Tokovi

$$\frac{d}{dt}\phi^{t}(\underset{\in\mathbb{R}^n}{x}) = F(\phi^t(x), t) \quad , \quad \phi^0 = id$$

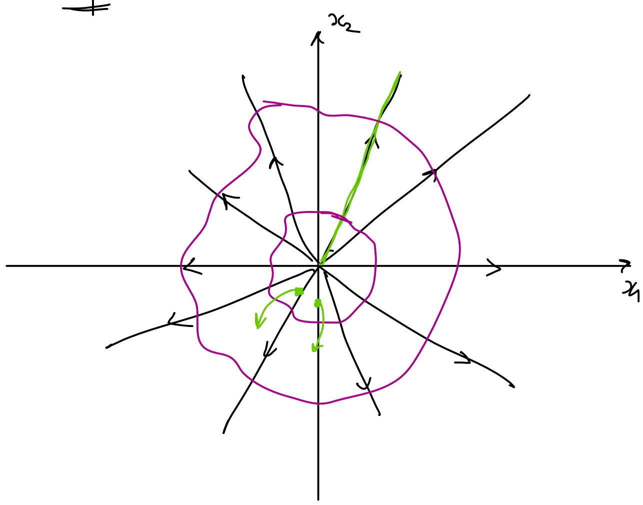
време

посматрамо ~~трансформацију~~ простора ($\mathbb{R}^n$)

ϕ^t називамо током век. поља F

$$\phi^t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(U \subseteq \mathbb{R}^n ,\ \phi^t\ U \to \mathbb{R}^n)$$

пр.



$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ОР $\quad x_1 = c_1 e^t$
$\quad\quad x_2 = c_2 e^t$

$$\phi^t\underbrace{(x_1, x_2)}_{\in\mathbb{R}^2} = (x_1 e^t, x_2 e^t) \quad , \quad \phi^t : \mathbb{R}^2 \to \mathbb{R}^2$$

пр $\quad X' = AX \rightsquigarrow \underbrace{\phi^t(X_0) = e^{tA} X_0 \text{ је ток}} \quad , \quad F(X,t) = A \cdot X$

доказ. $\quad$ 1) $\frac{d}{dt}\phi^t(X_0) = F(\phi^t(X_0), t)$

$$\left.\begin{aligned} &\frac{d}{dt}(e^{tA}X_0) = A e^{tA} \cdot X_0 \\ &F(\phi^t(X_0), t) = A \cdot \phi^t(X_0) = A\, e^{tA} \cdot X_0 \end{aligned}\right\} = \checkmark$$

2) $\phi^0 = id$

$$\phi^0(X_0) = e^{0A} X_0 = E \cdot X_0 = X_0 \quad \checkmark$$

① Наћи токове за ДЈ:

а) $x' = x + 2t$

б) $x' = -x$
$\quad\ y' = x^2 + y$

а) ОР: $x(t) = ce^{t} - 2t - 2$

$$x(0) = c - 2\cdot 0 - 2 = c - 2 = x_0$$

$$c = 2 + x_0$$

$$\phi^t(x_0) = (2+x_0)e^{t} - 2t - 2$$

провера: 1) $\frac{d}{dt}\phi^t(x_0) = \frac{d}{dt}\left((2+x_0)e^{t} - 2t - 2\right) = (2+x_0)e^{t} - 2$

$\qquad\qquad ||\ \checkmark$

$F(\phi^t(x_0), t) = \phi^t(x_0) + 2t = (2+x_0)e^{t} - 2t - 2 + 2t$

2) $\phi^0(x_0) = 2 + x_0 - 2 = x_0\ \checkmark$

б) $x' = -x \Rightarrow x(t) = c_1 e^{-t}$

$y' = x^2 + y = c_1^2 e^{-2t} + y \Rightarrow y(t) = c_2 e^{t} - \frac{1}{3}c_1^2 e^{-2t}$

$x(0) = x_0 \Rightarrow c_1 = x_0$

$y(0) = y_0 \Rightarrow y_0 = c_2 - \frac{1}{3}x_0^2 \Rightarrow c_2 = y_0 + \frac{1}{3}x_0^2$

$$\phi^t(x_0, y_0) = \left(x_0 e^{-t}, (y_0 + \tfrac{1}{3}x_0^2)e^{t} - \tfrac{1}{3}x_0^2 e^{-2t}\right)$$

Једнопараметарска фамилија пресликавања (ЈФП)

$\phi^t : M \to M,\ \forall t \in \mathbb{R}$
$(\mathbb{R}^n = M)$

1) $\phi^{t+s} = \phi^t \circ \phi^s$

2) $\phi^0 = id$

$\Longleftrightarrow$ дејство групе $(\mathbb{R}, +)$ на M

② Проверити да ли су дате ЈФП у $\mathbb{R}^n$

а) $\phi^t(x) = (t+1)\, x$

б) $\psi^t(x) = e^{t}\, x$

в) $\theta^t(x) = t \cdot \underbrace{(1, \dots, 1)}_{n} + x$

в) 1) $\theta^{t+s} = \theta^t \circ \theta^s$

$\theta^{t+s}(x) = (t+s)\,(1, \dots, 1) + x$

$\theta^t \circ \theta^s(x) = \theta^t\left(s(1, \dots, 1) + x\right) = t\,(1, \dots, 1) + s\cdot(1, \dots, 1) + x = (t+s)\,(1, \dots, 1) + x$ $\Big\} = \checkmark$

2) $\theta^0(x) = 0\,(1, \dots, 1) + x = x \Rightarrow \theta^0 = id\ \checkmark$

$\theta^t \circ \theta^s(x) = \theta^t(s(1,-1)+x) = t(1,-1) + s\cdot(1,-1) + x = (t+s)(1,-1)+x$ ✓

2) $\theta^0(x) = 0\,(1,-1) + x = x \Rightarrow \theta^0 = id$ ✓

③ Испитати да ли су токови из зад ① ЈФП:

а) $\phi^t(x_0) = (2+x_0)e^t - 2t - 2$

б) $\phi^t(x_0,y_0) = \left(x_0e^{-t},\ (y_0+\frac{1}{3}x_0^2)e^t - \frac{1}{3}x_0^2e^{-2t}\right)$ → домаћи (ДЗ)

а) $\phi^t \circ \phi^s(x_0) = (2+\phi^s(x_0))e^t - 2t - 2 = \left(2+(2+x_0)e^s - 2s - 2\right)e^t - 2t - 2 =$

$= \underbrace{2e^t}_{\times} + (2+x_0)\,e^s e^t - 2se^t \underbrace{- 2e^t}_{\times} - 2t - 2 =$

$= (2+x_0)e^{s+t} - 2se^t - 2t - 2$

$$\neq$$

$\phi^{t+s}(x_0) = (2+x_0)e^{t+s} - 2(t+s) - 2 = (2+x_0)e^{t+s} - 2t - 2s - 2$

$\Rightarrow$ није ЈФП

$\boxed{T}$ ϕ^t је ЈФП $\iff$ F аутономно

а) није аутономно

б) јесте

④ Наћи векторска поља за токове из задатка ①.

а) $\frac{d}{dt}\phi^t(x) = x = F(t, (t+1)\cdot x) = \frac{(t+1)\cdot x}{t+1}$

$F(t,x) = \frac{x}{t+1}$ (није аутономно)

б) $\frac{d}{dt}\psi^t(x) = e^t\cdot x = F(t, e^t\cdot x)$

$F(t,x) = x$ (јесте аутономно)

⑤ Наћи векторско поље за ток (у $\mathbb{R}^2$) $\psi^t(x,y) = (\underbrace{x\cos t + y\sin t}_{f_1},\ \underbrace{-x\sin t + y\cos t}_{f_2})$

$\frac{d}{dt}\psi^t(x,y) = (\underbrace{-x\sin t + y\cos t}_{f_2},\ \underbrace{-x\cos t - y\sin t}_{-f_1}) = F(t, \psi^t(x,y))$

$F(t, f_1, f_2) = (f_2, -f_1) = F(f_1, f_2)$