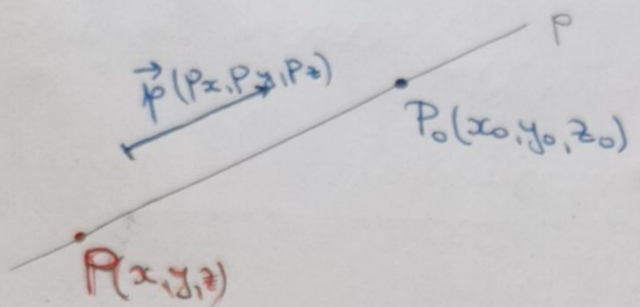
ПРАВА И РАВАН У ПРОСТОРУ

ПРАВА У ПРОСТОРУ



$P_0(x_0,y_0,z_0)$ - ТАЧКА СА ПРАВЕ p

$\vec{p}(p_x,p_y,p_z)$ - ВЕКТОР ПРАВЦА ПРАВЕ p

$P(x,y,z)$ - ПРОИЗВОЉНА ТАЧКА ПРАВЕ p

* ПАРАМЕТАРСКА Ј-НА ПРАВЕ У $\mathbb{R}^3$

$$\boxed{\begin{array}{l} x = \boxed{p_x}\, t + \boxed{x_0} \\ y = p_y\, t + y_0 \\ z = p_z\, t + z_0 \end{array}, \quad t \in \mathbb{R}}$$

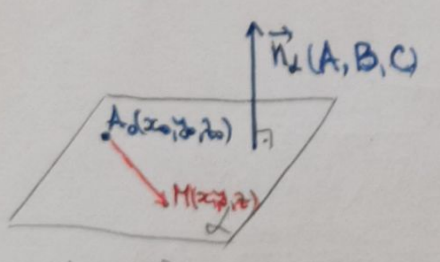
$$\uparrow \vec{p} \qquad \uparrow P_0$$

* КАНОНСКА Ј-НА ПРАВЕ

$$\boxed{\frac{x-x_0}{p_x} = \frac{y-y_0}{p_y} = \frac{z-z_0}{p_z} \; (= t)}$$

РАВАН У ПРОСТОРУ

ИМПЛИЦИТНА Ј-НА РАВНИ



$A_0(x_0,y_0,z_0)$ - ТАЧКА ИЗ РАВНИ α

$\vec{n}_\alpha(A,B,C)$ - ВЕКТОР НОРМАЛЕ НА РАВАН α

$M(x,y,z)$ - ПРОИЗВОЉНА ТАЧКА РАВНИ α

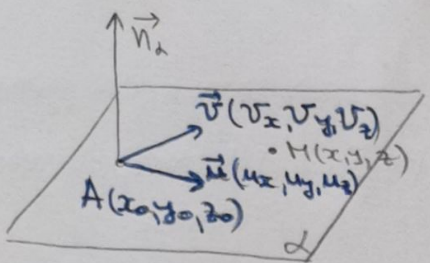
$$\vec{n}_\alpha \perp \alpha \Rightarrow \vec{n}_\alpha \perp \overrightarrow{AM} \Leftrightarrow \vec{n}_\alpha \circ \overrightarrow{AM} = 0$$

$$\boxed{\alpha : Ax + By + Cz + D = 0}$$

$$0 = \vec{n}_\alpha \circ \overrightarrow{AM} = (A,B,C) \circ (x-x_0, y-y_0, z-z_0) =$$

$$= A(x-x_0) + B(y-y_0) + C(z-z_0) \Rightarrow \boxed{\alpha : A(x-x_0) + B(y-y_0) + C(z-z_0) = 0}$$

ПАРАМЕТАРСКА Ј-НА РАВНИ



$A(x_0,y_0,z_0)$ – ТАЧКА ИЗ РАВНИ α

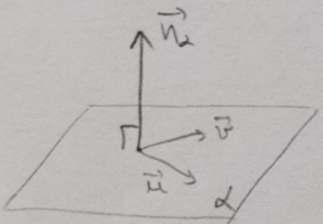
$\left.\begin{array}{l}\vec{u}(u_x,u_y,u_z)\\ \vec{v}(v_x,v_y,v_z)\end{array}\right\}$ ВЕКТОРИ ПАРАЛЕЛНИ РАВНИ α

$M(x,y,z)$ – ПРОИЗВОЉНА ТАЧКА РАВНИ α

$$
\boxed{\begin{array}{rcccccccl}
x & = & x_0 & + & u_x t & + & v_x s & & \\
y & = & y_0 & + & u_y t & + & v_y s & , & t, s \in \mathbb{R} \\
z & = & z_0 & + & u_z t & + & v_z s & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & A & & \vec{u} & & \vec{v} & &
\end{array}}
$$

(10) Равни $x-y+3z-2=0$ одредити параметарски једначину.

$\alpha: x-y+3z-2=0 \Rightarrow \underline{\vec{n}_\alpha(1,-1,3)}$



Потребно је одредити координате вектора $\vec{u}$ и $\vec{v}$ који су паралелни равни α.

$$\left.\begin{array}{l}\vec{u},\vec{v} \parallel \alpha\\ \vec{n}_\alpha \perp \alpha\end{array}\right\} \Rightarrow \vec{u},\vec{v} \perp \vec{n}_\alpha \iff \vec{u}\circ\vec{n}_\alpha = 0,\ \vec{v}\circ\vec{n}_\alpha=0$$

$$\iff \underset{\substack{\uparrow\\ \text{координате вектора паралелних равни } \alpha}}{(x,y,z)} \circ \underset{\vec{n}_\alpha}{(1,-1,3)} = 0$$

$$\iff \underline{x-y+3z=0} \quad (*)$$

Потребно је да одредимо векторе $\vec{u}$ и $\vec{v}$:

нпр. • $z=0 \rightsquigarrow (*) \Rightarrow x-y=0 \Rightarrow x=y \Rightarrow \boxed{\vec{u}(1,1,0)}$

• $x=0 \rightsquigarrow (*) \Rightarrow -y+3z=0 \Rightarrow y=3z \Rightarrow \boxed{\vec{v}(0,3,1)}$

+ провера: $\vec{u}\circ\vec{n}_\alpha = (1,1,0)\circ(1,-1,3) = 1-1=0 \Rightarrow \vec{u}\perp\vec{n}_\alpha$

$\vec{v}\circ\vec{n}_\alpha = (0,3,1)\circ(1,-1,3) = -3+3=0 \Rightarrow \vec{v}\perp\vec{n}_\alpha$

Тачка са равни α:

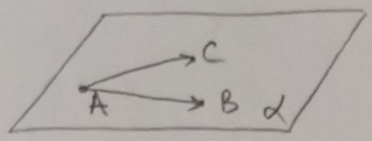
нпр. $\underline{y=z=0} \rightsquigarrow \alpha \Rightarrow x-2=0 \Rightarrow \underline{x=2} \Rightarrow \boxed{A(2,0,0)\in\alpha}$

$\Downarrow$

$$\boxed{\alpha: \begin{array}{l} x=2+t \\ y=t+3s, \quad t,s\in \\ z=s \end{array}}$$

(11) Одредити ј-ну равни која садржи тачке $A(1,2,3)$, $B(-1,2,-1)$ и $C(0,0,1)$.

ПАРАМЕТАРСКА:



$\vec{u} \sim \overrightarrow{AB} = (-2,0,-4) \Rightarrow \vec{u}(1,0,2)$

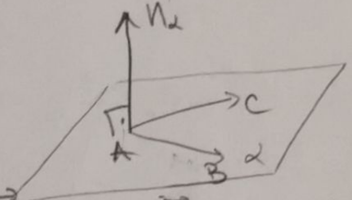
$\vec{v} \sim \overrightarrow{AC} = (-1,-2,-2) \Rightarrow \vec{v}(1,2,2)$

$C(0,0,1) \in \alpha$

$$\alpha: \begin{array}{l} x = \boxed{} + \boxed{1}\cdot t + \boxed{1} s \\ y = \boxed{} \qquad\quad + \boxed{2} s \\ z = \boxed{1} + \boxed{2} t + \boxed{2} s \end{array} \quad , t,s \in \mathbb{R}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ C & \vec{u} & v \end{array}$$

ИМПЛИЦИТНА:



$$\vec{n}_\alpha \sim \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -2 & 0 & -4 \\ -1 & -2 & -2 \end{vmatrix} = (-8,0,4) \Rightarrow \underline{\underline{\vec{n}_\alpha(-2,0,1)}}$$

$\underline{C(0,0,1) \in \alpha}$

$\alpha: \quad -2(x-0) + 0(y-0) + 1\cdot(z-1) = 0$ тј.

$$\boxed{\alpha: \ -2x + z - 1 = 0}$$

ПРАВА КАО ПРЕСЕК ДВЕ РАВНИ

$$\begin{array}{l} \alpha: Ax + By + Cz + D = 0 \\ \beta: A_1x + B_1y + C_1z + D_1 = 0 \end{array} \quad (*)$$

- Ако су $\vec{n}_\alpha, \vec{n}_\beta$ колинеарни вектори (тј. $\|\vec{n}_\alpha \times \vec{n}_\beta\| = 0$) равни су ПАРАЛЕЛНЕ или СЕ ПОКЛАПАЈУ.
- Ако су $\vec{n}_\alpha, \vec{n}_\beta$ линеарно независни, $\alpha \cap \beta = \{p\}$, p – ЈЕДИНСТВЕНА ПРЕСЕЧНА ПРАВА

$$\left.\begin{array}{l} p \in \alpha \Rightarrow \vec{p} \perp \vec{n}_\alpha \\ p \in \beta \Rightarrow \vec{p} \perp \vec{n}_\beta \end{array}\right\} \Rightarrow \vec{p} \sim \vec{n}_\alpha \times \vec{n}_\beta$$

Тачка $P \in p$ је једно решење система (*)

α: β:

Ако је права p дата каноском ј-ном $p: \dfrac{x-x_0}{p_x} = \dfrac{y-y_0}{p_y} = \dfrac{z-z_0}{p_z} = t$, две равни које се секу по правој p су НПР: $\alpha: \dfrac{x-x_0}{p_x} = \dfrac{y-y_0}{p_y}$, $\beta: \dfrac{y-y_0}{p_y} = \dfrac{z-z_0}{p_z}$

(13) Праву $p: x=t+4, y=-2t+1, z=3t-2, t\in\mathbb{R}$ записати као пресек две равни.

$$p: \begin{array}{l} x=t+4 \Rightarrow t=x-4 \\ y=-2t+1 \Rightarrow t=\frac{-y+1}{2}=\frac{y-1}{-2} \\ z=3t-2 \rightarrow t=\frac{z+2}{3} \end{array} \Bigg\} \Rightarrow \boxed{p: \frac{x-4}{1}=\frac{y-1}{-2}=\frac{z+2}{3} (=t)}$$

β

α

$$\alpha: \frac{x-4}{1}=\frac{y-1}{-2}$$

$$-2(x-4)=y-1$$

$$-2x-y+9=0 \quad \text{тј.}$$

$$\boxed{\alpha: 2x+y-9=0}$$

$$\beta: \frac{y-1}{-2}=\frac{z+2}{3}$$

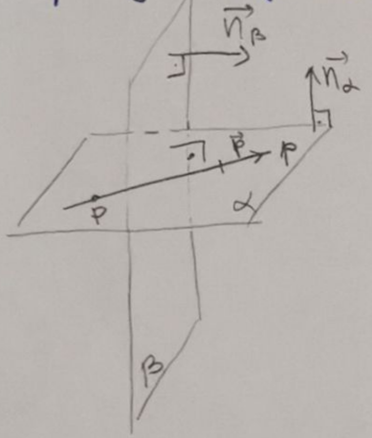
$$3(y-1)=-2(z+2) \quad \text{тј.}$$

$$\boxed{\beta: 3y+2z+1=0}$$

(14) Одредити једн. равни α која садржи праву $p: \frac{x-2}{1}=\frac{y-11}{5}=\frac{z-2}{1}$ и која је нормална на раван $\beta: y=0$.

P

$\vec{p}$



$$\left.\begin{array}{l} p\in\alpha: P(2,11,2)\in\alpha \\ p\in\alpha: \vec{p}(1,5,1) \quad \vec{n}_\alpha\perp\vec{p} \\ \beta\perp\alpha \Rightarrow \vec{n}_\beta\perp\vec{n}_\alpha, \ \vec{n}_\beta(0,1,0) \end{array}\right\} \Rightarrow \vec{n}_\alpha \sim \vec{p}\times\vec{n}_\beta$$

$$\vec{p}\times\vec{n}_\beta = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 5 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1\cdot\vec{e}_1 - 0\vec{e}_2 + 1\vec{e}_3 = \underline{(-1,0,1)}$$

$$\Rightarrow \vec{n}_\alpha \sim \vec{p}\times\vec{n}_\beta \Rightarrow \underline{\vec{n}_\alpha=(-1,0,1)}$$

$$\alpha: -1\cdot(x-2)+0(y-11)+1\cdot(z-2)=0$$

$$-x+2+z-2=0$$

$$\boxed{\alpha: -x+z=0}$$

(15) Праву $p: x-y-1=0, z-2x=0$ записати параметарски.

$\alpha\cap\beta=\{p\}$, $\alpha: x-y-1=0$, $\beta: z-2x=0$

$\vec{n}_\alpha(1,-1,0)$ $\quad$ $\vec{n}_\beta(-2,0,1)$

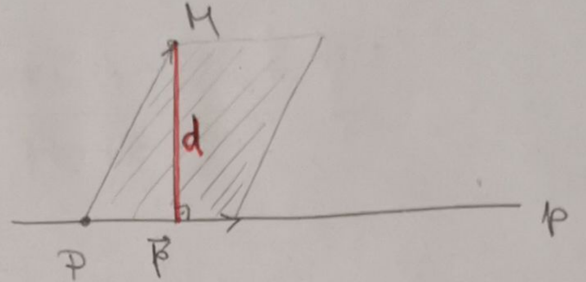
$$\left.\begin{array}{l} p\in\alpha \Rightarrow \vec{p}\perp\vec{n}_\alpha \\ p\in\beta \Rightarrow \vec{p}\perp\vec{n}_\beta \end{array}\right\} \Rightarrow \vec{p}\sim\vec{n}_\alpha\times\vec{n}_\beta = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = (-1,-1,-2) \Rightarrow \underline{\vec{p}=(1,1,2)}$$

$P\in p \Rightarrow$ нпр. $x=0 \Rightarrow y=-1 \Rightarrow z=0 \Rightarrow P(0,-1,0)\in p$

$$p: \begin{array}{l} x=1\cdot t+0 \\ y=1\cdot t-1 \\ z=2t+0 \end{array}, t\in\mathbb{R} \quad \text{тј.} \quad \boxed{p: \begin{array}{l} x=t \\ y=t-1 \\ z=2t \end{array}, t\in\mathbb{R}}$$

РАСТОЈАЊЕ ТАЧКЕ ОД ПРАВЕ

$$\boxed{(d=)\,d(M,p)=\frac{\|\overrightarrow{PM}\times\vec{p}\|}{\|\vec{p}\|}}$$



РАСТОЈАЊЕ ТАЧКЕ ОД РАВНИ

$\alpha: Ax+By+Cz+D=0$

$M(x_0,y_0,z_0)$

$$\boxed{(d=)\,d(M,\alpha)=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}}$$

16) Одредити растојање тачке $M(1,0,12)$ од:

а) праве $p: x-y-1=0,\ z-2x=0$

права из зад. 15

$p: \vec{p}(1,1,2)$, $P(0,-1,0)$ $\qquad \overrightarrow{PM}(1,1,12)$

$$d(M,p)=\frac{\|\overrightarrow{PM}\times\vec{p}\|}{\|\vec{p}\|} \quad (*)$$

$$\overrightarrow{PM}\times\vec{p}=\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 1 & 1 & 12\\ 1 & 1 & 2\end{vmatrix}=(2-12)\vec{e}_1-(2-12)\vec{e}_2+(1-1)\vec{e}_3=\underline{(-10,10,0)}$$

$$\|\overrightarrow{PM}\times\vec{p}\|=\sqrt{(-10)^2+10^2+0^2}=\sqrt{2\cdot100}=\underline{10\sqrt{2}}$$

$$\|\vec{p}\|=\sqrt{1^2+1^2+2^2}=\sqrt{1+1+4}=\underline{\sqrt{6}}$$

$$\Rightarrow d\overset{(*)}{=}\frac{10\sqrt{2}}{\sqrt{6}}=\frac{10\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\cdot\sqrt{3}}=\frac{10}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$\Rightarrow \boxed{d=\frac{10\sqrt{3}}{3}}$$

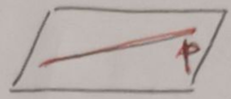
б) равни $\alpha: \overset{A}{x}\overset{B}{-y}\overset{C}{-4}z=0$.

$$d(M,\alpha)=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}} \Rightarrow$$

$$\underline{\underline{d(M,\alpha)}}=\frac{|1\cdot1-1\cdot0-4\cdot12+0|}{\sqrt{1^2+(-1)^2+(-4)^2}}=\frac{|1-48|}{\sqrt{1+1+16}}=\frac{|-47|}{\sqrt{18}}=\frac{47}{\sqrt{9}\cdot\sqrt{2}}=\frac{47}{3\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\underline{\underline{\frac{47\sqrt{2}}{6}}}$$

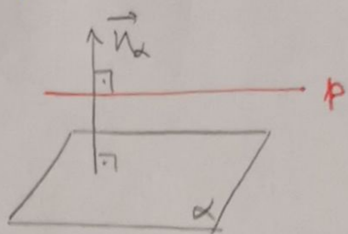
МЕЂУСОБНИ ПОЛОЖАЈ ПРАВЕ И РАВНИ

* ПРАВА ПРИПАДА РАВНИ, $p \in \alpha$



$\alpha \cap p = \{p\}$

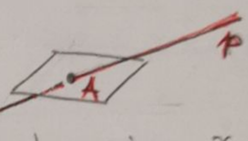
$\vec{p} \circ \vec{n}_\alpha = 0$



* ПАРАЛЕЛНЕ СУ, $p \parallel \alpha$

$\vec{p} \circ \vec{n}_\alpha = 0$

* ПРАВА И РАВАН СЕ СЕКУ



$p \cap \alpha = \{A\}$

p: $\vec{p}(p_x, p_y, p_z)$, $P(x_0, y_0, z_0)$

$\vec{n}_\alpha(A, B, C)$

p: $\dfrac{x-x_0}{p_x} = \dfrac{y-y_0}{p_y} = \dfrac{z-z_0}{p_z} = t$ тј. $x = p_x t + x_0$, $y = p_y t + y_0$, $z = p_z t + z_0$, $t \in \mathbb{R}$

α: $Ax + By + Cz + D = 0$

$P \rightsquigarrow \alpha$ тј.

$A(p_x t + x_0) + B(p_y t + y_0) + C(p_z t + z_0) + D = 0$

$Ap_x t + Bp_y t + Cp_z t + Ax_0 + By_0 + Cz_0 + D = 0$

$(Ap_x + Bp_y + Cp_z)t = -(Ax_0 + By_0 + Cz_0 + D)$

$$\Rightarrow \boxed{t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap_x + Bp_y + Cp_z} = -\frac{Ax_0 + By_0 + Cz_0 + D}{\vec{n}_\alpha \circ \vec{p}}} \ (*)$$

$\circ\ \vec{n}_\alpha \circ \vec{p} \neq 0$

$\vec{n}_\alpha \circ \vec{p} = (A, B, C) \circ (p_x, p_y, p_z) = Ap_x + Bp_y + Cp_z$

17) Дата је раван $\alpha: x - 2y + 5z - 1 = 0$. Одредити пресек те равни са правама:

a) $p: \dfrac{x-2}{3} = \dfrac{y-4}{7} = \dfrac{z}{1} = t$

$p: x = 3t + 2$, $y = 7t + 4$, $z = t$, $t \in \mathbb{R}$

$\left.\begin{array}{l} \vec{n}_\alpha(1, -2, 5) \\ \vec{p}(3, 7, 1) \\ P(2, 4, 0) \end{array}\right\} \Rightarrow \vec{p} \circ \vec{n}_\alpha = (3, 7, 1) \circ (1, -2, 5) = 3 - 14 + 5 = -6 \neq 0 \Rightarrow$

$p \cap \alpha = \{A\} \Rightarrow p \rightsquigarrow \alpha \overset{(*)}{\Rightarrow} t = -\dfrac{Ax_0 + By_0 + Cz_0 + D}{\vec{p} \circ \vec{n}_\alpha}$, $P(x_0, y_0, z_0)$

$t = -\dfrac{1 \cdot 2 - 2 \cdot 4 + 5 \cdot 0 - 1}{-6} = -\dfrac{2 - 8 - 1}{-6} = -\dfrac{-7}{-6} = \underline{\underline{-\dfrac{7}{6}}} \rightsquigarrow p$

$\{A\} = p \cap \alpha$, A: $x = 3 \cdot \left(-\frac{7}{6}\right) + 2 = -\frac{7}{2} + 2 = -\frac{3}{2}$

$y = 7\left(-\frac{7}{6}\right) + 4 = -\frac{49}{6} + 4 = -\frac{25}{6}$

$z = -\frac{7}{6}$

тј. $\boxed{A\left(-\frac{3}{2}, -\frac{25}{6}, -\frac{7}{6}\right)}$

д) $q: \frac{x}{3} = \frac{y-2}{-1} = \frac{z-1}{-1}$ $\qquad \alpha: x-2y+5z-1=0$

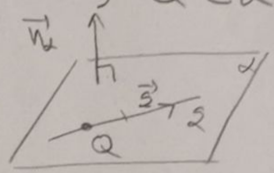
$Q(0,2,1)$

$$\left.\begin{array}{l}\vec{q}(3,-1,-1)\\ \vec{n}_\alpha(1,-2,5)\end{array}\right\} \Rightarrow \underline{\vec{q}\circ\vec{n}_\alpha} = (3,-1,-1)\circ(1,-2,5) = 3+2-5 = \underline{\underline{0}}$$

$\vec{q}\circ\vec{n}_\alpha = 0 \Leftrightarrow q \parallel \alpha$

Дали $q \in \alpha$?

$Q \rightsquigarrow \alpha: \quad 0-2\cdot2+5\cdot1-1 = -4+5-1 = 0 \Rightarrow Q\in\alpha \Rightarrow \underline{\underline{q\in\alpha}}$



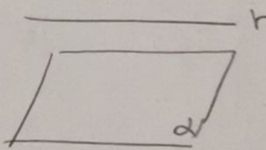
е) $r: \frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{-1}$

$\vec{r}(3,-1,-1)$, $R(1,2,3)$

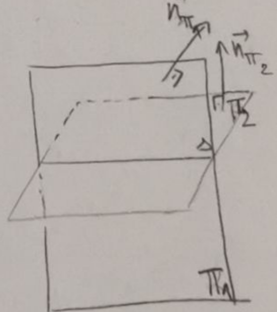
$\vec{r}\circ\vec{n}_\alpha = (3,-1,-1)\circ(1,-2,5) = 3+2-5 = 0$

$\underline{q \overset{?}{\in} \alpha}$: $R \rightsquigarrow \alpha$ $\quad 1-2\cdot2+5\cdot3-1 = 1-4+15-1 = 11 \neq 0 \Rightarrow$

$$\left.\begin{array}{l}R\notin\alpha\\ \vec{r}\perp\vec{n}_\alpha\end{array}\right\} \Rightarrow \underline{\underline{r\parallel\alpha}}$$



ж) $s: x-y=1,\ x+y-z+5=0$



II начин

Решаваме систе

$s: \begin{array}{l} x-y-1=0\\ x+y-z+5=0\end{array}$

$\alpha: x-2y+5z-1=0$

I начин $\quad \{s\} = \pi_1 \cap \pi_2$

$\pi_1: x-y-1=0$, $\quad \vec{n}_{\pi_1}(1,-1,0)$

$\pi_2: x+y-z+5=0$, $\quad \vec{n}_{\pi_2}(1,1,-1)$

$$\left.\begin{array}{l}s\in\pi_1 \Rightarrow \vec{s}\perp\vec{n}_{\pi_1}\\ s\in\pi_2 \Rightarrow \vec{s}\perp\vec{n}_{\pi_2}\end{array}\right\} \Rightarrow \vec{s} \sim \vec{n}_{\pi_1}\times\vec{n}_{\pi_2}$$

$$\vec{n}_{\pi_1}\times\vec{n}_{\pi_2} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 1 & -1 & 0\\ 1 & 1 & -1\end{vmatrix} = \vec{e}_1+\vec{e}_2+2\vec{e}_3 = (1,1,2) \Rightarrow \underline{\underline{\vec{s}(1,1,2)}}$$

$\underline{S\in s}$: нпр. $y=0 \Rightarrow x=1 \Rightarrow z=6$ т.е. $\underline{\underline{S(1,0,6)}}$

$\vec{s}\circ\vec{n}_\alpha = (1,1,2)\circ(1,-2,5) = 1-2+10 = 9 \neq 0 \Rightarrow \underline{\underline{s\cap\alpha = \{A\}}}$

$$\circledast \Rightarrow t = -\frac{Ax_0+By_0+Cz_0+D}{\vec{s}\circ\vec{n}_\alpha} = -\frac{1\cdot1-2\cdot0+5\cdot6-1}{9} = -\frac{1+30-1}{9} = -\frac{30}{9} = \underline{\underline{-\frac{10}{3}}}$$

$t=-\frac{10}{3} \rightsquigarrow s: \begin{array}{l} x = t+1\\ y = t\\ z = 2t+6\end{array}, t\in\mathbb{R} \Rightarrow A: \begin{array}{l} x = -\frac{10}{3}+1 = -\frac{7}{3}\\ y = -\frac{10}{3}\\ z = 2\cdot\left(-\frac{10}{3}\right)+6 = -\frac{20}{3}+6 = -\frac{2}{3}\end{array}$ $\boxed{A\left(-\frac{7}{3},-\frac{10}{3},-\frac{2}{3}\right)}$

e) $t: x-z+2=0,\ -y+3z+2=0$ $\qquad \alpha: x-2y+5z-1=0$

$\pi_1: x-z+2=0,\quad \vec{n}_{\pi_1}(1,0,-1)$

$\pi_2: -y+3z+2=0,\quad \vec{n}_{\pi_2}(0,-1,3)$

$\{t\}=\pi_1\cap\pi_2$

$$\left.\begin{array}{l} t\in\pi_1 \Rightarrow \vec{t}\perp\vec{n}_{\pi_1} \\ t\in\pi_2 \Rightarrow \vec{t}\perp\vec{n}_{\pi_2}\end{array}\right\} \Rightarrow \vec{t}\sim\vec{n}_{\pi_1}\times\vec{n}_{\pi_2}=\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 1 & 0 & -1\\ 0 & -1 & 3\end{vmatrix}=-\vec{e}_1-3\vec{e}_2-\vec{e}_3=(-1,-3,-1)$$

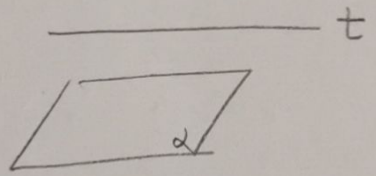
$\Rightarrow \underline{\vec{t}(1,3,1)}$

$\underline{T\in t}$: нпр. $z=0 \Rightarrow x=-2 \Rightarrow y=2$ тј. $\underline{T(-2,2,0)}$

$\vec{n}_\alpha\circ\vec{t}=(1,-2,5)\circ(1,3,1)=1-6+5=0$

$\underline{T\overset{?}{\in}\alpha}$: $T \rightsquigarrow \alpha$: $-2-2\cdot2+5\cdot0-1=-2-4-1=-7\neq0 \Rightarrow T\notin\alpha$

$\Rightarrow \underline{t\parallel\alpha}$



18 Одредити тачку продора праве $p: \frac{x+1}{3}=\frac{y+3}{2}=\frac{z-5}{-4}=t$ кроз раван $\alpha: 3x+y+5z-7=0$.

$p: \vec{p}(3,2,-4),\quad P(-1,-3,5)$

$\vec{n}_\alpha(3,1,5)$

$\vec{n}_\alpha\circ\vec{p}=(3,1,5)\circ(3,2,-4)=9+2-20=-9\neq0 \Rightarrow p\cap\alpha=\{A\}$

$$\overset{(*)}{\Longrightarrow} t=-\frac{Ax_0+By_0+Cz_0+D}{\vec{n}_\alpha\circ\vec{p}}=-\frac{3\cdot(-1)+1\cdot(-3)+5\cdot5-7}{-9}=-\frac{-3-3+25-7}{-9}=$$

$\Rightarrow t=-\frac{12}{-9}=\frac{4}{3}$

$t=\frac{4}{3} \rightsquigarrow p:\quad \begin{aligned}x&=3t-1\\ y&=2t-3\\ z&=-4t+5\end{aligned} \quad\Rightarrow\quad A:\ \begin{aligned}x&=\cancel{3}\cdot\frac{4}{\cancel{3}}-1=3\\ y&=2\cdot\frac{4}{3}-3=-\frac{1}{3}\\ z&=-4\cdot\frac{4}{3}+5=-\frac{1}{3}\end{aligned}$ тј.

$$\boxed{A\left(3,-\frac{1}{3},-\frac{1}{3}\right)}$$

ЈЕДНАЧИНА РАВНИ ИЗ ИСТОГ ПРАМЕНА

$\alpha: \overbrace{Ax+By+Cz+D}^{f_1}=0$

$\beta: \underbrace{\bar{A}x+\bar{B}y+\bar{C}z+\bar{D}}_{f_2}=0$

$\gamma \in \mathcal{X}_{\alpha,\beta} \Rightarrow \boxed{\gamma: f_1+\lambda f_2=0 \quad \vee \quad \gamma: f_2=0}$

тј. $\gamma: \mu f_1+\eta \cdot f_2=0 \Rightarrow \underline{\gamma=f_2} \quad \vee \quad \gamma \neq f_2$

$\Updownarrow$

$\mu \neq 0 \Rightarrow \gamma: f_1+\underbrace{\frac{\eta}{\mu}}_{=\lambda} f_2=0$ тј. $\underline{\gamma: f_1+\lambda f_2=}$

ТЕОРЕМА: Скуп свих равни које садрже праву p дату као пресек две равни тј. $p: \begin{cases} \alpha: Ax+By+Cz+D=0 \\ \beta: \bar{A}x+\bar{B}y+\bar{C}z+\bar{D}=0 \end{cases}$ је дат једначином:

$$\gamma: \lambda_1(Ax+By+Cz+D)+\lambda_2(\bar{A}x+\bar{B}y+\bar{C}z+\bar{D})=0.$$

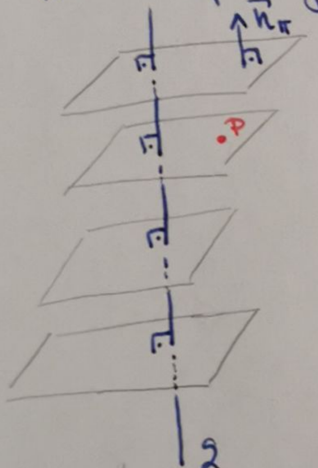
19 Одредити једначину фамилије свих равни које садрже тачку $P(5,-2,1)$ и

а) нормалне

б) паралелне

су на праву $q: \frac{x+1}{2}=\frac{y+2}{-3}=\frac{z-4}{2}$.

а)



$\pi: \pi \perp q,\ \pi \ni P(5,-2,1)$

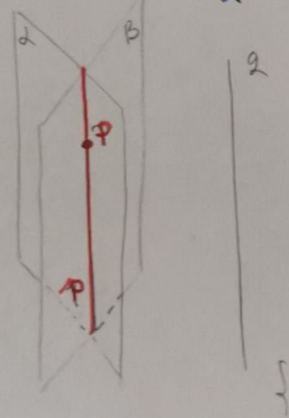
$\pi \perp q \Rightarrow \vec{n}_\pi \sim \vec{q}=(2,-3,2)$

$\pi: 2\cdot(x-5)-3(y+2)+2(z-1)=0$

$2x-10-3y-6+2z-2=0$

$\boxed{\pi: 2x-3y+2z-18=0}$

б)



$\pi \parallel q$ је фамилија равни које садрже праву

$p: p\parallel q,\ p \ni P(5,-2,1)$

$p\parallel q \Rightarrow \vec{p}=\vec{q}=(2,-3,2) \Rightarrow p: \frac{x-5}{2}=\frac{y+2}{-5}=\frac{z-1}{2}$

$\alpha: \frac{x-5}{2}=\frac{y+2}{-3} \Leftrightarrow -3(x-5)=2(y+2) \Rightarrow \alpha: \underbrace{3x+2y-11}_{f_1}=0$

$\beta: \frac{y+2}{-3}=\frac{z-1}{2} \Leftrightarrow 2(y+2)=-3(z-1) \Rightarrow \beta: \underbrace{2y+3z+1}_{f_2}=0$

$\{p\}=\alpha\cap\beta \Rightarrow 1^\circ\ \pi: f_1+\lambda f_2=0 \quad \vee \quad 2^\circ\ \pi: f_2=0$

$1^\circ \Rightarrow \pi: 3x+2y-11+\lambda(2y+3z+1)=0 \Leftrightarrow 1^\circ\ \boxed{\pi: 3x+2(1+\lambda)+3\lambda z-11+\lambda=0}$ $\quad 2^\circ\ \boxed{\pi: 2y+3z+1=0}$

(20) Одредити једну равни која садржи праву $p: \begin{cases} x+y+z=0 \\ 2x-2z+3=0 \end{cases}$ и са равни $\alpha: x-4y-8z+12=0$ гради угао од $\frac{\pi}{4}$.

$\{p\} = \Pi_1 \cap \Pi_2 \qquad \Pi_1: \overbrace{x+y+z}^{f_1}=0$

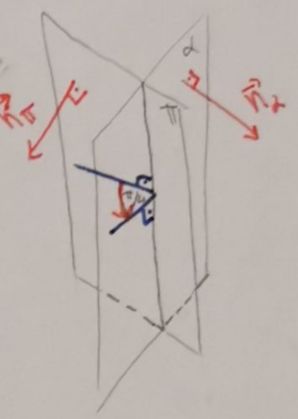
$\qquad\qquad\qquad\quad \Pi_2: \underbrace{2x-2z+3}_{f_2}=0$

Како раван Π садржи праву $p \Rightarrow \Pi \in \mathcal{X}_{\Pi_1,\Pi_2} \Rightarrow$

1° $\Pi: f_1+\lambda f_2=0$

$\Pi \quad x+y+z+\lambda(2x-2z+3)=0$

$\boxed{\Pi: (1+2\lambda)x+y+(1-2\lambda)z+3\lambda=0}$



$\left.\begin{array}{l} \measuredangle(\alpha,\Pi)=\frac{\pi}{4} \\ \measuredangle(\alpha,\Pi)=\measuredangle(\vec{n}_\alpha,\vec{n}_\pi) \end{array}\right\} \Rightarrow \measuredangle(\vec{n}_\alpha,\vec{n}_\pi)=\frac{\pi}{4}$

(ПОСМАТРАНО ДА ЈЕ УГАО ИЗМЕЂУ РАВНИ ОШТАР)

$\boxed{|\vec{n}_\alpha \circ \vec{n}_\pi| = \|\vec{n}_\alpha\| \, \|\vec{n}_\pi\| \cdot \cos \underbrace{\measuredangle(\alpha,\Pi)}_{\frac{\pi}{4}}}$ $\circledast$

$\vec{n}_\alpha(1,-4,-8)$

$\vec{n}_\pi(1+2\lambda,1,1-2\lambda)$

$\left.\begin{array}{l} \|\vec{n}_\alpha\| = \sqrt{1+16+64} = 9 \\ \|\vec{n}_\pi\| = \sqrt{(1+2\lambda)^2+1+(1-2\lambda)^2} = \sqrt{1+\cancel{4\lambda}+4\lambda^2+1+1-\cancel{4\lambda}+4\lambda^2} \\ \quad = \sqrt{8\lambda^2+3} \end{array}\right\} \rightsquigarrow \circledast$

$\cos\frac{\pi}{4} = \frac{|\vec{n}_\alpha \circ \vec{n}_\pi|}{\|\vec{n}_\alpha\| \cdot \|\vec{n}_\pi\|}$

$\frac{\sqrt{2}}{2} = \frac{|(1,-4,-8)\circ(1+2\lambda,1,1-2\lambda)|}{9\cdot\sqrt{8\lambda^2+3}}$

$\frac{\sqrt{2}}{2} = \frac{|1+2\lambda-4-8(1-2\lambda)|}{9\sqrt{8\lambda^2+3}}$

$\frac{\sqrt{2}}{2} = \frac{|18\lambda-11|}{9\cdot\sqrt{8\lambda^2+3}} \quad \Big/^2$

$\frac{1}{2} = \frac{(18\lambda-11)^2}{81(8\lambda^2+3)}$

$2(18\lambda-11)^2 = 81(8\lambda^2+3)$

$2(324\lambda^2-396\lambda+121) = 648\lambda^2+243$

$792\lambda+1=0 \Rightarrow \boxed{\lambda=-\frac{1}{792}} \rightsquigarrow \vec{n}_\pi \Rightarrow \vec{n}_\pi\left(1+2\cdot\left(-\frac{1}{792}\right), 1, 1-2\cdot\left(-\frac{1}{792}\right)\right)$ тј.

$\vec{n}_\pi\left(\frac{395}{396}, 1, \frac{397}{396}\right) \sim (395,396,397)$

$\Rightarrow \Pi: \frac{395}{396}x+y+\frac{397}{396}z+3\cdot\left(-\frac{1}{792}\right)=0 \quad \Big/\cdot 396$ тј. $\boxed{\Pi: 395x+396y+397z-\frac{3}{2}=0}$

2° $\overline{\overline{\Pi}}: f_2=0$

$\overline{\overline{\Pi}}: 2x-2z+3=0$

$\vec{n}_{\overline{\overline{\Pi}}}(2,0,-2)$

$$\cos\measuredangle(\alpha,\overline{\overline{\Pi}}) = \frac{|\vec{n}_\alpha \circ \vec{n}_{\overline{\overline{\Pi}}}|}{\|\vec{n}_\alpha\|\cdot\|\vec{n}_{\overline{\overline{\Pi}}}\|} = \frac{|(1,-4,-8)\circ(2,0,-2)|}{\sqrt{1+16+64}\ \sqrt{4+4}} = \frac{|2+16|}{\sqrt{81}\cdot\sqrt{8}} = \frac{|18|}{9\cdot 2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$\Rightarrow \measuredangle(\alpha,\overline{\overline{\Pi}}) = \frac{\pi}{4} \Rightarrow \overline{\overline{\Pi}}$ заклапа са α угао од $\frac{\pi}{4} \Rightarrow$

решење су $\boxed{\begin{array}{l}\overline{\Pi}: 395x+396y+397z-\frac{3}{2}=0\\ \overline{\overline{\Pi}}: 2x-2z+3=0\end{array}}$

(21) Одредити једначину равни α која садржи праву $p: \frac{x-2}{1}=\frac{y-11}{5}=\frac{z-2}{1}$ и чије је растојање од тачке $M(1,1,1)$ једнако $\frac{5}{\sqrt{14}}$.

$\{p\}=\Pi_1\cap\Pi_2$ $\quad \Pi_1: \frac{x-2}{1}=\frac{y-11}{5}$ $\quad \Pi_2: \frac{y-11}{5}=\frac{z-2}{1}$

тј. $\Pi_1: 5x-y+1=0$

$\Pi_2: y-5z-1=0$

$\Pi \ni p \Rightarrow \Pi \in \mathcal{X}_{\Pi_1,\Pi_2}$

1° $\Pi: 5x-y+1+\lambda(y-5z-1)=0$

$\Pi: 5x+(\lambda-1)y-5\lambda z+1-\lambda=0 \quad (*)$

$$d(M,\Pi)=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

$M(\overset{x_0}{1},\overset{y_0}{1},\overset{z_0}{1})$

$$d(M,\Pi)=\frac{|5\cdot1+(\lambda-1)1-5\lambda\cdot1+1-\lambda|}{\sqrt{5^2+(\lambda-1)^2+(-5\lambda)^2}} = \quad \underline{\underline{(1)}}$$

$$=\frac{|5+\cancel{\lambda}-\cancel{1}-5\lambda+\cancel{1}-\cancel{\lambda}|}{\sqrt{25+\lambda^2-2\lambda+1+25\lambda^2}}=\frac{5|1-\lambda|}{\sqrt{26\lambda^2-2\lambda+26}}$$

$d(M,\Pi)=\frac{5}{\sqrt{14}} \quad \underline{\underline{(2)}}$

$(1),(2) \rightsquigarrow \frac{\cancel{5}}{\sqrt{14}}=\frac{\cancel{5}|1-\lambda|}{\sqrt{26\lambda^2-2\lambda+26}}$

$\cancel{\sqrt{2}}\cdot\sqrt{13\lambda^2-\lambda+13}=\cancel{\sqrt{2}}\cdot\sqrt{7}|1-\lambda| \quad /^2$

$13\lambda^2-\lambda+13=7(1-\lambda)^2$

$13\lambda^2-\lambda+13=7(1-2\lambda+\lambda^2)$

$13\lambda^2-\lambda+13-7+14\lambda-7\lambda^2=0$

$\boxed{6\lambda^2+13\lambda+6=0} \Rightarrow \lambda_{1/2}=\frac{-13\pm\sqrt{169-4\cdot36}}{12}=\frac{-13\pm\sqrt{25}}{12}=\frac{-13\pm5}{12}$ $\nearrow -\frac{3}{2}$ $\searrow -\frac{2}{3}$

2° $\Pi': y-5z-1=0$

$d(M,\Pi')=\frac{|1-5\cdot1-1|}{\sqrt{1+25}}=\frac{5}{\sqrt{26}}$

$d(M,\Pi')=\frac{5}{\sqrt{14}}$ $\Rightarrow$

$\frac{5}{\sqrt{26}}\neq\frac{5}{\sqrt{14}} \Rightarrow \Pi'$ није

раван чије је растојање од M једнако $\frac{5}{\sqrt{14}}$.

1° $\lambda = -\frac{3}{2} \rightsquigarrow (*)$

$5x + (-\frac{3}{2}-1)y - 5(-\frac{3}{2})z + 1 - (-\frac{3}{2}) = 0$

$5x - \frac{5}{2}y + \frac{15}{2}z + \frac{5}{2} = 0 \quad / \cdot \frac{2}{5}$

$\boxed{\alpha_1 : 2x - y + 3z + 1 = 0}$

2° $\lambda = -\frac{2}{3} \rightsquigarrow (*)$

$5x + (-\frac{2}{3}-1)y - 5(-\frac{2}{3})z + 1 - (-\frac{2}{3}) = 0$

$5x - \frac{5}{3}y + \frac{10}{3}z + \frac{5}{3} = 0 \quad | \cdot \frac{3}{5}$

$\boxed{\alpha_2 : 3x - y + 2z + 1 = 0}$

Равни α_1, α_2 садрже праву p и њихово растојање од тачке M је $\frac{5}{\sqrt{14}}$.

МЕЂУСОБНИ ПОЛОЖАЈ ДВЕ ПРАВЕ У ПРОСТОРУ

$p : \vec{p}, P \qquad q : \vec{q}, Q$

* ПОКЛАПАЈУ СЕ $\vec{p}, \vec{q}, \vec{PQ}$ колинеарни
* ПАРАЛЕЛНЕ СУ $\vec{p}, \vec{q}, \vec{PQ}$ НИСУ колинеарни

} КОЛИНЕАРНЕ ПРАВЕ $\vec{p} \times \vec{q} = \vec{0}$

* СЕКУ СЕ У ЈЕДНОЈ ТАЧКИ $\vec{p} \times \vec{q} \neq \vec{0}$

} КОМПЛАНАРНЕ ПРАВЕ $[\vec{p}, \vec{q}, \vec{PQ}] = 0$

* МИМОИЛАЗНЕ СУ $[\vec{p}, \vec{q}, \vec{PQ}] \neq 0$

22 Одредити међусобни положај правих (и пресечну тачку ако постоји):

a) $p : \frac{x-2}{1} = \frac{y-2}{0} = \frac{z-2}{-1} = t \qquad q : 2x = y,\ 3x = z$

$x = \frac{y}{2} \quad x = \frac{z}{3} \Rightarrow \underline{q : \frac{x}{1} = \frac{y}{2} = \frac{z}{3}}$

$\vec{p}(1, 0, -1)$

$P(2,2,2)$

$\vec{q}(1,2,3) \qquad Q(0,0,0)$

$\vec{PQ}(-2,-2,-2)$

$$[\vec{p}, \vec{q}, \vec{PQ}] = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \\ -2 & -2 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 3 \\ -2 & -2 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} =$$

$$= 1(-4+6) - (-2+4) = 2 - 2 = 0$$

$\Rightarrow p, q$ су компланарне праве.

$$\vec{p} \times \vec{q} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix}\vec{e}_1 - \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}\vec{e}_2 + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}\vec{e}_3 =$$

$$= 2\vec{e}_1 - 4\vec{e}_2 + 2\vec{e}_3 = \underline{(2,-4,2)} \neq \vec{0}$$

$\Rightarrow$ <u>p, q се секу у једној тачки</u>, $p \cap q = \{X\}$

Решимо систем: p: $x = t+2$, $\boxed{y = 2}$, $z = -t+2$ $\qquad q$: $2x = y$, $3x = z$ $\Rightarrow 2x = 2 \Rightarrow \boxed{x = 1}$, $\boxed{z = 3}$

Проверимо да ли је систем сагласан: $1 = t+2,\ 3 = -t+2 \Rightarrow \boxed{t = -1}$ ✓ $\Rightarrow$

$p \cap q = \{X\}, \ \boxed{X(1,2,3)}$

б) $p: \frac{x-2}{1} = \frac{y-2}{0} = \frac{z-2}{-1}$ $\qquad$ $q: \frac{x-1}{1} = \frac{y-2}{0} = \frac{z-3}{-1} = s$

$\vec{p}(1,0,-1)$ $\qquad$ $\vec{q}(1,0,-1)$

$P(2,2,2)$ $\qquad$ $Q(1,2,3)$

$\underline{\vec{p} = \vec{q}} \Rightarrow p \parallel q \quad \vee \quad p = q \quad (1)$

$\underline{P \overset{?}{\in} q}:$ $\quad q: \begin{array}{l} x = s+1 \\ y = 2 \\ z = -s+3 \end{array}$ $\qquad P(2,2,2) \rightsquigarrow q: \quad \begin{array}{l} 2 = s+1 \Rightarrow s = 1 \\ 2 = 2 \checkmark \\ 2 = -s+3 \Rightarrow s = 1 \end{array} \checkmark$

систем је сагласан

$\Rightarrow \underline{P \in q} \quad (2)$

$\overset{(1)}{\underset{(2)}{\Longrightarrow}} \quad \underline{p = q}$, ПОКЛАПАЈУ СЕ

в) $p: \frac{x-2}{1} = \frac{y-2}{0} = \frac{z-2}{-1} = t$ $\qquad$ $q: \frac{x+1}{-2} = \frac{y-2}{0} = \frac{z+7}{2} = s$

$\vec{p}(1,0,-1)$ $\qquad$ $\vec{q}(-2,0,2) \sim \vec{p}$

$P(2,2,2)$ $\qquad$ $Q(-1,2,-7)$

$\vec{q} = -2\vec{p} \Rightarrow p = q \quad \vee \quad p \parallel q \quad (1)$

$P(2,2,2) \overset{?}{\in} q: \quad \begin{array}{l} x = -2s-1 \\ y = 2 \\ z = 2s-7 \end{array}$ $\qquad P \rightsquigarrow q: \quad \begin{array}{l} 2 = -2s-1 \Rightarrow s = -\frac{3}{2} \\ 2 = 2 \\ 2 = 2s-7 \Rightarrow s = \frac{9}{2} \end{array} \Rightarrow$

систем није сагласан

$\Rightarrow \underline{P \notin q} \quad (2)$

$(1), (2) \Rightarrow \underline{p \parallel q}$, ПАРАЛЕЛНЕ СУ

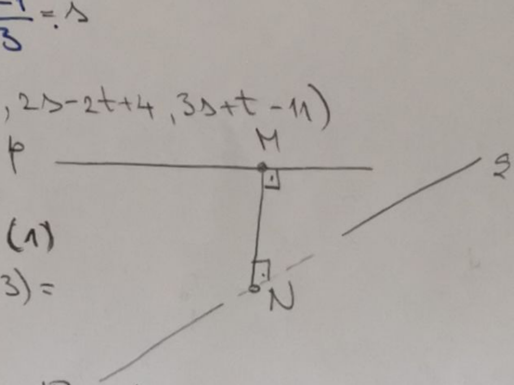
23 Одреди заједничку нормалу и растојање између мимоилазних правих $p: \frac{x-4}{1} = \frac{y+3}{2} = \frac{z-12}{-1} = t$ и $q: \frac{x-3}{-7} = \frac{y-1}{2} = \frac{z-1}{3} = s$

$M \in p: \quad M(t) = (t+4,\ 2t-3,\ -t+12)$

$N \in q: \quad N(s) = (-7s+3,\ 2s+1,\ 3s+1)$ $\quad \Bigg\} \Rightarrow \overrightarrow{MN}(-7s-t-1,\ 2s-2t+4,\ 3s+t-11)$

$\overrightarrow{MN} \perp \vec{p}: \quad \underline{0} = \overrightarrow{MN} \circ \vec{p} = (-7s-t-1, 2s-2t+4, 3s+t-11) \circ (1,2,-1) =$

$= -7s-t-1+4s-4t+8-3s-t+11 = \underline{-6s-6t+18} \quad (1)$



$\overrightarrow{MN} \perp \vec{q}: \quad \underline{0} = \overrightarrow{MN} \circ \vec{q} = (-7s-t-1, 2s-2t+4, 3s+t-11) \circ (-7,2,3) =$

$= (-7s-t-1)(-7) + 2 \cdot (2s-2t+4) + 3(3s+t-11) =$

$= 49s+4s+9s+7t-4t+3t+7+8-33 = 62s+6t-18 \quad (2)$

$$\left.\begin{array}{l}(1)\\(2)\end{array}\right\} \Rightarrow \begin{array}{l} -6s-6t+18=0 \quad /: \quad 6(-s-t+3)=0 \\ 62s+6t-18=0 \quad /: \quad 2(31s+3t-9)=0 \end{array} \Bigg\} \Rightarrow t = 3-s$$

$$31s+\cancel{9}-3s-\cancel{9}=0$$

$$\Rightarrow \boxed{\begin{array}{l} s=0 \\ t=3 \end{array}} \begin{array}{l} \rightsquigarrow N(s) \Rightarrow N(0)=(3,1,1) \\ \longrightarrow M(t) \Rightarrow M(3)=(7,3,9) \end{array}$$

$$\overrightarrow{MN}\ (-4,-2,-8) \sim (2,1,4)$$

$$d(p,q) = \|\overrightarrow{MN}\| = \sqrt{(-4)^2+(-2)^2+(-8)^2} = \sqrt{16+4+64} = 2\sqrt{4+1+16} = \underline{\underline{2\sqrt{21}}}$$

$$\boxed{MN: \quad \frac{x-3}{2} = \frac{y-1}{1} = \frac{z-1}{4}}$$

II начин за растојање

$$\left.\begin{array}{ll} \vec{p}\,(1,2,-1) & P(4,-3,12) \\ \vec{q}\,(-7,2,3) & Q(3,1,1) \end{array}\right\} \overrightarrow{PQ}(-1,4,-11)$$

РАСТОЈАЊЕ ИЗМЕЂУ МИМОИЛАЗНИХ ПРАВИХ

$$V = B \cdot h = \|\vec{p}\times\vec{q}\| \cdot \|\overrightarrow{MN}\|$$

$h = \|\overrightarrow{MN}\|$ јер је MN заједничка нормала правих p и q

$$\| \quad |[\vec{p},\vec{q},\overrightarrow{PQ}]|$$

$$\boxed{d(p,q) = \|\overrightarrow{MN}\| = \frac{|[\vec{p},\vec{q},\overrightarrow{PQ}]|}{\|\vec{p}\times\vec{q}\|}}$$

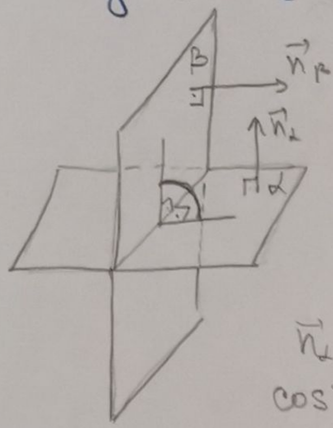
$$[\vec{p},\vec{q},\overrightarrow{PQ}] = \begin{vmatrix} 1 & 2 & -1 \\ -7 & 2 & 3 \\ -1 & 4 & -11 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 3 \\ 4 & -11 \end{vmatrix} - 2 \cdot \begin{vmatrix} -7 & 3 \\ -1 & -11 \end{vmatrix} + (-1) \cdot \begin{vmatrix} -7 & 2 \\ -1 & 4 \end{vmatrix} =$$

$$= (-22-12) - 2(77+3) - 1(-28+2) = -34-160+26 = -168$$

$$\vec{p}\times\vec{q} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 2 & -1 \\ -7 & 2 & 3 \end{vmatrix} = (6+2)\vec{e}_1 - (3-7)\vec{e}_2 + (2+14)\vec{e}_3 = (8,4,16) = 4(2,1,4)$$

$$d = \frac{|-168|}{\sqrt{4^2}\cdot\sqrt{4+1+16}} = \frac{\cancelto{42}{168}}{\cancel{4}\cdot\sqrt{21}} = \frac{42}{\sqrt{21}}\cdot\frac{\sqrt{21}}{\sqrt{21}} = \frac{42\sqrt{21}}{21} = \underline{\underline{2\sqrt{21}}}$$

(24) Израчунати (користећи дигитрон за arccos) угао између равни $\alpha: x+2y-3z-1=0$ и равни $\beta: 2x-3y+4z+2=0$.



Да бисмо одредили угао између две равни по деф. потребно је да одредимо угао између правих $p\in\alpha$, $q\in\beta$ т.д. $p, q \perp a$, $\{a\}=\alpha\cap\beta$

Такође, угао $\varphi = \measuredangle(\alpha,\beta) = \measuredangle(\vec{n}_\alpha, \vec{n}_\beta)$

$$\left.\begin{aligned}\vec{n}_\alpha\circ\vec{n}_\beta &= \|\vec{n}_\alpha\|\|\vec{n}_\beta\|\cdot\cos\measuredangle(\alpha,\beta)\\ \cos\varphi &= \cos\measuredangle(\alpha,\beta)\end{aligned}\right\}\Rightarrow \cos\varphi = \frac{\vec{n}_\alpha\circ\vec{n}_\beta}{\|\vec{n}_\alpha\|\|\vec{n}_\beta\|}$$

$\vec{n}_\alpha(1,2,-3)$
$\vec{n}_\beta(2,-3,4)$

$$\Rightarrow \cos\varphi = \frac{(1,2,-3)\circ(2,-3,4)}{\sqrt{1^2+2^2+(-3)^2}\cdot\sqrt{2^2+(-3)^2+4^2}} = \frac{1\cdot 2+2(-3)+(-3)\cdot 4}{\sqrt{1+4+9}\cdot\sqrt{4+9+16}} = \frac{2-6-12}{\sqrt{14}\cdot\sqrt{29}}$$

$$\boxed{\cos\varphi = -\frac{16}{\sqrt{406}}}$$

$$\frac{\alpha\ \text{rad}}{\pi} = \frac{\varphi \leftarrow \alpha^\circ}{180^\circ}$$

$$\varphi = \arccos\left(-\frac{16}{\sqrt{406}}\right) \approx 2{,}49\ \text{rad} \approx 142{,}5671887^\circ$$

ТУП УГАО

ОДГОВАРАЈУЋИ ОШТАР УГАО ИЗМЕЂУ α и β је $\pi-\varphi = \pi - 2{,}49\ \text{rad} = 0{,}65\ \text{rad}$

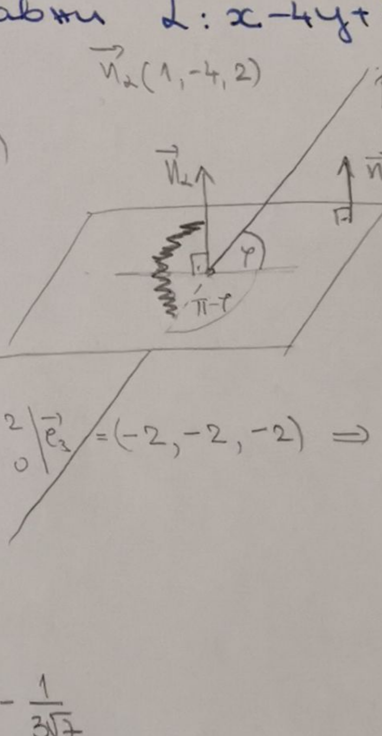
$\Rightarrow \pi-\varphi = 37{,}4328113^\circ$

(25) Израчунати (користећи дигитрон за arccos) угао између праве $p: x+2y-3z-1=0,\ x-z+2=0$ и равни $\alpha: x-4y+2z+2=0$.

$\{p\} = \beta\cap\gamma \qquad \beta: x+2y-3z-1=0, \quad \vec{n}_\beta(1,2,-3)$

$\qquad\qquad\quad \gamma: x-z+2=0, \quad \vec{n}_\gamma(1,0,-1)$

$\vec{n}_\alpha(1,-4,2)$



$$\left.\begin{aligned}p\in\beta &\Rightarrow \vec{p}\perp\vec{n}_\beta\\ p\in\gamma &\Rightarrow \vec{p}\perp\vec{n}_\gamma\end{aligned}\right\}\Rightarrow \vec{p}\sim\vec{n}_\beta\times\vec{n}_\gamma \Rightarrow$$

$$\vec{n}_\beta\times\vec{n}_\gamma = \begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 1 & 2 & -3\\ 1 & 0 & -1\end{vmatrix} = \begin{vmatrix}2 & -3\\ 0 & -1\end{vmatrix}\vec{e}_1 - \begin{vmatrix}1 & -3\\ 1 & -1\end{vmatrix}\vec{e}_2 + \begin{vmatrix}1 & 2\\ 1 & 0\end{vmatrix}\vec{e}_3 = (-2,-2,-2) \Rightarrow \underline{\vec{p}(1,1,1)}$$

$$\measuredangle(\alpha,p) = \frac{\pi}{2} - \measuredangle(\vec{p},\vec{n}_\alpha)$$

$$\cos\measuredangle(\vec{p},\vec{n}_\alpha) = \frac{\vec{p}\circ\vec{n}_\alpha}{\|\vec{p}\|\|\vec{n}_\alpha\|} = \frac{(1,1,1)\circ(1,-4,2)}{\sqrt{1^2+1^2+1^2}\ \sqrt{1^2+(-4)^2+2^2}} = \frac{1-4+2}{\sqrt{3}\cdot\sqrt{21}} = \frac{-1}{3\sqrt{7}} = -\frac{1}{3\sqrt{7}}$$

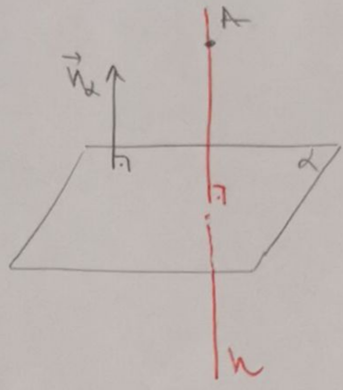
$\Rightarrow \measuredangle(\vec{p},\vec{n}_\alpha) = \arccos\left(-\frac{1}{3\sqrt{7}}\right) \approx 1{,}7\ \text{rad} \leftarrow$ ТУП УГАО ИЗМЕЂУ $\vec{p}$ и $\vec{n}_\alpha$

$\Rightarrow \varphi = \measuredangle(\alpha,p) = \measuredangle(\vec{p},\vec{n}_\alpha) - \frac{\pi}{2} = 1{,}7\ \text{rad} - \frac{\pi}{2} \approx \underline{0{,}13\ \text{rad}}$

ОШТАР УГАО ИЗМЕЂУ ПРАВЕ И РАВНИ.

26) Одреди ј-ну нормале из тачке $A(2,3,-1)$ на раван $\alpha: 2x+y-4z+5=0$

$n: n \ni A, \ \vec{n} \perp \alpha$



$$\left.\begin{array}{l}\vec{n}_\alpha \perp \alpha \\ \vec{n} \perp \alpha\end{array}\right\} \Rightarrow \underline{\vec{n} = \vec{n}_\alpha(2,1,-4)}$$

$$\Rightarrow \boxed{n: \ \frac{x-2}{2} = \frac{y-3}{1} = \frac{z+1}{-4}}$$

27) Одреди λ тако да се праве $p: \frac{x-2}{3} = \frac{y+4}{5} = \frac{z-1}{-2}$ и $q: \frac{x-\lambda}{2} = \frac{y-3}{1} = \frac{z+5}{0}$ секу. Које су координате пресечне тачке?

$p: \ \vec{p}(3,5,-2) \quad P(2,-4,1)$

$q: \ \vec{q}(2,1,0) \quad Q(\lambda,3,-5)$

$\Rightarrow \overrightarrow{PQ}(\lambda-2,7,-6)$

I начин:

p сече q $\Leftrightarrow$ $\vec{p}, \vec{q}, \overrightarrow{PQ}$ компланарни вектори $\Leftrightarrow \underline{[\vec{p}, \vec{q}, \overrightarrow{PQ}] = 0}$

$$\underline{0} = [\vec{p}, \vec{q}, \overrightarrow{PQ}] = \begin{vmatrix} 3 & 5 & -2 \\ 2 & 1 & 0 \\ \lambda-2 & 7 & -6 \end{vmatrix} = -2\begin{vmatrix} 2 & 1 \\ \lambda-2 & 7 \end{vmatrix} - 0 + (-6)\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = -2(14-\lambda+2) - 6(3-10) =$$

$$= -2(16-\lambda) - 6(-7) = -32 + 2\lambda + 42 = \underline{10 + 2\lambda}$$

$10 + 2\lambda = 0 \Rightarrow \boxed{\lambda = -5} \rightsquigarrow q$

$\underline{p \cap q = \{S\}} \Rightarrow S \in p: \ x = 3t+2, \ y = 5t-4, \ z = -2t+1 \qquad S \in q: \ x = 2s-5, \ y = s+3, \ z = -5 \qquad \Rightarrow$ решимо систем

$$\left.\begin{array}{l} 3t+2 = 2s-5 \\ 5t-4 = s+3 \\ -2t+1 = -5 \Rightarrow \boxed{t=3} \end{array}\right] \Rightarrow \left.\begin{array}{l} 2s-5 = 11 \\ s+3 = 11 \Rightarrow \boxed{s=8} \end{array}\right] \Rightarrow 16-5 = 11 \ \checkmark$$ систем је сагласан

$\Rightarrow S: \ x = 2 \cdot 8 - 5 = 11, \ y = 8+3 = 11, \ z = -5 \qquad \boxed{S(11,11,-5)}$

II начин: $\{S\} = p \cap q \Rightarrow S \in p: \ x = 3t+2, \ y = 5t-4, \ z = -2t+1 \qquad S \in q: \ x = 2s+\lambda, \ y = s+3, \ z = -5 \ \}$ решимо систем

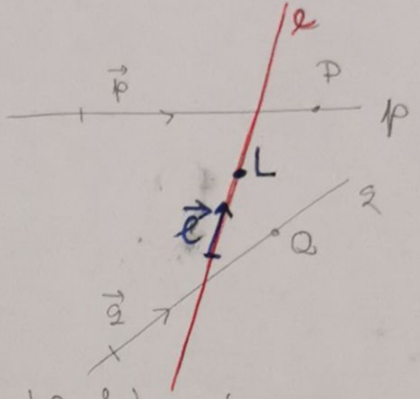
$$\left.\begin{array}{l} 3t+2 = 2s+\lambda \\ 5t-4 = s+3 \\ -2t+1 = -5 \Rightarrow \boxed{t=3} \end{array}\right] \Rightarrow \left.\begin{array}{l} 11 = 2s+\lambda \\ 11 = s+3 \Rightarrow \boxed{s=8} \end{array}\right] \Rightarrow \boxed{\lambda = -5}$$

$s = 8, \ \lambda = -5 \rightsquigarrow q \Rightarrow S: \ x = 16-5 = 11, \ y = 8+3 = 11, \ z = -5$

$\boxed{S(11,11,-5)}$

(28) Одредити ј-ну праве која садржи тачку $L(2,-1,7)$ и сече праве $p: \frac{x-1}{2} = \frac{y-4}{-3} = \frac{z-3}{1}$ и $q: \frac{x-2}{-1} = \frac{y-11}{-3} = \frac{z+2}{0}$.

$l: l \ni L(2,-1,7) \quad \vec{l}(a,b,c)$



$p: \vec{p}(2,-3,1)$
$P(1,4,3)$
$q: \vec{q}(-1,-3,0)$
$Q(2,11,-2)$

I начин:

p и l се секу да су компланарне $\Leftrightarrow$

$\vec{p}, \vec{l}, \vec{LP}$ су компланарни вектори $\Leftrightarrow$

$[\vec{p}, \vec{l}, \vec{LP}] = 0 \qquad \vec{LP} = (-1,5,-4)$

$$\underline{0} = \begin{vmatrix} 2 & -3 & 1 \\ a & b & c \\ -1 & 5 & -4 \end{vmatrix} = 2 \cdot \begin{vmatrix} b & c \\ 5 & -4 \end{vmatrix} - (-3)\begin{vmatrix} a & c \\ -1 & -4 \end{vmatrix} + 1 \cdot \begin{vmatrix} a & b \\ -1 & 5 \end{vmatrix} = 2(-4b-5c) + 3(-4a+c) + 5a + b =$$

$$= -8b - 10c - 12a + 3c + 5a + b = \underline{-7a - 7b - 7c}$$

$-7(a+b+c) = 0$ тј. $\boxed{a+b+c=0}$ (1)

q и l се секу да су компланарне $\Leftrightarrow [\vec{q}, \vec{l}, \vec{LQ}] = 0 \qquad \vec{LQ}(0,12,-9)$

$$0 = \begin{vmatrix} -1 & -3 & 0 \\ a & b & c \\ 0 & 12 & -9 \end{vmatrix} = -1 \begin{vmatrix} b & c \\ 12 & -9 \end{vmatrix} - (-3)\begin{vmatrix} a & c \\ 0 & -9 \end{vmatrix} = -(-9b-12c) + 3(-9a) =$$

$$= 9b + 12c - 27a$$

$3(-9a+3b+4c) = 0$ тј. $\boxed{-9a+3b+4c=0}$ (2)

$\underline{(1), (2)}$:

$a+b+c=0 \quad / \cdot (-3)$
$-9a+3b+4c=0$

$a+b+c=0$
$-9a+\cancel{3b}+4c-3a-\cancel{3b}-3c=0$

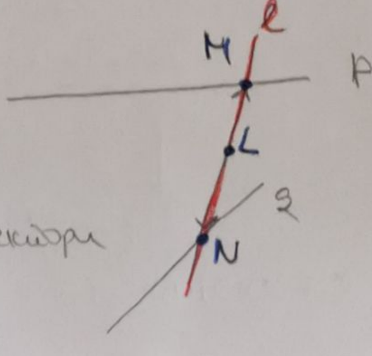
$a+b+c=0$
$-12a+c=0 \Rightarrow \boxed{c=12a}$

$b = -a - c = -a - 12a = -13a$ тј. $\boxed{b=-13a}$

$\Rightarrow \vec{l}(a,b,c) = (a,-13a,12a) \sim \underline{(1,-13,12)}$

$$\boxed{l: \frac{x-2}{1} = \frac{y+1}{-13} = \frac{z-7}{12}}$$

II начин



$M \in p \Rightarrow M(2t+1, -3t+4, t+3)$

$N \in q \Rightarrow N(-s+2, -3s+11, -2)$

M, L, N су колинеарне $\Rightarrow \vec{LM}, \vec{LN}$ су колинеарни вектори

$\Rightarrow \exists k \in \mathbb{R} \setminus \{0\}$ тј. $\underline{\vec{LM} = k \cdot \vec{LN}}$

$$\left.\begin{array}{l}\overrightarrow{LM}(2t-1,\ -3t+5,\ t-4)\\ \overrightarrow{LN}(-s,\ -3s+12,\ -9)\\ \overrightarrow{LM}=k\cdot\overrightarrow{LN}\end{array}\right\}\Rightarrow$$

$$(2t-1,\ -3t+5,\ t-4)=k(-s,\ -3s+12,\ -9)$$

тј.
$$\begin{aligned} 2t-1&=-ks \quad /\cdot(-3)\\ -3t+5&=k(-3s+12)\\ t-4&=-9k\end{aligned}$$

$$\begin{aligned} \boxed{-6t+3=3ks}&\\ -3t+5&=-3ks+12k \quad +\\ t-4&=-9k\end{aligned}$$

$$\begin{aligned} \boxed{-6t+3=3ks}&\\ -9t+8&=12k \quad +\\ t-4&=-9k \quad \cdot 9\end{aligned}$$

$$\begin{aligned} -6t+3&=3ks\\ t-4&=-9k\\ -\cancel{9t}+8+\cancel{9t}-36&=12k-81k\\ -28&=-69k \Rightarrow \boxed{k=\frac{28}{69}}\end{aligned}$$

$$\Rightarrow t=-9\cdot\frac{28}{69}+4=\frac{-9\cdot28+4\cdot69}{69}=\frac{4(-9\cdot7+69)}{69}=\frac{4\cdot6}{69}$$

тј. $\boxed{t=\frac{24}{69}\overset{|:3}{=}\frac{8}{23}}$

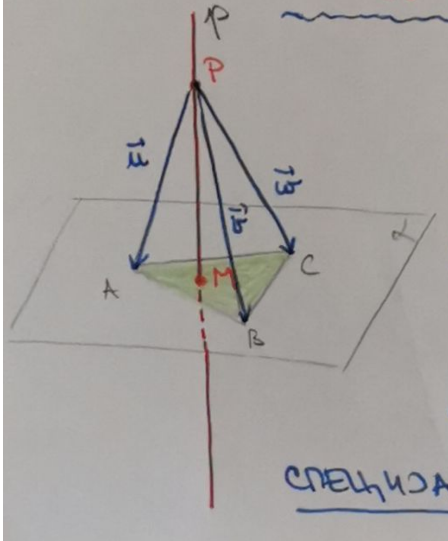
$$\Rightarrow -6\cdot\frac{8}{23}+3=3\cdot\frac{28}{69}\cdot s \Rightarrow s=\frac{\frac{-6\cdot8+3\cdot23}{\cancel{23}}}{\frac{\cancel{3}\cdot28}{\cancel{69}\ \cancel{3}}}=\frac{-48+69}{28}=\frac{21}{28}=\frac{3}{4}$$

$$\boxed{s=\frac{3}{4}}$$

$$\Rightarrow \overrightarrow{LN}(-s,\ -3s+12,\ -9)=\left(-\frac{3}{4},\ -3\cdot\frac{3}{4}+12,\ -9\right)=\left(-\frac{3}{4},\ \frac{39}{4},\ -9\right)\Big|\cdot\frac{4}{3}\sim\underline{(-1,\ 13,\ -12)}$$

$$\Rightarrow \boxed{l:\ \frac{x-2}{-1}=\frac{y+1}{13}=\frac{z-7}{-12}}$$

ПРОДОР ПРАВЕ КРОЗ ТРОУГАО



p СЕЧЕ $\triangle ABC$ у тачки M $\Leftrightarrow$ p ПРИПАДА ТРИЕДРУ $PABC$

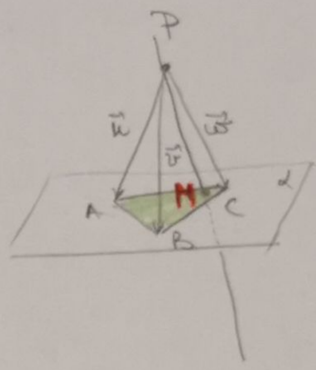
$\Leftrightarrow$ БАЗЕ $(\vec{u},\vec{v},\vec{p}),(\vec{v},\vec{w},\vec{p}),(\vec{w},\vec{u},\vec{p})$ СУ ИСТЕ ОРЈЕНТАЦИЈЕ

ПОСЛЕДИЦА: Вектори $(\vec{u},\vec{v},\vec{w})$ простора, чине базу „+" орјентације ако је $[\vec{u},\vec{v},\vec{w}]>0$, а „−" орјентације ако је $[\vec{u},\vec{v},\vec{w}]<0$.

$$\Leftrightarrow \operatorname{sign}[\vec{u},\vec{v},\vec{p}]=\operatorname{sign}[\vec{v},\vec{w},\vec{p}]=\operatorname{sign}[\vec{w},\vec{u},\vec{p}]$$

СПЕЦИЈАЛНО: нпр. $[\vec{u},\vec{v},\vec{p}]=0 \Rightarrow M\in AB$ правој

29) Испитати да ли права $p: \frac{x-1}{2} = \frac{y}{-6} = \frac{z+1}{-4}$ сече троугао $\triangle ABC$, ако је $A(2,4,6)$, $B(-4,2,0)$, $C(6,4,-2)$. У случају да сече, одредити координате пресечне тачке.



$p: \vec{p}(2,-6,-4) \sim \underline{(1,-3,-2)}$ $\qquad P(1,0,-1)$

$\vec{PA}(1,4,7)$
$\vec{PB}(-5,2,1)$
$\vec{PC}(5,4,-1)$

$$[\vec{PA},\vec{PB},\vec{p}] = \left|\begin{array}{ccc|cc} 1 & 4 & 7 & 1 & 4 \\ -5 & 2 & 1 & -5 & 2 \\ 1 & -3 & -2 & 1 & -3 \end{array}\right| = 2\cdot(-2)+4\cdot1\cdot1+7\cdot(-5)(-3) - 4(-5)(-2) - 1\cdot1\cdot(-3) - 7\cdot2\cdot1 =$$
$$= -4+4+105-40+3-14 = \underline{54>0}$$

$$[\vec{PB},\vec{PC},\vec{p}] = \left|\begin{array}{ccc|cc} -5 & 2 & 1 & -5 & 2 \\ 5 & 4 & -1 & 5 & 4 \\ 1 & -3 & -2 & 1 & -3 \end{array}\right| = 40-2\cancel{-15}+20\cancel{+15}-4 = \underline{54>0}$$

$$[\vec{PC},\vec{PA},\vec{p}] = \left|\begin{array}{ccc|cc} 5 & 4 & -1 & 5 & 4 \\ 1 & 4 & 7 & 1 & 4 \\ 1 & -3 & -2 & 1 & -3 \end{array}\right| = -40+28+3+8+105+4 = \underline{108>0}$$

Мешовити производи су истог знака ⇒ базе су исте оријентације тј. p СЕЧЕ $\triangle ABC$

ПРЕСЕЧНА ТАЧКА $\quad M = P + t\vec{p},\ t\in\mathbb{R}$

$\vec{u}=\vec{PA},\ \vec{v}=\vec{PB},\ \vec{w}=\vec{PC}$
$M-P = t\vec{p}$ тј. $\boxed{\vec{PM}=t\vec{p}}$

$M \in$ равни $\triangle ABC \iff [\vec{AB},\vec{AC},\vec{AM}]=0 \iff$

$$\underset{\sim}{0} = [\vec{PB}-\vec{PA}, \vec{PC}-\vec{PA}, \vec{PM}-\vec{PA}] = [\vec{v}-\vec{u}, \vec{w}-\vec{u}, t\vec{p}-\vec{u}] =$$
$$= [\vec{v}, \vec{w}-\vec{u}, t\vec{p}-\vec{u}] - [\vec{u}, \vec{w}-\vec{u}, t\vec{p}-\vec{u}] =$$
$$= [\vec{v},\vec{w},t\vec{p}-\vec{u}] - [\vec{v},\vec{u},t\vec{p}-\vec{u}] - [\vec{u},\vec{w},t\vec{p}-\vec{u}] - \cancelto{0}{[\vec{u},-\vec{u},t\vec{p}-\vec{u}]} =$$
$$= [\vec{v},\vec{w},t\vec{p}] - [\vec{v},\vec{w},\vec{u}] - [\vec{v},\vec{u},t\vec{p}] + \cancelto{0}{[\vec{v},\vec{u},\vec{u}]} - [\vec{u},\vec{w},t\vec{p}] + \cancelto{0}{[\vec{u},\vec{w},\vec{u}]} =$$
$$= t[\vec{v},\vec{w},\vec{p}] - [\vec{u},\vec{v},\vec{w}] - t[\vec{v},\vec{u},\vec{p}] - t[\vec{u},\vec{w},\vec{p}] =$$
$$= -[\vec{u},\vec{v},\vec{w}] + t[\vec{v},\vec{w},\vec{p}] + t[\vec{u},\vec{v},\vec{p}] + t[\vec{w},\vec{u},\vec{p}] =$$
$$= -[\vec{u},\vec{v},\vec{w}] + t\left([\vec{u},\vec{v},\vec{p}] + [\vec{v},\vec{w},\vec{p}] + [\vec{w},\vec{u},\vec{p}]\right)$$

$$\boxed{t = \frac{[\vec{u},\vec{v},\vec{w}]}{[\vec{u},\vec{v},\vec{p}] + [\vec{v},\vec{w},\vec{p}] + [\vec{w},\vec{u},\vec{p}]}}$$

$\{M\} = p \cap \alpha$:
$$[\vec{u},\vec{v},\vec{w}] = \left|\begin{array}{ccc|cc} 1 & 4 & 7 & 1 & 4 \\ -5 & 2 & 1 & -5 & 2 \\ 5 & 4 & -1 & 5 & 4 \end{array}\right| = -2\cancel{+20}-140\cancel{-20}-4-70 = -216 \qquad t = \frac{-216}{54+54+108} = -1$$

$t=-1 \Rightarrow M = (1,0,-1) - 1\cdot(1,-3,-2) = (0,3,1)$ $\qquad \boxed{M(0,3,1)}$

ПРЕСЕК ТРОУГЛА И РАВНИ

ПРЕСЕК МОЖЕ БИТИ:
* $\emptyset$
* ТЕМЕ ТРОУГЛА
* ДУЖ (КОЈА НИЈЕ ИВИЦА ТРОУГЛА)
* ИВИЦА
* САМ ТРОУГАО

(30) Одредити пресек равни $\alpha: x-y+2z-3=0$ и троугла ABC, ако је $A(1,-2,0)$, $B(-1,2,3)$, $C(2,1,3)$.

ПРОВЕРИМО ШТА ЈЕ ПРЕСЕК СВАКЕ ОД ИВИЦА ТРОУГЛА СА РАВНИ!

• ДУЖ $[AB]$: $M = A + t\vec{AB}$, $t \in [0,1]$ ← реп сваке тачке са дужи AB

$\vec{AB}(-2,4,3)$ $\quad$ $[AB]: \begin{array}{l} x=-2t+1 \\ y=4t-2 \\ z=3t \end{array}$ $, t\in[0,1] \rightsquigarrow \alpha$

$-2t+1-(4t-2)+2\cdot 3t-3=0$

$-\cancel{2t}+1-\cancel{4t}+2+\cancel{6t}-3=0$

$0=0 \quad \checkmark \quad \forall t\in[0,1] \Rightarrow \boxed{[AB]\in\alpha}$

• ДУЖ $[AC]$: $N = A + s\cdot\vec{AC}$, $s\in[0,1]$

$\vec{AC}(1,3,3)$ $\quad$ $[AC]: \begin{array}{l} x=s+1 \\ y=3s-2 \\ z=3s \end{array}$ $, s\in[0,1] \rightsquigarrow \alpha$

$s+1-(3s-2)+2\cdot 3s-3=0$

$s+1-3s+2+6s-3=0$

$4s=0 \Rightarrow s=0 \Rightarrow N=A$ тј. $\boxed{A\in\alpha}$

⇒ C и све остале тачке са дужи AC изузев A НЕ ПРИПАДАЈУ равни α. (*)

$\overset{(*)}{\Rightarrow}$ не морамо проверавати за $[BC]$ дуж.

$\Rightarrow \quad \alpha \cap \triangle ABC = [AB]$