Шта је то диференцијална (ДЈ) једначина? $f(t, x(t), x'(t), \dots, x^{(n)}(t)) = 0$

пр $t^2 + x^{(III)} \cdot t + x \cdot x' = 0$ → 3 реда

$\Downarrow$

$x(t) = \dots$

ред ДЈ = ред највећег извода у једначини

$x'(t) = \dots$

$x^{(17)} - 3x'' + x = t^3$ → 17. реда

ДЈ → изрази као једначина (+ систем) 1. реда $x'(t) = f(t, x)$

$x'(t) = \dot{x}(t) = \frac{dx}{dt} = x' = \dot{x}$

- $x(t)$
- $y(x)$

① Решити ДЈ $x' = t^2 + t$ / $\int$

↓ наћи сва решења } обично: ∞ решења

$x(t) = \frac{t^3}{3} + \frac{t^2}{2} + C$, $C \in \mathbb{R}$

② $\int_0^x f(u)\,du = f(x)$ / $'$, $f: \mathbb{R} \to \mathbb{R}$ диференцијабилна

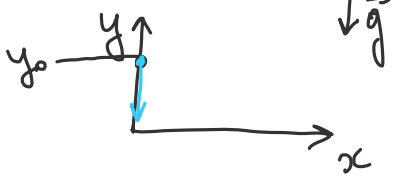
$f(x) = f'(x) \rightsquigarrow f(x) = c \cdot e^x$, $c \in \mathbb{R}$

видећемо само касније

- ОПШТЕ РЕШЕЊЕ (ОР) облик који описује сва решења ДЈ — $f(x) = c\,e^x$, $c \in \mathbb{R}$
- ПАРТИКУЛАРНО РЕШЕЊЕ (ПР) једно конкретно решење ДЈ — $f(x) = 2e^x$.

ДЈ → физика

③ Слободан пад:



$\vec{g}$

$\ddot{x} = 0$

$\ddot{y} = -g$

почетни услови

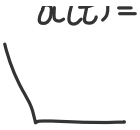
$x(t)$ → положај

$v(t) = \dot{x}(t)$ → брзина

$a(t) = \dot{v}(t) = \ddot{x}(t)$ → убрзање

почетни условия

$$\begin{cases} x(0)=0 & x'(0)=0 \\ y(0)=y_0 & y'(0)=0 \end{cases}$$



→ удрзање

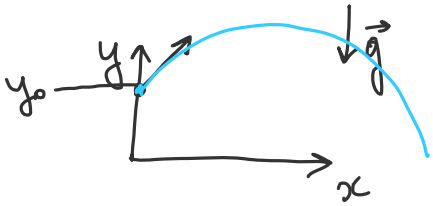
$x(t) = c_1 t + c_2$

$\dot{y}(t) = -gt + c_3 \rightarrow y(t) = -\frac{gt^2}{2} + c_3 t + c_4$ $\quad c_i \in \mathbb{R}$

$x(0)=0 \Rightarrow c_2=0$
$\dot{x}(0)=0 \Rightarrow c_1=0$ $\Big\}$ $x(t)=0$

$y(0)=y_0 \Rightarrow c_4=y_0$
$\dot{y}(0)=0 \Rightarrow c_3=0$ $\Big\}$ $y(t) = -\frac{gt^2}{2} + y_0$

④ Коси хитац:



$$\left.\begin{aligned} \ddot{x} &= 0 \\ \ddot{y} &= -g \end{aligned}\right\}$$

почетни условия

$$\begin{cases} x(0)=0 & x'(0)=v_x \\ y(0)=y_0 & y'(0)=v_y \end{cases}$$

$x(t) = c_1 t + c_2$

$\dot{y}(t) = -gt + c_3 \rightarrow y(t) = -\frac{gt^2}{2} + c_3 t + c_4$ $\quad c_i \in \mathbb{R}$

$x(0)=0 \Rightarrow c_2=0$
$\dot{x}(0)=v_x \Rightarrow c_1=v_x$ $\Big\}$ $x(t) = v_x t$

$y(0)=y_0 \Rightarrow c_4=y_0$
$\dot{y}(0)=v_y \Rightarrow c_3=v_y$ $\Big\}$ $y(t) = -\frac{gt^2}{2} + v_y t + y_0$

$t = \frac{x}{v_x}$

$y = -\frac{g}{2v_x^2}x^2 + \frac{v_y}{v_x}x + y_0$

⑤ $x' = kx$, $k \in \mathbb{R}$

$\frac{dx}{dt} = kx$, $x \neq 0$

$\frac{dx}{x} = k\,dt \;/\int$

$\ln|x| = k \cdot t + c$, $c \in \mathbb{R}$

$k<0$ → радиоактивни распад

$k=0 \rightarrow x' = 0 \rightarrow x = c \in \mathbb{R}$

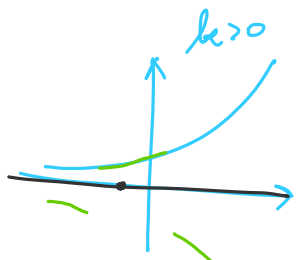
$k>0$ → експоненцијални раст

$(\ln|x|)' = \frac{1}{|x|} \cdot \operatorname{sgn}(x) = \frac{1}{x}$

$k>0$

$e^{\cdot}$ | $\ln|x| = kt + C$, $C \in \mathbb{R}$

$|x| = e^{kt+C} = \underbrace{e^{C}}_{C_1} e^{kt} = \boxed{C_1 e^{kt}}$, $C_1 > 0$



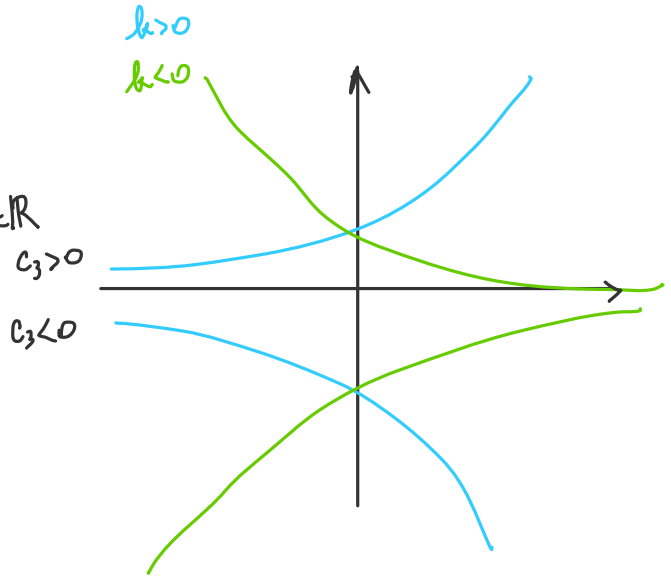
$x = C_2 \cdot e^{kt}$, $C_2 \in \mathbb{R}\setminus\{0\}$ $(C_2 = \pm C_1)$

$x = ? \rightarrow x$ диф $\Rightarrow x$ непр.

Линарова Т (касицде) } решена се не сеч

$x(t) = 0 \rightarrow$ решение

ОР: $x = C_3 \cdot e^{kt}$, $C_3 \in \mathbb{R}$



Разделяне променливите.

$x'(t) = \dfrac{f(t)}{g(x)}$, $f \in C(a,b)$, $g \in C(c,d)$, $g \neq 0$

ОР $\int_{x_0}^{x} g(u)\,du = \int_{t_0}^{t} f(v)\,dv$ $\left(\int g(x)\,dx = \int f(t)\,dt\right)$

$x' = \dfrac{dx}{dt} = \dfrac{f(t)}{g(x)}$

$g(x)\,dx = f(t)\,dt \;/\int$

• Коцидев проблем: $x' = f(x,t)$

$x(t_0) = x_0 \leftarrow$ Коцидев услов

n-ти реда. $x(t_0) = x_0$

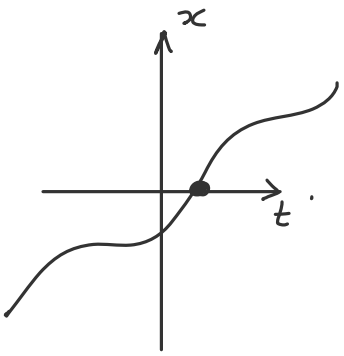
$x'(t_0) = x_1$

:

пр. задачи 3 и 4

$x^{(n-1)}(t_0) = x_{n-1}$

⑥ $tx' = x$, ОР и ПР под $x(-3) = \frac{1}{3}$.



$$t \cdot \frac{dx}{dt} = x \Rightarrow \frac{dx}{x} = \frac{dt}{t} \quad / \int$$

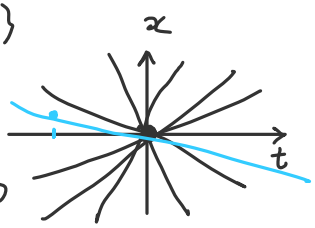
$$\ln|x| = \ln|t| + C, \quad C \in \mathbb{R}$$

$$|x| = |t| \cdot \underbrace{e^C}_{c_1}, \quad c_1 > 0$$

$$x = c_2 t, \quad c_2 \in \mathbb{R} \setminus \{0\}$$

$x = 0 \rightarrow$ решение $\longrightarrow$ $\boxed{x = c_3 t, c_3 \in \mathbb{R}}$ ОР

$t = 0$ у почетку $\quad 0 \cdot x'(0) = x(0) \Rightarrow x(0) = 0$



ПР: $x(-3) = \frac{1}{3}$

$$c_3 \cdot (-3) = \frac{1}{3} \Rightarrow c_3 = -\frac{1}{9} \Rightarrow \boxed{x(t) = -\frac{t}{9}}$$

⑦ $x' = \frac{2xt}{t^2 - 1}$ $\quad |t| \neq 1$

Решити ДЈ, скицирати решења
↳ интегралне криве

Коши ПР: а) $x(0) = 1$

б) $x(2) = 1$

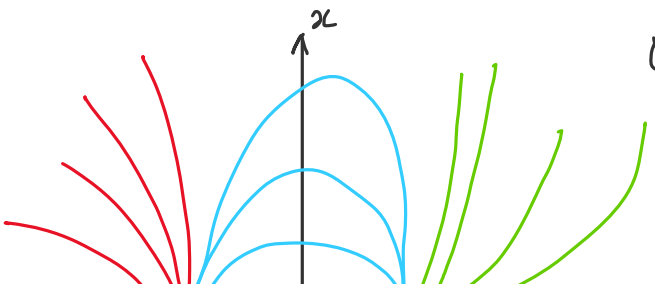
$$\frac{dx}{dt} = x \cdot \frac{2t}{t^2-1} \Rightarrow \frac{dx}{x} = \frac{2t\,dt}{t^2-1} \quad / \int$$

$$\ln|x| = \ln|t^2 - 1| + C, \quad C \in \mathbb{R}$$

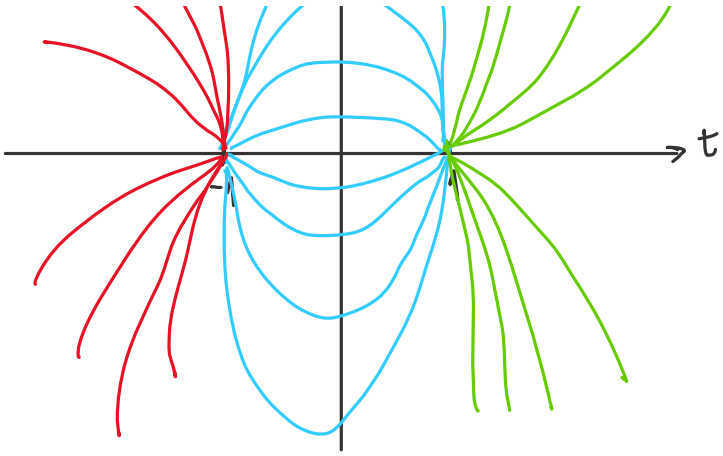
$$e^{\square} = e^{\square}$$

$$|x| = |t^2 - 1| \cdot c_1, \quad c_1 > 0$$

$$\boxed{x = c_2 \cdot (t^2 - 1)} \text{ ОР}, \quad c_2 \in \mathbb{R}$$



10 + 2(-1,1)



$x = c_2(t-1)$;

1° $t \in (-1,1)$
2° $t \in (1,+\infty)$
3° $t \in (-\infty,-1)$

а) $x(0)=1$, $0 \in (-1,1)$

$1 = c_2(0^2-1) = -c_2 \Rightarrow c_2 = -1$

$\Rightarrow x(t) = 1-t^2$, $t \in (-1,1)$

б) $x(2)=1$, $2 \in (1,+\infty)$

$1 = c_2(2^2-1) = 3c_2 \Rightarrow c_2 = \frac{1}{3}$

$\Rightarrow x(t) = \frac{t^2-1}{3}$, $t \in (1,+\infty)$

За домашно $x' = kx^2$, $k>0$
→ едномерна експлозия } решете, покажете да решенье
няма деф. на $\mathbb{R}$ и да не
може да се продължи

(8) Кои све C^1 функције $f: \mathbb{R} \to \mathbb{R}$ тд $f(0)=1$ и:

површина испод графика од 0 до x_0 од f } $\forall x_0 > 0$
=
дужина лука од 0 до x_0 од f

$$\int_0^{x_0} f(u)\,du = \int_0^{x_0} \sqrt{1+f'(u)^2}\,du \quad \Big/'_{x_0}$$

$$f(x_0) = \sqrt{1+f'(x_0)^2} \quad /^2 \longrightarrow f>0 \longrightarrow \boxed{f \geq 1}$$

$$f(x_0)^2 - 1 = f'(x_0)^2$$

$$f'(x_0) = \sqrt{f(x_0)^2-1} \qquad f' = \frac{df}{dx_0}$$

$$\frac{df}{\sqrt{f^2-1}} = dx_0 \quad / \int \longrightarrow f=1\ ?$$

0 :

$\sqrt{f^2-1}$? ⋮ → $f=1$!

1 $x' = f(\alpha t+\beta x+\gamma)$, $f\in C(a,b)$, $\alpha,\beta\in\mathbb{R}\setminus\{0\}$, $\gamma\in\mathbb{R}$

смяна. $x(t) \rightsquigarrow y(t)$

$y(t) = \alpha t+\beta x(t)+\gamma$

$y' = \alpha+\beta x' \Rightarrow x' = \frac{y'-\alpha}{\beta}$

} $\rightsquigarrow$ РП

(3) а) $x' = x+2t-3$

б) $x' = (x+t)^2$ $\quad [f(\bullet) = \bullet^2+1]$

б) $y = x+t$

$y' = x'+1$

$y'-1 = x' = (x+t)^2 = y^2 \Rightarrow y' = y^2+1 \Rightarrow \frac{dy}{y^2+1} = dt \,/\int$

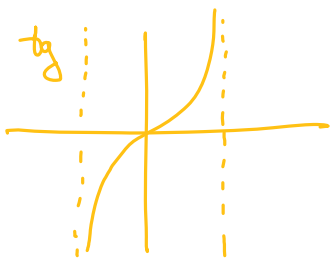
tg | $\operatorname{arctg} y = t+c$, $c\in\mathbb{R}$

$y = \operatorname{tg}(t+c)$, $c\in\mathbb{R}$

$x+t = \operatorname{tg}(t+c)$

$\boxed{x = \operatorname{tg}(t+c)-t \ , \ c\in\mathbb{R}}$

нап.



$-\frac{\pi}{2} < t+c < \frac{\pi}{2}$

$-\frac{\pi}{2}-c < t < \frac{\pi}{2}-c$

∞ реш, а свако деф на другачијем интервалу

+ решења не могу да се продуже, јер $\to+\infty$

(Т о продужењу $\rightsquigarrow$ предавања)

2 $x' = f\left(\frac{x}{t}\right)$, $f\in C(a,b)$ (хомогена)

смена $y(t) = \frac{x(t)}{t}$

$yt = x \,/' \Rightarrow x' = (yt)' = y'\cdot t + y\cdot 1 = y+y't$

} $\rightsquigarrow$ РП

$\frac{x}{}$

$yt = x \,/ \Rightarrow x' = (yt)' = y' \cdot t + y \cdot 1 = y + yt'$

(10) a) $x' = e^{\frac{x}{t}} + \frac{x}{t}$

б) $x' = -\frac{x^2+t^2}{2xt}$

в) $x' = \frac{x^2-2tx-t^2}{x^2+2tx-t^2}$

г) $t \sin\frac{x}{t} \cdot x' = x \cdot \sin\frac{x}{t} + t$

$f(\cdot) = \frac{\cdot^2-2\cdot-1}{\cdot^2+2\cdot-1}$

в) $x' = \frac{x^2-2tx-t^2}{x^2+2tx-t^2} \overset{:t^2}{=} \frac{\left(\frac{x}{t}\right)^2-2\frac{x}{t}-1}{\left(\frac{x}{t}\right)^2+2\frac{x}{t}-1}$

$y = \frac{x}{t} \Rightarrow x = yt \,/' \Rightarrow x' = y't + y$

$y't + y = x' = \frac{\left(\frac{x}{t}\right)^2-2\frac{x}{t}-1}{\left(\frac{x}{t}\right)^2+2\frac{x}{t}-1} = \frac{y^2-2y-1}{y^2+2y-1}$

$y't = \frac{y^2-2y-1}{y^2+2y-1} - y = \frac{y^2-2y-1-y(y^2+2y-1)}{y^2+2y-1} = \frac{-y^3-y^2-y-1}{y^2+2y-1}$

$y' = \frac{dy}{dt}$

$\frac{y^2+2y-1}{y^3+y^2+y+1}\,dy = -\frac{dt}{t} \,/ \int$

$=0?$

$\int \frac{y^2+2y-1}{y^3+y^2+y+1}\,dy = -\int \frac{dt}{t} = -\ln|t| + C, \quad C \in \mathbb{R}$

$\frac{y^2+2y-1}{y^3+y^2+y+1} = \frac{y^2+2y-1}{(y+1)(y^2+1)} = \frac{A}{y+1} + \frac{By+C}{y^2+1} = \frac{A(y^2+1)+(y+1)(By+C)}{(y+1)(y^2+1)}$

$y^2: \; 1 = A + B$

$y: \; 2 = B + C$

$1: \; -1 = A + C$

$(1)+(2)-(3) \quad 2B = (1+2)-(-1) = 4$

$B = 2$

$C = 0$

$A = -1$

$$-\ln|t| + C = \int - = \int \frac{-1}{y+1}\,dy + \int \frac{2y}{y^2+1}\,dy = -\ln|y+1| + \ln|y^2+1| = \ln\left|\frac{y^2+1}{y+1}\right|$$

$$\frac{e^C}{|t|} = \left|\frac{y^2+1}{y+1}\right| \Rightarrow \frac{y^2+1}{y+1} = \frac{c_1}{t}, \quad c_1 \in \mathbb{R}\setminus\{0\}$$

$$\frac{\left(\frac{x}{t}\right)^2+1}{\frac{x}{t}+1} = \frac{c_1}{t} \Rightarrow \boxed{\frac{x^2+t^2}{x+t} = c_1}$$

(*) $x+t \neq 0$

имплицитно зададено

експлицитно зад $x(t) = \ldots$

$(y+1)(y^2+1) = 0$? $y+1=0$ ($y^2+1 \neq 0$)

$y=-1 \Rightarrow \frac{x}{t} = -1 \Rightarrow \boxed{x=-t}$ решение?

$$x' = -1 = \frac{t^2+2t^2-t^2}{t^2-2t^2-t^2} = \frac{2t^2}{-2t^2} = -1 \quad \checkmark$$

Напр. ко почетну дешиво и са $x^2+2xt-t^2$

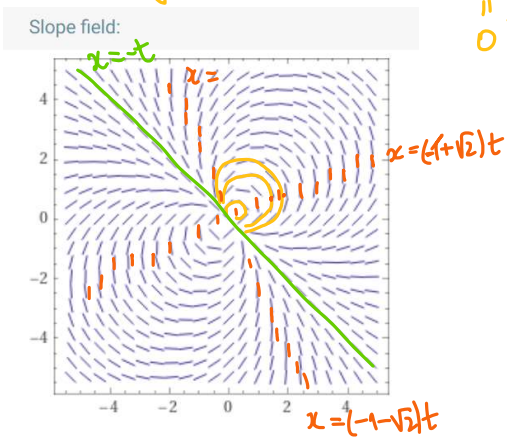
$\overset{!!}{0}$? $\Rightarrow x' = \infty$? $\rightsquigarrow$ наклб вертикалан

$x^2+2xt-t^2=0$, $y=\frac{x}{t}$

$y^2+2y-1=0$

$y=-1\pm\sqrt{2}$

$x=(-1\pm\sqrt{2})t$



https://tinyurl.com/linkKaWA1

ако желимо ОР, не морамо овим да се бавимо