Vektorska polja i komutator

$F=(F_1,F_2,\dots,F_n) \longleftrightarrow F=F_1\frac{\partial}{\partial x_1}+F_2\frac{\partial}{\partial x_2}+\dots+F_n\frac{\partial}{\partial x_n}$

$F: C^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)$

$F(f)=F_1\frac{\partial f}{\partial x_1}+\dots+F_n\frac{\partial f}{\partial x_n}$

$F:\mathbb{R}^n\to\mathbb{R}^n$

$F(x_1,\dots,x_n)=(F_1(x_1,\dots,x_n),\dots,F(x_1,\dots,x_n))$

- $F(\lambda f+\mu g)=\lambda F(f)+\mu F(g)$
- $F(fg)=F(f)g+fF(g)$

komutator: $[F,G]=F\circ G-G\circ F$

$[F,G]=0 \Leftrightarrow F\circ G=G\circ F$

vektorska polja

$[F,G](f)=F(\underbrace{G(f)}_{C^\infty})-G(\underbrace{F(f)}_{C^\infty})$ ($f\in C^\infty$, $F(f)\in C^\infty$, $G(f)\in C^\infty$)

smene promenljive: $\varphi:(x_1,\dots,x_n)\mapsto(y_1,\dots,y_n)$

$(\forall k)\quad \frac{\partial}{\partial y_k}=\sum_{l=1}^{n}\frac{\partial x_l}{\partial y_k}\cdot\frac{\partial}{\partial x_l}$

$\varphi_* F=G$ → y koord $y_1,\dots,y_n$

push forward (guranje) ↕ y koord $x_1,\dots,x_n$

1) $X=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$ u $\mathbb{R}^2$ $\quad X\longleftrightarrow(-y,x)$

Naći tok ϕ^t koji ide kroz $(1,0)$

$\frac{d\phi^t}{dt}=X(\phi^t)$ ($\phi^t\in C^\infty$), $\phi^0=id$

$\phi^t(x)=(u(t),v(t))$ → X u tački ϕ^t

$(u'(t),v'(t))=(-v(t),u(t))$

$u'=-v$, $v'=u$ $\quad\dots\quad u=c_1\sin t+c_2\cos t$, $v=-c_1\cos t+c_2\sin t$

$$\left.\begin{aligned} u' &= -v \\ v' &= u \end{aligned}\right\} \dots \qquad \begin{aligned} u &= c_1 \sin t + c_2 \cos t \\ v &= -c_1 \cos t + c_2 \sin t \end{aligned}$$

$\phi^0(x_0) = x_0 = (x_1, x_2)$

$u(0) = c_2$ $\qquad c_2 = \pi_1 x_0 = x_1$

$v(0) = -c_1$ $\qquad c_1 = -\pi_2 x_0 = -x_2$

$$\phi^t(\underbrace{x_1, x_2}_{\in \mathbb{R}^2}) = (-x_2 \sin t + x_1 \cos t,\ x_2 \cos t + x_1 \sin t)$$

② Найти с-ти за кое $\phi^t(x,y) = (\underbrace{x\cos t + y \sin t}_{f_1}, \underbrace{-x\sin t + y\cos t}_{f_2})$

$$\frac{d\phi^t(x,y)}{dt} = \underbrace{X}_{?}(\phi^t(x,y))$$

$$\frac{d\phi^t(x,y)}{dt} = (-x\sin t + y\cos t,\ -x\cos t - y\sin t) = (f_2, -f_1) = X(\phi^t(x,y)) = X(f_1, f_2)$$

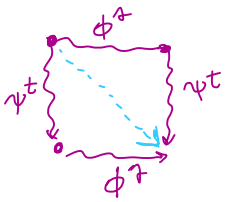
$$\Rightarrow X(x,y) = (y, -x)$$

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

③ $X = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 2x_3\frac{\partial}{\partial x_3}$, $\phi^t(x_1, x_2, x_3) = (-x_2 \sin t + x_1 \cos t,\ x_2 \cos t + x_1 \sin t,\ e^t x_3)$

а) ψ^t ?

б) ψ и ϕ да ли комутират? $\left(\psi^t \circ \phi^s = \phi^s \circ \psi^t\right)$



а) $\psi^t = (u, v, w)$

$$\frac{d\psi^t}{dt} = X(\psi^t) \Rightarrow \begin{aligned} u' &= u \Rightarrow u(t) = c_1 e^t \\ v' &= v \Rightarrow v(t) = c_2 e^t \\ w' &= 2w \Rightarrow w(t) = c_3 e^{2t} \end{aligned}$$

$$\left.\begin{aligned} \psi^t(x_1, x_2, x_3)\big|_{t=0} &= (x_1, x_2, x_3) \\ \| \end{aligned}\right\} \Rightarrow \psi^t(x_1, x_2, x_3) = (x_1 e^t, x_2 e^t, x_3 e^{2t})$$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \Rightarrow \psi^t(x_1,x_2,x_3) = (x_1 e^t, x_2 e^t, x_3 e^{2t})$

(c_1,c_2,c_3)

б) $\psi^t(x_1,x_2,x_3) = (x_1 e^t, x_2 e^t, x_3 e^{2t})$

$\phi^s(x_1,x_2,x_3) = (-x_2 \sin s + x_1 \cos s,\ x_2 \cos s + x_1 \sin s,\ e^s x_3)$

$\psi^t \circ \phi^s(x_1,x_2,x_3) = \psi^t(\ldots) = \left((-x_2 \sin s + x_1 \cos s)e^t,\ (x_2 \cos s + x_1 \sin s)e^t,\ (e^s x_3)\cdot e^{2t}\right)$))

$\phi^s \circ \psi^t(x_1,x_2,x_3) = \phi^s(\ldots) = (-x_2 e^t \sin s + x_1 e^t \cos s,\ x_2 e^t \cos s + x_1 e^t \sin s,\ e^s \cdot x_3 e^{2t})$

$\Rightarrow \psi$ и ϕ комутират

да смята Катя в ц. 4 за ϕ^t и изразяват $[X,Y]$

Т $F \rightsquigarrow \phi^t$, $G \rightsquigarrow \psi^t$ $\quad [F,G]=0 \Leftrightarrow \phi$ и ψ комутират

④ $X_1 = \frac{\partial}{\partial x}$, $Y_1 = \frac{\partial}{\partial y}$

$X_2 = \frac{\partial}{\partial x}$, $Y_2 = (1+x^2)\frac{\partial}{\partial y}$

Катя $[X_1,Y_1]$, $[X_2,Y_2]$

$$[X_1,Y_1](f) = X_1(Y_1(f)) - Y_1(X_1(f)) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f\right) = \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} = 0$$

↳ 2 пъти извади комутират

$$[X_2,Y_2](f) = \frac{\partial}{\partial x}\left((1+x^2)\frac{\partial f}{\partial y}\right) - (1+x^2)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{2(1+x^2)}{\partial x}\frac{\partial f}{\partial y} + (1+x^2)\frac{\partial^2 f}{\partial x \partial y} - (1+x^2)\frac{\partial^2 f}{\partial y \partial x} = 2x\frac{\partial f}{\partial y}$$

$\Rightarrow [X_2,Y_2] = 2x\frac{\partial}{\partial y} = (0, 2x)$

⑤ $[X, fY] = X(f)Y + f[X,Y]$, $f \in C^\infty(\mathbb{R}^n)$

$$[X,fY](g) = X(fY(g)) - fY(X(g)) = X(f)\cdot Y(g) + f\,X(Y(g)) - f\cdot Y(X(g)) = X(f)\,Y(g) + f\left(X(Y(g)) - Y(X(g))\right) =$$

$$= \left(X(f)\cdot Y + f\cdot[X,Y]\right)(g)$$

произв. фун. f и $Y(g)$

$\Rightarrow [X,fY] = X(f)\,Y + f\,[X,Y]$

$\Rightarrow [X, fY] = X(f)\,Y + f\,[X,Y]$

Дополнительно: доказать $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

⑥ $\{xy > 0\} \subseteq \mathbb{R}^2$, $X = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $Y = 2\sqrt{xy}\,\frac{\partial}{\partial x}$

а) $[X,Y]$

б) $x = uv^2$, $y = u$

выразить X и Y у новыми коорд

$\Rightarrow \frac{x}{y} = v^2 \Rightarrow v = \sqrt{\frac{x}{y}}$, $u = y$

а) $[X,Y] = \left(\frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(2\sqrt{xy}\,\frac{\partial}{\partial x}\right) - 2\sqrt{xy}\,\frac{\partial}{\partial x}\left(\frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$

$= \frac{x}{y}\frac{\partial}{\partial x}\left(2\sqrt{xy}\,\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(2\sqrt{xy}\,\frac{\partial}{\partial x}\right) - 2\sqrt{xy}\,\frac{\partial}{\partial x}\left(\frac{x}{y}\frac{\partial}{\partial x}\right) - 2\sqrt{xy}\,\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) =$

$= \frac{x}{y}\cdot\left(\sqrt{\frac{y}{x}}\,\frac{\partial}{\partial x} + 2\sqrt{xy}\,\frac{\partial^2}{\partial x^2}\right) + \left(\sqrt{\frac{x}{y}}\,\frac{\partial}{\partial x} + 2\sqrt{xy}\,\frac{\partial^2}{\partial y \partial x}\right) - 2\sqrt{xy}\left(\frac{1}{y}\frac{\partial}{\partial x} + \frac{x}{y}\frac{\partial^2}{\partial x^2}\right) + 2\sqrt{xy}\,\frac{\partial^2}{\partial x \partial y} =$

$= 2\sqrt{\frac{x}{y}}\,\frac{\partial}{\partial x} - 2\sqrt{\frac{x}{y}}\,\frac{\partial}{\partial x} = 0$

б) $\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} = v^2\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = X$

$\frac{\partial x}{\partial u} = \frac{\partial(uv^2)}{\partial u} = v^2$

$\frac{\partial y}{\partial u} = \frac{\partial u}{\partial u} = 1$

$\frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} = 2uv\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} = 2\sqrt{xy}\,\frac{\partial}{\partial x} = Y$

$\varphi(x,y) = \left(y, \sqrt{\frac{x}{y}}\right)$, $\varphi_* X = \frac{\partial}{\partial u}$, $\varphi_* Y = \frac{\partial}{\partial v}$

$\varphi_*[X,Y] \overset{\text{пред}}{=} [\varphi_* X, \varphi_* Y] = \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right] = 0$

$\Rightarrow [X,Y] = 0$

⑦ Да ли постоји смена, тд X_2, Y_2 из зад ⑤ будут координатна?

$X_2 = \frac{\partial}{\partial$

$X_2 = \frac{\partial}{\partial x}$

$Y_2 = (1+x^2)\frac{\partial}{\partial y}$

$u(x,y)$ } ?
$v(x,y)$ }

$\frac{\partial}{\partial x} = X_2 = \frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}$

$(1+x^2)\frac{\partial}{\partial y} = Y_2 = \frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y}$

$\frac{\partial x}{\partial u} = 1, \ \frac{\partial y}{\partial u} = 0$

$\frac{\partial x}{\partial v} = 0, \ \frac{\partial y}{\partial v} = (1+x^2)$

ТЕШКО

$\varphi_* X_2 = \frac{\partial}{\partial u}$

$\varphi_* Y_2 = \frac{\partial}{\partial v}$

$\Rightarrow \varphi_*[X_2, Y_2] = 0 \overset{(\varphi_*)^{-1}}{\Rightarrow} [X_2, Y_2] = 0$ ↯

$\varphi_*[X_2, Y_2] = 0$: $y(u,v)$ коорд

$[X_2, Y_2] = 0$: $y(x,y)$ коорд

⑧ $X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$

$Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$

$Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

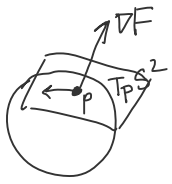
$y \ \mathbb{R}^3$

сфера $S^2 = \{x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$

а) X, Y, Z тангентни на сферу

б) $[X,Y], [X,Z], [Y,Z]$ тангентни на сферу

а)



$x^2+y^2+z^2=1$

$F(x,y,z) = C \Rightarrow (\nabla F)_p \perp T_p S^2$

X тангентна на $S^2 \Leftrightarrow X \perp \nabla F$

$\nabla F = (2x, 2y, 2z)$

$\langle \nabla F, X \rangle = \langle (2x, 2y, 2z), (0, -z, y) \rangle = 0$ ✓

y, z - симпл

б) $[X,Y] = \ldots = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

6) $[X,Y] = \cdots = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

$\langle \nabla F, [X,Y] \rangle = 0$