0$

$\boxed{\alpha: -x+z=0}$

15) Праву $p: x-y-1=0, z-2x=0$ записати параметарски.

$\alpha\cap\beta=\{p\}$, $\alpha: x-y-1=0$, $\beta: z-2x=0$

$\vec{n}_\alpha(1,-1,0)$ $\quad$ $\vec{n}_\beta(-2,0,1)$

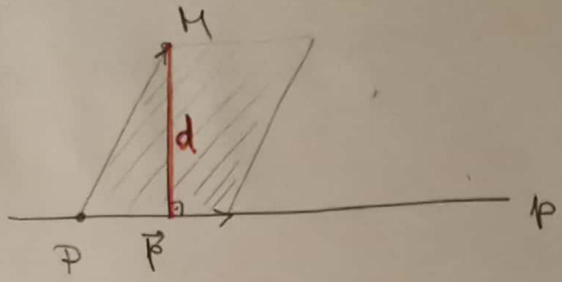
$$\left.\begin{array}{l} p\in\alpha \Rightarrow \vec{p}\perp\vec{n}_\alpha \\ p\in\beta \Rightarrow \vec{p}\perp\vec{n}_\beta \end{array}\right\} \Rightarrow \vec{p}\sim\vec{n}_\alpha\times\vec{n}_\beta = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = (-1,-1,-2) \Rightarrow \underline{\underline{\vec{p}=(1,1,2)}}$$

$P\in p$ $\Rightarrow$ нпр. $x=0 \Rightarrow y=-1 \Rightarrow z=0$ $\Rightarrow \underline{P(0,-1,0)\in p}$

$$p: \begin{array}{l} x=1\cdot t+0 \\ y=1\cdot t-1 \\ z=2t+0 \end{array}, t\in\mathbb{R} \quad \text{тј.} \quad \boxed{p: \begin{array}{l} x=t \\ y=t-1 \\ z=2t \end{array}, t\in\mathbb{R}}$$

РАСТОЈАЊЕ ТАЧКЕ ОД ПРАВЕ

$$\boxed{(d=)\,d(M,p)=\frac{\|\overrightarrow{PM}\times\vec{p}\|}{\|\vec{p}\|}}$$



РАСТОЈАЊЕ ТАЧКЕ ОД РАВНИ

$\alpha: Ax+By+Cz+D=0$

$M(x_0,y_0,z_0)$

$$\boxed{(d=)\,d(M,\alpha)=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}}$$

(16) Одредити растојање тачке $M(1,0,12)$ од:

а) праве $p: x-y-1=0,\ z-2x=0$

права из зад. (15)

p: $\vec{p}(1,1,2)$, $P(0,-1,0)$ $\qquad \overrightarrow{PM}(1,1,12)$

$$d(M,p)=\frac{\|\overrightarrow{PM}\times\vec{p}\|}{\|\vec{p}\|} \quad (*)$$

$$\overrightarrow{PM}\times\vec{p}=\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 1 & 1 & 12\\ 1 & 1 & 2\end{vmatrix}=(2-12)\vec{e}_1-(2-12)\vec{e}_2+(1-1)\vec{e}_3=\underline{(-10,10,0)}$$

$$\left.\begin{aligned}\|\overrightarrow{PM}\times\vec{p}\| &= \sqrt{(-10)^2+10^2+0^2}=\sqrt{2\cdot 100}=\underline{10\sqrt{2}}\\ \|\vec{p}\| &= \sqrt{1^2+1^2+2^2}=\sqrt{1+1+4}=\underline{\sqrt{6}}\end{aligned}\right\}\Rightarrow d\overset{(*)}{=}\frac{10\sqrt{2}}{\sqrt{6}}=\frac{10\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\cdot\sqrt{3}}=\frac{10}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$\Rightarrow \boxed{d=\frac{10\sqrt{3}}{3}}$$

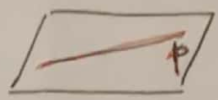
б) равни $\alpha: \underset{A}{x}\underset{B}{-y}\underset{C}{-4}z=0.$

$$d(M,\alpha)=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}} \Rightarrow$$

$$\underline{\underline{d(M,\alpha)}}=\frac{|1\cdot 1-1\cdot 0-4\cdot 12+0|}{\sqrt{1^2+(-1)^2+(-4)^2}}=\frac{|1-48|}{\sqrt{1+1+16}}=\frac{|-47|}{\sqrt{18}}=\frac{47}{\sqrt{9}\cdot\sqrt{2}}=\frac{47}{3\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\underline{\underline{\frac{47\sqrt{2}}{6}}}$$

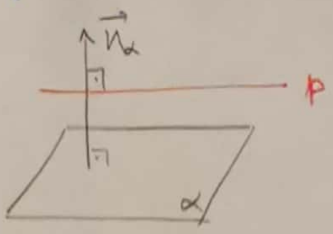
МЕЂУСОБНИ ПОЛОЖАЈ ПРАВЕ И РАВНИ

* ПРАВА ПРИПАДА РАВНИ, $p \subset \alpha$



$\alpha \cap p = \{p\}$

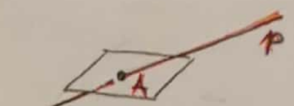
$\vec{p} \circ \vec{n}_\alpha = 0$



* ПАРАЛЕЛНЕ СУ, $p \parallel \alpha$

$\vec{p} \circ \vec{n}_\alpha = 0$

* ПРАВА И РАВАН СЕ СЕКУ



$p \cap \alpha = \{A\}$

$p: \vec{p}(p_x, p_y, p_z),\ P(x_0, y_0, z_0)$

$\vec{n}_\alpha(A, B, C)$

$$p: \frac{x-x_0}{p_x} = \frac{y-y_0}{p_y} = \frac{z-z_0}{p_z} = t \quad \text{тј.} \quad \begin{aligned} x &= p_x t + x_0 \\ y &= p_y t + y_0 \\ z &= p_z t + z_0 \end{aligned}, \ t \in \mathbb{R}$$

$$\alpha: Ax + By + Cz + D = 0$$

$p \rightsquigarrow \alpha$ тј.

$$A(p_x t + x_0) + B(p_y t + y_0) + C(p_z t + z_0) + D = 0$$

$$Ap_x t + Bp_y t + Cp_z t + Ax_0 + By_0 + Cz_0 + D = 0$$

$$(Ap_x + Bp_y + Cp_z)t = -(Ax_0 + By_0 + Cz_0 + D)$$

$$\Rightarrow \boxed{t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap_x + Bp_y + Cp_z} = -\frac{Ax_0 + By_0 + Cz_0 + D}{\vec{n}_\alpha \circ \vec{p}}} \quad (*)$$

$\circ\ \vec{n}_\alpha \circ \vec{p} \neq 0$

$$\vec{n}_\alpha \circ \vec{p} = (A, B, C) \circ (p_x, p_y, p_z) = Ap_x + Bp_y + Cp_z$$

(17) Дата је раван $\alpha: x - 2y + 5z - 1 = 0$. Одредити пресек те равни са правама:

а) $p: \dfrac{x-2}{3} = \dfrac{y-4}{7} = \dfrac{z}{1} = t$ $\qquad p: \begin{aligned} x &= 3t + 2 \\ y &= 7t + 4 \\ z &= t \end{aligned}, \ t \in \mathbb{R}$

$\left.\begin{aligned} &\vec{n}_\alpha(1, -2, 5) \\ &\vec{p}(3, 7, 1) \end{aligned}\right\} \Rightarrow \vec{p} \circ \vec{n}_\alpha = (3, 7, 1) \circ (1, -2, 5) = 3 - 14 + 5 = -6 \neq 0 \Rightarrow$

$P(2, 4, 0)$

$p \cap \alpha = \{A\} \Rightarrow p \rightsquigarrow \alpha \overset{(*)}{\Rightarrow} t = -\dfrac{Ax_0 + By_0 + Cz_0 + D}{\vec{p} \circ \vec{n}_\alpha}$ $\quad P(x_0, y_0, z_0)$

$$t = -\frac{1 \cdot 2 - 2 \cdot 4 + 5 \cdot 0 - 1}{-6} = -\frac{2 - 8 - 1}{-6} = -\frac{-7}{-6} = -\frac{7}{6} \rightsquigarrow p$$

$\{A\} = p \cap \alpha$, $\quad A: \begin{aligned} x &= 3 \cdot \left(-\tfrac{7}{6}\right) + 2 = -\tfrac{7}{2} + 2 = -\tfrac{3}{2} \\ y &= 7\left(-\tfrac{7}{6}\right) + 4 = -\tfrac{49}{6} + 4 = -\tfrac{25}{6} \\ z &= -\tfrac{7}{6} \end{aligned}$

тј. $\boxed{A\left(-\frac{3}{2}, -\frac{25}{6}, -\frac{7}{6}\right)}$

д) $g: \frac{x}{3} = \frac{y-2}{-1} = \frac{z-1}{-1}$ $\qquad \alpha: x-2y+5z-1=0$

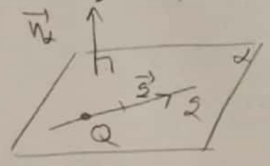
$Q(0,2,1)$

$\left.\begin{array}{l}\vec{g}(3,-1,-1) \\ \vec{n}_\alpha(1,-2,5)\end{array}\right\} \Rightarrow \underline{\vec{g}\circ\vec{n}_\alpha} = (3,-1,-1)\circ(1,-2,5) = 3+2-5 = \underline{0}$

$\vec{g}\circ\vec{n}_\alpha = 0 \iff g \parallel \alpha$

Дали $g \in \alpha$?

$Q \rightsquigarrow \alpha$: $0-2\cdot2+5\cdot1-1 = -4+5-1=0 \Rightarrow Q\in\alpha \Rightarrow \underline{\underline{g\in\alpha}}$



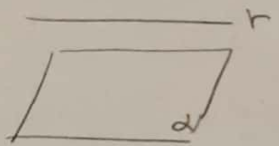
е) $r: \frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{-1}$

$\vec{r}(3,-1,-1)$, $R(1,2,3)$

$\vec{r}\circ\vec{n}_\alpha = (3,-1,-1)\circ(1,-2,5) = 3+2-5=0$

$\underline{g \overset{?}{\in} \alpha}$: $R \rightsquigarrow \alpha$ $\quad 1-2\cdot2+5\cdot3-1 = 1-4+15-1 = 11 \neq 0 \Rightarrow$

$\left.\begin{array}{l} R\notin\alpha \\ \vec{r}\perp\vec{n}_\alpha\end{array}\right\} \Rightarrow \underline{\underline{r\parallel\alpha}}$

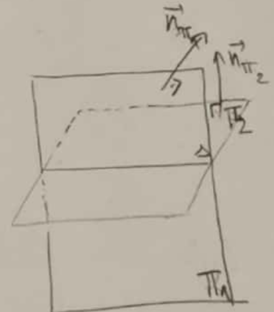


ж) $s: x-y=1,\ x+y-z+5=0$

I начин $\qquad \{s\} = \pi_1 \cap \pi_2$

$\pi_1: x-y-1=0$ $\qquad , \vec{n}_{\pi_1}(1,-1,0)$

$\pi_2: x+y-z+5=0$ $\qquad , \vec{n}_{\pi_2}(1,1,-1)$



II начин

Решаваме системата

$s: \ x-y-1=0$
$\quad\ \ x+y-z+5=0$

$\alpha: x-2y+5z-1=0$

$\left.\begin{array}{l} s\in\pi_1 \Rightarrow \vec{s}\perp\vec{n}_{\pi_1} \\ s\in\pi_2 \Rightarrow \vec{s}\perp\vec{n}_{\pi_2}\end{array}\right\} \Rightarrow \vec{s} \sim \vec{n}_{\pi_1}\times\vec{n}_{\pi_2}$

$\vec{n}_{\pi_1}\times\vec{n}_{\pi_2} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = \vec{e}_1+\vec{e}_2+2\vec{e}_3 = (1,1,2) \Rightarrow \underline{\vec{s}(1,1,2)}$

$S\in s$: нпр $y=0 \Rightarrow x=1 \Rightarrow z=6$ т.е. $\underline{\underline{S(1,0,6)}}$

$\vec{s}\circ\vec{n}_\alpha = (1,1,2)\circ(1,-2,5) = 1-2+10 = 9\neq0 \Rightarrow \underline{\underline{s\cap\alpha=\{A\}}}$

$\overset{*}{\Rightarrow}\ t = -\frac{Ax_0+By_0+Cz_0+D}{\vec{s}\circ\vec{n}_\alpha} = -\frac{1\cdot1-2\cdot0+5\cdot6-1}{9} = -\frac{1+30-1}{9} = -\frac{30}{9} = \underline{\underline{-\frac{10}{3}}}$

$t=-\frac{10}{3} \rightsquigarrow s: \begin{array}{l} x=t+1 \\ y=t \\ z=2t+6\end{array}$ $, t\in\mathbb{R}$ $\Rightarrow A: \begin{array}{l} x=-\frac{10}{3}+1=-\frac{7}{3} \\ y=-\frac{10}{3} \\ z=2\cdot\left(-\frac{10}{3}\right)+6 = -\frac{20}{3}+6 = -\frac{2}{3}\end{array}$ $\boxed{A\left(-\frac{7}{3}, -\frac{10}{3}, -\frac{2}{3}\right)}$

e) $t: x-z+2=0,\ -y+3z+2=0$ $\qquad \alpha: x-2y+5z-1=0$

$\pi_1: x-z+2=0, \quad \vec{n}_{\pi_1}(1,0,-1)$

$\pi_2: -y+3z+2=0, \quad \vec{n}_{\pi_2}(0,-1,3)$

$\{t\}=\pi_1\cap\pi_2$

$$\left.\begin{array}{l} t\in\pi_1 \Rightarrow \vec{t}\perp\vec{n}_{\pi_1} \\ t\in\pi_2 \Rightarrow \vec{t}\perp\vec{n}_{\pi_2}\end{array}\right\} \Rightarrow \vec{t}\sim\vec{n}_{\pi_1}\times\vec{n}_{\pi_2} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{vmatrix} = -\vec{e}_1-3\vec{e}_2-\vec{e}_3 = (-1,-3,-1)$$

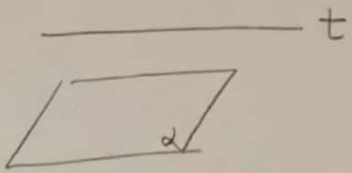
$\Rightarrow \underline{\vec{t}(1,3,1)}$

$\underline{T\in t}$: нпр $z=0 \Rightarrow x=-2 \Rightarrow y=2$ тј. $\underline{T(-2,2,0)}$

$\vec{n}_\alpha\circ\vec{t}=(1,-2,5)\circ(1,3,1)=1-6+5=0$

$\underline{T\overset{?}{\in}\alpha}$: $T \rightsquigarrow \alpha$: $-2-2\cdot2+5\cdot0-1=-2-4-1=-7\neq0 \Rightarrow T\notin\alpha$

$\Rightarrow \underline{t\parallel\alpha}$



⑱ Одредити тачку продора праве $p: \frac{x+1}{3}=\frac{y+3}{2}=\frac{z-5}{-4}=t$ кроз раван $\alpha: 3x+y+5z-7=0$.

$p: \vec{p}(3,2,-4), \quad P(-1,-3,5)$

$\vec{n}_\alpha(3,1,5)$

$\vec{n}_\alpha\circ\vec{p}=(3,1,5)\circ(3,2,-4)=9+2-20=-9\neq0 \Rightarrow p\cap\alpha=\{A\}$

$$\overset{(*)}{\Rightarrow} t=-\frac{Ax_0+By_0+Cz_0+D}{\vec{n}_\alpha\circ\vec{p}}=-\frac{3\cdot(-1)+1\cdot(-3)+5\cdot5-7}{-9}=-\frac{-3-3+25-7}{-9}=$$

$\Rightarrow t=-\frac{12}{-9}=\frac{4}{3}$

$t=\frac{4}{3} \rightsquigarrow p$: $\begin{array}{l} x=3t-1 \\ y=2t-3 \\ z=-4t+5 \end{array} \Rightarrow A: \begin{array}{l} x=3\cdot\frac{4}{3}-1=3 \\ y=2\cdot\frac{4}{3}-3=-\frac{1}{3} \\ z=-4\cdot\frac{4}{3}+5=-\frac{1}{3} \end{array}$ тј.

$$\boxed{A\left(3,-\frac{1}{3},-\frac{1}{3}\right)}$$

ЈЕДНАЧИНА РАВНИ ИЗ ИСТОГ ПРАМЕНА

$\alpha: \overbrace{Ax+By+Cz+D}^{f_1}=0$

$\beta: \underbrace{\bar{A}x+\bar{B}y+\bar{C}z+\bar{D}}_{f_2}=0$

$\gamma \in \mathcal{X}_{\alpha,\beta} \Rightarrow \boxed{\gamma: f_1+\lambda f_2=0 \quad \vee \quad \gamma: f_2=0}$

Д. $\gamma: \mu f_1+\eta f_2=0 \Rightarrow \underline{\gamma=f_2} \quad \vee \quad \gamma \neq f_2$

$\Updownarrow$

$\mu\neq 0 \Rightarrow \gamma: f_1+\underbrace{\frac{\eta}{\mu}}_{=\lambda} f_2=0 \Rightarrow \underline{\gamma: f_1+\lambda f_2=0}$

ТЕОРЕМА: Скуп свих равни које садрже праву p дату као пресек две равни тј. $p: \begin{cases} \alpha: Ax+By+Cz+D=0 \\ \beta: \bar{A}x+\bar{B}y+\bar{C}z+\bar{D}=0 \end{cases}$ је дат једначином:

$$\gamma: \lambda_1(Ax+By+Cz+D)+\lambda_2(\bar{A}x+\bar{B}y+\bar{C}z+\bar{D})=0.$$

19) Одредити једначину фамилије свих равни које садрже тачку $P(5,-2,1)$ и

а) нормалне

б) паралелне

су на праву $q: \frac{x+1}{2}=\frac{y+2}{-3}=\frac{z-4}{2}$.

а)

$\pi: \pi\perp q,\ \pi \ni P(5,-2,1)$

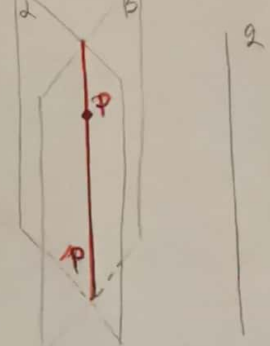
$\pi \perp q \Rightarrow \underline{\vec{n}_\pi \sim \vec{q}=(2,-3,2)}$

$\pi: \ 2\cdot(x-5)-3(y+2)+2(z-1)=0$

$2x-10-3y-6+2z-2=0$

$\boxed{\pi: \ 2x-3y+2z-18=0}$

б)



$\pi \parallel q$ је фамилија равни које садрже праву

$p: p\parallel q,\ p\ni P(5,-2,1)$

$p\parallel q \Rightarrow \vec{p}=\vec{q}=(2,-3,2) \Rightarrow p: \frac{x-5}{2}=\frac{y+2}{-3}=\frac{z-1}{2}$

$\alpha: \frac{x-5}{2}=\frac{y+2}{-3} \iff -3(x-5)=2(y+2) \Rightarrow \alpha: \underbrace{3x+2y-11}_{f_1}=0$

$\beta: \frac{y+2}{-3}=\frac{z-1}{2} \iff 2(y+2)=-3(z-1) \Rightarrow \beta: \underbrace{2y+3z+1}_{f_2}=0$

$\{p\}=\alpha\cap\beta \Rightarrow 1^\circ\ \pi: f_1+\lambda f_2=0 \quad 2^\circ\ \pi: f_2=0$

$1^\circ \Rightarrow \pi: 3x+2y-11+\lambda(2y+3z+1)=0 \iff \boxed{\pi: 3x+2(1+\lambda)+3\lambda z-11+\lambda=0}$

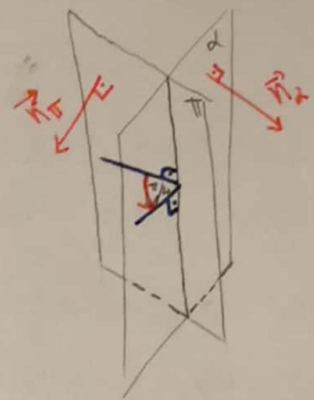
$2^\circ\ \boxed{\pi: 2y+3z+1=0}$

(20) Одредити једну раван која садржи праву $p: \begin{cases} x+y+z=0 \\ 2x-2z+3=0 \end{cases}$ и са равни $\alpha: x-4y-8z+12=0$ гради угао од $\frac{\pi}{4}$.

$$\{p\} = \pi_1 \cap \pi_2 \qquad \pi_1: \overbrace{x+y+z}^{f_1}=0$$
$$\pi_2: \underbrace{2x-2z+3}_{f_2}=0$$

Како раван π садржи праву $p \Rightarrow \pi \in \mathcal{X}_{\pi_1,\pi_2} \Rightarrow$

1° $\pi: f_1+\lambda f_2=0$



$\pi: x+y+z+\lambda(2x-2z+3)=0$

$$\boxed{\pi: (1+2\lambda)x + y + (1-2\lambda)z + 3\lambda = 0}$$

$$\left.\begin{aligned} &\measuredangle(\alpha,\pi)=\frac{\pi}{4} \\ &\measuredangle(\alpha,\pi)=\measuredangle(\vec{n}_\alpha,\vec{n}_\pi) \end{aligned}\right\} \Rightarrow \measuredangle(\vec{n}_\alpha,\vec{n}_\pi)=\frac{\pi}{4}$$

(ПОСМАТРАНО ДА ЈЕ УГАО ИЗМЕЂУ РАВНИ ОШТАР)

$$\boxed{|\vec{n}_\alpha \circ \vec{n}_\pi| = \|\vec{n}_\alpha\|\,\|\vec{n}_\pi\| \cdot \cos\underbrace{\measuredangle(\alpha,\pi)}_{\frac{\pi}{4}}} \ (*)$$

$\vec{n}_\alpha(1,-4,-8)$

$\vec{n}_\pi(1+2\lambda,1,1-2\lambda)$

$$\left.\begin{aligned} &\|\vec{n}_\alpha\| = \sqrt{1+16+64} = 9 \\ &\|\vec{n}_\pi\| = \sqrt{(1+2\lambda)^2+1+(1-2\lambda)^2} = \sqrt{1+\cancel{4\lambda}+4\lambda^2+1+1-\cancel{4\lambda}+4\lambda^2} \\ &\quad = \sqrt{8\lambda^2+3} \end{aligned}\right\} \rightsquigarrow (*)$$

$$\cos\frac{\pi}{4} = \frac{|\vec{n}_\alpha \circ \vec{n}_\pi|}{\|\vec{n}_\alpha\|\cdot\|\vec{n}_\pi\|}$$

$$\frac{\sqrt{2}}{2} = \frac{|(1,-4,-8)\circ(1+2\lambda,1,1-2\lambda)|}{9\cdot\sqrt{8\lambda^2+3}}$$

$$\frac{\sqrt{2}}{2} = \frac{|1+2\lambda-4-8(1-2\lambda)|}{9\sqrt{8\lambda^2+3}}$$

$$\frac{\sqrt{2}}{2} = \frac{|18\lambda-11|}{9\cdot\sqrt{8\lambda^2+3}} \quad \Big/^2$$

$$\frac{1}{2} = \frac{(18\lambda-11)^2}{81(8\lambda^2+3)}$$

$$2(18\lambda-11)^2 = 81(8\lambda^2+3)$$

$$2(324\lambda^2-396\lambda+121) = 648\lambda^2+243$$

$$792\lambda+1=0 \Rightarrow \boxed{\lambda=-\frac{1}{792}} \rightsquigarrow \vec{n}_\pi \Rightarrow \vec{n}_\pi\left(1+2\cdot\left(-\frac{1}{792}\right), 1, 1-2\cdot\left(-\frac{1}{792}\right)\right) \text{ tj.}$$

$$\vec{n}_\pi\left(\frac{395}{396}, 1, \frac{397}{396}\right) \sim (395,396,397)$$

$$\Rightarrow \pi: \frac{395}{396}x+y+\frac{397}{396}z+3\cdot\left(-\frac{1}{792}\right)=0 \ \Big/\cdot 396 \quad \text{tj.} \ \boxed{\pi: 395x+396y+397z-\frac{3}{2}=0}$$

2° $\overline{\pi}: f_2=0$

$\overline{\pi}: 2x-2z+3=0$

$\vec{n}_{\overline{\pi}}(2,0,-2)$

$$\underline{\cos \measuredangle(\alpha,\overline{\pi})} = \frac{|\vec{n}_\alpha \circ \vec{n}_{\overline{\pi}}|}{\|\vec{n}_\alpha\| \cdot \|\vec{n}_{\overline{\pi}}\|} = \frac{|(1,-4,-8)\circ(2,0,-2)|}{\sqrt{1+16+64}\ \sqrt{4+4}} = \frac{|2+16|}{\sqrt{81}\ \sqrt{8}} = \frac{|18|}{9\cdot 2\sqrt{2}} = \frac{1}{\sqrt{2}} = \underline{\frac{\sqrt{2}}{2}}$$

$\Rightarrow \measuredangle(\alpha,\overline{\pi}) = \frac{\pi}{4} \Rightarrow \overline{\pi}$ заклапа са α угао од $\frac{\pi}{4} \Rightarrow$

решење су: $\boxed{\begin{array}{l} \pi: 395x+396y+397z-\frac{3}{2}=0 \\ \overline{\pi}: 2x-2z+3=0 \end{array}}$