O smenama u dj

Смене

1 $x(t) \rightsquigarrow x(u)$

(1) $2t\,x''(t) + (1+\sqrt{t})\,x'(t) - x(t) = t + \sin(\sqrt{t})$, $t>0$

Доказати да се сменом $u=\sqrt{t}$ своди на $x'' + x' - 2x = 2u^2 + 2\sin u$.

ову јну ћемо касније знати да решимо

$$\frac{df}{dt} = \frac{du}{dt}\,\frac{df}{du}$$

$x(t) \to x(u)$

$\frac{dx}{dt} \rightsquigarrow \frac{dx}{du}$

$\frac{d^2x}{dt^2} \rightsquigarrow \frac{d^2x}{du^2}$

$u=\sqrt{t}$

$t=u^2$

$\frac{du}{dt} = \frac{d(\sqrt{t})}{dt} = \frac{1}{2\sqrt{t}} = \frac{1}{2u}$

$$\frac{dx}{dt} = \frac{dx}{du}\cdot\frac{du}{dt} = x'_u\cdot\frac{1}{2u}$$

$$\frac{d}{dt}(x'_u)\,\frac{1}{2u} + x'_u\cdot\frac{d}{dt}\left(\frac{1}{2u}\right) =$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(x'_u\,\frac{1}{2u}\right) =$$

$$= \frac{du}{dt}\,\frac{d}{du}\left(x'_u\,\frac{1}{2u}\right) = \frac{1}{2u}\left(\frac{d}{du}(x'_u)\cdot\frac{1}{2u} + x'_u\,\frac{d}{du}\left(\frac{1}{2u}\right)\right) =$$

$$= \frac{1}{2u}\cdot\left(x''_{uu}\cdot\frac{1}{2u} + x'_u\left(-\frac{1}{2u^2}\right)\right)$$

$$2u^2\left(\frac{x''_{uu}}{4u^2} - \frac{x'_u}{4u^3}\right) + (1+u)\,x'_u\,\frac{1}{2u} - x = u^2 + \sin u$$

$$\frac{x''_{uu}}{2} \underbrace{- \frac{x'_u}{2u}}_{\times} \underbrace{+ \frac{x'_u}{2u}}_{\times} + \frac{x'_u}{2} - x = u^2 + \sin u \quad /\cdot 2$$

$$x''_{uu} + x'_u - 2x = 2u^2 + 2\sin u$$

2 $x(t) \rightsquigarrow y(t)$

$$y=f(x) \rightsquigarrow y' = \underline{f'(x)}\;x'$$

(2) а) $t\,\boxed{x^2x'} + \boxed{x^3} = t\cos t$

$y(t) = x(t)^3 \to y' = 3x^2x'$. $\frac{y'}{3}\cdot t + y = t\cos t$ (лин)

б) $\boxed{x'\cos x} = \frac{\boxed{\sin x}}{t} - \boxed{\sin^2 x}$

$y(t) = \sin x(t) \to y' = \cos x\cdot x'$: $y' = \frac{y}{t} - y^2$ (Рик, Бер $\alpha=2$)

в) $\boxed{x'\operatorname{tg}x} + 4t^3\boxed{\cos^3 x} = 2t$

$x'\operatorname{tg}x = x'\cdot\frac{\sin x}{\cos x}$, $y(t) = \cos x(t) \to y' = -\sin x\cdot x'$: $\frac{-y'}{y} + 4t^3y^3 = 2t \Rightarrow y' - 4t^3y^4 = -2ty$

(Бер $\alpha=4$)

$\cos x$ … $= -2ty$

(БЕР $\alpha = 4$)

Г) $t\boxed{e^{x}x'} - 2t\boxed{e^{x/2}} = 4\boxed{e^{x}}$

$y(t) = e^{x(t)} \rightarrow y' = e^{x}\cdot x'$ $\qquad ty' - 2t\sqrt{y} = 4y$ $\qquad$ (БЕР $\alpha = \frac{1}{2}$)

$(e^{x(t)/2})$

$\boxed{3}$ $x(t) \rightsquigarrow y(u)$

(3) $x' = \dfrac{x((\ln x)^2 + t)}{2t^{3/2}}$ $\quad , t>0, \; x>0$

$\dfrac{dx}{dt} \rightsquigarrow \dfrac{dy}{du}$

Решити уводећи: $u = \sqrt{t} \Rightarrow t = u^2$

$y = \ln x \Rightarrow x = e^{y}$

$$\frac{dx}{dt} = \boxed{\frac{dx}{dy}}\cdot\boxed{\frac{dy}{du}}\cdot\boxed{\frac{du}{dt}} = \frac{dy}{du}\,\frac{e^{y}}{2u}$$

$$\frac{du}{dt} = \frac{d(\sqrt{t})}{dt} = \frac{1}{2\sqrt{t}} = \frac{1}{2u}$$

$$\frac{dx}{dy} = \frac{d(e^{y})}{dy} = e^{y}$$

$y = \ln x$

$x = e^{y}$

$$\frac{dy}{du}\cdot\frac{e^{y}}{2u} = \frac{e^{y}(y^2+u^2)}{2u^3} \Rightarrow \frac{dy}{du} = \frac{y^2+u^2}{u^2} = \left(\frac{y}{u}\right)^2 + 1 \quad \text{(ХОМ)}$$

⋮

$y(u) \rightsquigarrow x(t) = e^{y(t)} = e^{y(u^2)}$

коначно решење: $y(u) = \frac{1}{2}\left(\sqrt{3}\,u\,tg\left(\frac{\sqrt{3}}{2}(c+u)\right) + u\right)$

$x(t) = e^{\frac{1}{2}\left(\sqrt{3t}\; tg\left(\frac{\sqrt{3}}{2}(c+\sqrt{t})\right) + \sqrt{t}\right)}$ $\quad , c\in\mathbb{R}$

(4) Решити ј-ј. $(x+2t-2)x' = x-t-1$ уводећи смену $x = v + \frac{4}{3}$, $t = u + \frac{1}{3}$.

$$\frac{dv}{du} = \underset{\shortparallel\; 1}{\frac{dv}{dx}}\;\frac{dx}{dt}\cdot\underset{\shortparallel\; 1}{\frac{dt}{du}} = \frac{dx}{dt}$$

$$v' = \frac{(v+\frac{4}{3}) - (u+\frac{1}{3}) - 1}{(v+\frac{4}{3}) + 2(u+\frac{4}{3}) - 2} = \frac{v-u}{v+2u} \quad \text{(ХОМ)}$$

$$\frac{v}{u} = w(u)$$

$v = u\,w\,/' \Rightarrow v' = w + u\cdot w'$

$$w + u \cdot w' = \frac{w-1}{w+2} \Rightarrow u\, w' = \frac{w-1}{w+2} - w = \frac{w-1-w^2-2w}{w+2} = -\frac{w^2+w+1}{w+2}$$

$$w^2+w+1 = 0 \,?$$

$$\times$$

$$\left(w+\tfrac{1}{2}\right)^2 + \tfrac{3}{4} > 0$$

$$\frac{w'(w+2)}{w^2+w+1} = -\frac{1}{u} \;\Big/ \int du \;\ldots$$

$$w(u) \rightsquigarrow v(u) \rightsquigarrow x(t)$$