Nehomogeni sistemi

$$y'(x) = \underbrace{A \cdot y(x)}_{\text{хомогени део}} + \underbrace{g(x)}_{\text{нехомогени део}} \qquad (y' = p(x)y + g(x))$$

$$\text{ОР}: \; y(x) = \underbrace{y_H(x)}_{\text{ОР хомогене}} + \underbrace{y_p(x)}_{\text{ПР нехомогене}}$$

$$\circledast \; g(x) = P_s[x] \cdot e^{\mu x}, \quad \mu \in \mathbb{R}$$

$$s \in \mathbb{N}_0, \; s = st(P_s[x]) = \deg(P_s[x])$$

$$\xrightarrow[\text{у облику}]{\text{тражимо}} \; y_p(x) = Q_{m+s}[x] \, e^{\mu x}$$

m = вишеструкост броја μ као решења карактеристичне јне A

① а)

$$y_1' = 2y_1 + y_2 + xe^x$$
$$y_2' = -y_1 + 2y_2 - e^x$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad y' = Ay + g$$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} xe^x \\ -e^x \end{bmatrix} = \begin{bmatrix} x \\ -1 \end{bmatrix} \cdot e^x$$

ХОМОГЕНА. $y' = Ay$

$$\det(A - \lambda E) = 0 \Rightarrow \lambda_{1/2} = 2 \pm i$$

$$\lambda_1 = 2 + i \rightsquigarrow (A - \lambda_1 E)\gamma_1 = 0 \rightsquigarrow \gamma_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\psi(x) = e^{\lambda_1 x} \gamma_1 = e^{2x} \cdot \begin{bmatrix} \cos x + i \sin x \\ i \cos x - \sin x \end{bmatrix} \rightsquigarrow \begin{matrix} \text{Re} \\ \text{Im} \end{matrix}$$

$$\text{ОРХ.} \; y_H(x) = c_1 e^{2x} \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix} + c_2 e^{2x} \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

НЕХОМОГЕНА

$$g(x) = \begin{bmatrix} x \\ -1 \end{bmatrix} e^x = P_s[x] \, e^{\mu x}$$

$$\begin{matrix} \mu = 1 \\ s = 1 \end{matrix} \xrightarrow[\lambda_{1/2} = 2 \pm i]{} \mu \text{ није сопс. бр. } A \Rightarrow m = 0$$

$$y_p(x) = Q_{0+1}[x]\, e^{1x} = Q_1[x]\cdot e^x = \begin{bmatrix} a_1x+b_1 \\ a_2x+b_2 \end{bmatrix} e^x \rightsquigarrow y_p' = Ay_p + g$$

$$\begin{bmatrix} a_1x+b_1+a_1 \\ a_2x+b_2+a_2 \end{bmatrix}\cdot e^x = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a_1x+b_1 \\ a_2x+b_2 \end{bmatrix} e^x + \begin{bmatrix} x \\ -1 \end{bmatrix}\cdot e^x$$

$$\left.\begin{aligned} a_1x+b_1+a_1 &= 2a_1x+2b_1+a_2x+b_2+x \\ a_2x+b_2+a_2 &= -a_1x-b_1+2a_2x+2b_2-1 \end{aligned}\right\} \forall x \Rightarrow$$

$$\begin{aligned} a_1 &= 2a_1+a_2+1 \quad \Rightarrow a_1 = a_2 = -\tfrac{1}{2} \\ b_1+a_1 &= 2b_1+b_2 \\ a_2 &= -a_1+2a_2 \rightsquigarrow a_2 = a_1 \\ b_2+a_2 &= -b_1+2b_2-1 \end{aligned}$$

$$\left.\begin{aligned} -\tfrac{1}{2} &= b_1+b_2 \\ \tfrac{1}{2} &= -b_1+b_2 \end{aligned}\right\} b_2 = 0,\ b_1 = -\tfrac{1}{2}$$

$$y_p(x) = \begin{bmatrix} -\frac{1}{2}x-\frac{1}{2} \\ -\frac{1}{2}x \end{bmatrix}\cdot e^x$$

ОР: $y(x) = y_H(x) + y_p(x)$

б) $y_1' = 2y_1+y_2+2e^x$

$y_2' = y_1+2y_2-3e^{4x}$

$y' = Ay+g$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $g(x) = \begin{bmatrix} 2e^x \\ -3e^{4x} \end{bmatrix}$

ХОМ: $\lambda_1 = 1,\ \lambda_2 = 3$

$\gamma_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \gamma_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\varphi_1(x) = e^x\cdot\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \varphi_2(x) = e^{3x}\cdot\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

НЕХОМ: $g(x) = \begin{bmatrix} 2e^x \\ -3e^{4x} \end{bmatrix} \neq P_s[x]\, e^{mx}$

$$g(x) = g_1(x) + g_2(x)$$

$$g_1(x) = \begin{bmatrix} 2e^x \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ -3e^{4x} \end{bmatrix} \quad \Bigg\} \quad y_p(x) = y_{p_1}(x) + y_{p_2}(x)$$

$$\underbrace{}_{y_{p_1}(x)} \qquad\qquad \underbrace{}_{y_{p_2}(x)}$$

$\underline{g_1(x):}$ $\quad g_1(x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot e^x = P_s[x]\, e^{\mu x}$

$$\begin{matrix} \mu = 1 \\ s = 0 \end{matrix} \xrightarrow{\lambda_1 = 1} m = 1 \Rightarrow y_{p_1}(x) = Q_1[x] \cdot e^x = \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} e^x$$

$s + m = 1$

$$y_{p_1}' = A y_{p_1} + g_1(x)$$

$$y_{p_1}(x) \underset{\text{нтр.}}{=} \begin{bmatrix} x-1 \\ -x \end{bmatrix} \cdot e^x$$

$\underline{g_2(x):}$ $\quad g_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot e^{4x} \Rightarrow \begin{matrix} s = 0 \\ \mu = 4 \rightsquigarrow m = 0 \end{matrix} \Rightarrow y_{p_2}(x) = Q_0[x]\, e^{4x} = \begin{bmatrix} a \\ b \end{bmatrix} e^{4x}$

$$y_{p_2}' = A y_{p_2} + g_2(x)$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot 4\cancel{e^x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \cdot \cancel{e^{4x}} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot \cancel{e^{4x}}$$

$$\left.\begin{matrix} 4a = 2a + b \\ 4b = a + 2b - 3 \end{matrix}\right\} \quad \begin{matrix} a = -1 \\ b = -2 \end{matrix}$$

$$y_{p_2}(x) = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \cdot e^{4x}$$

$$\text{ОР}: \; y(x) = c_1 \cdot \begin{bmatrix} e^x \\ -e^x \end{bmatrix} + c_2 \cdot \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix} + \begin{bmatrix} x-1 \\ -x \end{bmatrix} \cdot e^x + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \cdot e^{4x}, \quad c_1, c_2 \in \mathbb{R}$$

$y' = A(x)\, y$ не мора обавезно да буде са кк

ОР: $y(x) = \Phi(x) \cdot c, \quad c \in \mathbb{R}^n$

$$\Phi(x) = [\varphi_1 \downarrow \; \dots \; \varphi_n \downarrow]$$

$y' = A(x) \cdot y + g(x)$

$\Phi(x)$ – фунд матр хомогеног система $y' = A(x)\, y$

$$\text{ОР}: \; y(x) = \Phi(x) \left(c + \int \Phi^{-1}(x) \cdot g(x)\, dx \right) = \Phi(x) \cdot c + \Phi(x) \int \Phi^{-1}(x)\, g(x)\, dx$$

$\Phi(x)$ – фунд. матр. [illegible]

ОР: $y(x) = \Phi(x)\left(C + \int \Phi^{-1}(x)\cdot g(x)\,dx\right) = \underbrace{\Phi(x)\cdot C}_{\text{ОРХ}} + \underbrace{\Phi(x)\int \Phi^{-1}(x)\,g(x)\,dx}_{y_p(x)\text{ од нехомогене}}$

Извођење: хомогена има ОР $\cdot\ X(t) = \Phi(t)\,c,\ c\in\mathbb{R}^n$

варијација константе: $c \rightsquigarrow c(t)$

$$X(t) = \Phi(t)\,c(t)$$

$$X'(t) = \Phi'(t)\cdot c(t) + \Phi(t)\,c'(t)$$

$X'(t) = A(t)\,X(t) + g(t) \Rightarrow \underbrace{\Phi'(t)}_{\Phi'=A\Phi}\,c(t) + \Phi(t)\,c'(t) = A(t)\,(\Phi(t)\cdot c(t)) + g(t)$

$$\Rightarrow \quad \underbrace{A(t)\,\Phi(t)}\cdot c(t) + \Phi(t)\,c'(t) = \underbrace{A(t)\,\Phi(t)}\cdot c(t) + g(t)$$

$W(t)\neq 0 \Rightarrow \Phi(t)$ инвертибилно

$\Phi^{-1}(t)\cdot$ $\Phi(t)\cdot c'(t) = g(t)$

$$c'(t) = \Phi^{-1}(t)\cdot g(t)$$

$$c(t) = \int \Phi^{-1}(t)\,g(t)\,dt + C$$

→ ово је интеграл по компонентама

$$X(t) = \Phi(t)\,c(t) = \Phi(t)\left(C + \int \Phi^{-1}(t)\,g(t)\,dt\right) =$$

$$= \boxed{\Phi(t)\,C} + \boxed{\Phi(t)\cdot\int \Phi^{-1}(t)\,g(t)\,dt}$$

општа реш хомогене — партикуларно решење нехомогене

② $y_1' = y_2 + \operatorname{tg}^2 x + 1$

$y_2' = -y_1 + \operatorname{tg} x$

$y' = Ay + g(x)$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ g(x) = \begin{bmatrix} \operatorname{tg}^2 x + 1 \\ \operatorname{tg} x \end{bmatrix}$? ✗ $\neq P_s[x]\,e^{\mu x}$

$\Phi(x) = ?$

$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned}\right\}\ y_1'' = y_2' = -y_1 \Rightarrow y_1'' + y_1 = 0 \rightsquigarrow y_1(x) = c_1\cos x + c_2\sin x$

$y_2(x) = -c_1\sin x + c_2\cos x$

$\begin{bmatrix} u_1(x) & \cos x & \sin x \end{bmatrix}$

$y_2' = -y_1$

$$y_2(x) = -c_1 \sin x + c_2 \cos x$$

$$y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1 \cdot \underbrace{\begin{bmatrix} \cos x \\ -\sin x \end{bmatrix}}_{\varphi_1(x)} + c_2 \underbrace{\begin{bmatrix} \sin x \\ \cos x \end{bmatrix}}_{\varphi_2(x)}$$

$$\Phi(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

$\varphi_1(x)$ $\varphi_2(x)$

$$W(x) = \det \Phi(x) = \cos^2 x - (-\sin^2 x) = 1 \neq 0$$

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \rightsquigarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Rightarrow \Phi^{-1}(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$$\int \Phi^{-1}(x)\, g(x)\, dx = \int \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} tg^2 x + 1 \\ tg\, x \end{bmatrix} dx = \int \begin{bmatrix} \frac{\sin^2 x}{\cos x} + \cos x - \frac{\sin^2 x}{\cos x} \\ \frac{\sin^3 x}{\cos^2 x} + \sin x + \sin x \end{bmatrix} dx =$$

↳ план то план

$$= \int \begin{bmatrix} \cos x \\ \frac{\sin^3 x}{\cos^2 x} + 2\sin x \end{bmatrix} dx = \begin{bmatrix} \int \cos x \, dx \\ \int \left(\frac{\sin^3 x}{\cos^2 x} + 2\sin x \right) dx \end{bmatrix} = \begin{bmatrix} \sin x \\ \cos x + \frac{1}{\cos x} - 2\cos x \end{bmatrix} = \begin{bmatrix} \sin x \\ -\cos x + \frac{1}{\cos x} \end{bmatrix}$$

$$\int \frac{\sin^3 x}{\cos^2 x} dx = \int \frac{(1-\cos^2 x) \sin x \, dx}{\cos^2 x} = \int \frac{\cos^2 x - 1}{\cos^2 x} (-\sin x) dx = \int \frac{t^2 - 1}{t^2} dt = t - \int \frac{dt}{t^2} =$$

$\cos x = t$

$dt = (-\sin x) dx$

$$= t - \left(-\frac{1}{t}\right) = t + \frac{1}{t} = \cos x + \frac{1}{\cos x}$$

$$y_p(x) = \Phi(x) \int \Phi^{-1}(x) g(x)\, dx = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \sin x \\ -\cos x + \frac{1}{\cos x} \end{bmatrix} = \begin{bmatrix} \sin x \cos x - \sin x \cos x + \frac{\sin x}{\cos x} \\ \underbrace{-\sin^2 x - \cos^2 x}_{-1} + 1 \end{bmatrix} = \begin{bmatrix} tg\, x \\ 0 \end{bmatrix}$$

$$OP: \; y(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} tg\, x \\ 0 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$