3 Linearna DJ 1. reda

$p, g : (a,b) \to \mathbb{R}$ neprekidne

$$x' + p(t)x = g(t)$$

$$\text{predavanja} \rightarrow x(t) = e^{-\int p\,dt}\left(c + \int e^{\int p\,dt} \cdot g\,dt\right), \quad c \in \mathbb{R}$$

$$\int p\,dt = \int_{t_0}^{t} p(u)\,du$$

↳ isti na oba mesta

Izvođenje I – metoda varijacije konstante:

$$x' + p(t)x = g(t)$$

1) rešimo prvo homogenu

$$x' + p(t)x = 0 \quad \Rightarrow \quad \frac{x'}{x} = -p(t) \,/ \int dt$$

$$\ln|x| = -\int p(t)\,dt \quad \Rightarrow \quad x(t) = C \cdot e^{-\int p(t)\,dt}, \quad C \in \mathbb{R}$$

2) varijramo konstantu C kao fju $c(t)$

$$x(t) = c(t) \cdot e^{-\int p(t)\,dt}$$

3) uvrstimo rešenje u tom obliku

$$x' = c' \cdot e^{-\int p(t)\,dt} + c \cdot e^{-\int p(t)\,dt} \cdot (-p(t))$$

$$x' + p(t)x = g(t) \rightsquigarrow c' e^{-\int p(t)\,dt} \underbrace{- p(t) \cdot c \cdot e^{-\int p(t)\,dt}}_{\times} + \underbrace{p(t)\, c e^{-\int p(t)\,dt}}_{\times} = g(t)$$

$$\Rightarrow c' = g(t) e^{\int p(t)\,dt} \,/ \int dt$$

$$c(t) = \int g(t) e^{\int p(t)\,dt}\,dt + \tilde{c}, \quad \tilde{c} \in \mathbb{R}$$

$$\text{OP}: \quad x(t) = e^{-\int p(t)\,dt} \cdot \left(\tilde{c} + \int g(t) \cdot e^{\int p(t)\,dt}\,dt\right), \quad \tilde{c} \in \mathbb{R}$$

Izvođenje II – integracioni faktor

ideja: pomnožimo obe strane jne sa nečim tako da je levi izvod nečega

$$(x' + p(t)x) \cdot \square = (\quad)'$$

↳ uzmemo $e^{\int p(t)\,dt}$

$$\underbrace{x' \cdot e^{\int p(t)\,dt} + p(t) e^{\int p(t)\,dt} \cdot x}_{\left(x \cdot e^{\int p(t)\,dt}\right)'} = g(t) \cdot e^{\int p(t)\,dt} \quad / \int dt$$

$$x \cdot e^{\int p(t)\,dt} = \int g(t) \cdot e^{\int p(t)\,dt}\,dt + C, \quad C \in \mathbb{R}$$

$$x(t) = e^{-\int p(t)\,dt} \cdot \left(C + \int g(t) \cdot e^{\int p(t)\,dt} + C\right), \quad C \in \mathbb{R}$$

Izvođenje III – partikularno rešenje

1) Tvrdimo da je $x_p(t) = \int g(t) e^{\int p(t)\,dt}$ partikularno rešenje

Извођење III – варијација[illegible]

1) показати да је $x_p(t) = \int q(t)\, e^{\int p(t)\,dt}$ партикуларно решење

2) увести смену $y(t) = x(t) - x_p(t)$ и показати да y задовољава $y' + p(t)y = 0$

(1) а) $tx' - x = t^3$

б) $x' + x = \frac{1}{1+e^{2t}}$

в) $x' - 2xt = 6te^{t^2}$

г) $tx' + ax + t^n = 0$, $a \in \mathbb{R}, n \in \mathbb{N}$

$q(t) \equiv 0 \to$ хом ЛДЈ1.р

$q(t) \not\equiv 0 \to$ нехомогена ЛДЈ1р

б) $x' + x = \boxed{\frac{1}{1+e^{2t}}} = q(t)$

$p(t) = 1$

$\int p\,dt = \int dt = t\,(+C)$

$\int e^{\int p\,dt} \cdot q\,dt = \int e^t \cdot \frac{1}{1+e^{2t}}\,dt = \int \frac{du}{1+u^2} = \operatorname{arctg} u\,(+C) = \operatorname{arctg}(e^t)\,(+C)$

$e^t = u$

$du = e^t dt$

ОР $x(t) = e^{-t}\left(C + \operatorname{arctg}(e^t)\right)$, $C \in \mathbb{R}$

$= C \cdot e^{-t} + e^{-t}\operatorname{arctg}(e^t)$

↳ реш хом дела

а) $tx' - x = t^3 \;/\cdot t$

$x' \boxed{-\frac{1}{t}} x = \boxed{t^2}$

$p(t) = -\frac{1}{t}$ итд

$q(t) = t^2$

(2) Наћи реш ДЈ $x' \sin t - x\cos t = -\frac{\sin^2 t}{t^2}$ тј. $\lim_{t\to\infty} x(t) = 0$

I начин. линеарна $x' - \operatorname{ctg} t\, x = -\frac{\sin t}{t^2}$ ...

II начин. $x'\sin t - x\cos t = x' \cdot \sin t - x\,(\sin t)'$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$\frac{x'\sin t - x\cos t}{(\sin t)^2} = \left(\frac{x}{\sin t}\right)'$

$x' \cdot \sin t - x\cos t = -\frac{\sin^2 t}{t^2} \;/\cdot \sin^2 t$

$(g)' = \frac{}{g^2}$ $\qquad (\sin t)^2 = (\sin t)$

$x' \sin t - x\cos t = -\frac{\sin^2 t}{t^2} \quad / : \sin^2 t$

$\left(\frac{x}{\sin t}\right)' = -\frac{1}{t^2} \quad / \int$

$\frac{x}{\sin t} = \frac{1}{t} + C \Rightarrow \boxed{x = C\sin t + \frac{\sin t}{t}, \ C \in \mathbb{R}}$

$\lim_{x\to\infty} x(t) = 0 \cdot \quad \frac{\sin t}{t} \xrightarrow[t\to\infty]{} 0$

$C \sin t \to 0 \ ? \Rightarrow \boxed{C = 0}$

пр. $\boxed{x(t) = \frac{\sin t}{t}}$

4. Бернулијева ДЈ

$x' + p(t)x = g(t)x^\alpha, \ \alpha \in \mathbb{R} \qquad p, g : (a,b) \to \mathbb{R}$ непр.

за $\alpha = 1$: тоа е линеарна (или РП)

радиме за $\alpha \neq 1$:

смена $\downarrow \quad y(t) = x(t)^{1-\alpha}$

$y' = (x^{1-\alpha})' = (1-\alpha)\, x^{-\alpha} \cdot x' \Rightarrow x^{-\alpha} x' = \frac{y'}{1-\alpha}$

$\cdot x^{-\alpha}$

$x' x^{-\alpha} + p(t)\, x^{1-\alpha} = g(t)$

$\frac{y'}{1-\alpha} + p(t) y = g(t)$

$y' + (1-\alpha) p(t) y = (1-\alpha) g(t) \to$ лин

③ а) $x' = \frac{x}{t} - x^2$

б) $x' + \frac{x}{t} = x^2 \frac{\ln t}{t}$

в) $tx' - 2t\sqrt{x} = 4x$

б) $x' + \frac{x}{t} = x^2 \frac{\ln t}{t} \quad / : x^2$ $\qquad \alpha = 2$

$t > 0$

$y = x^{1-2} = x^{-1} = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

б) $x + \frac{x}{t} = x^2 \frac{\ln t}{t} \quad / : x^2$

$t > 0$

$\frac{x' x^{-2}}{} + \frac{x^{-1}}{t} = \frac{\ln t}{t}$

$-y' + \frac{y}{t} = \frac{\ln t}{t} \quad / \cdot (-1)$

$y' - \frac{1}{t} y = -\frac{\ln t}{t}$ → лин.

$p(t) = -\frac{1}{t}$

$q(t) = -\frac{\ln t}{t}$

$y = x^{1-2} = x^{-1} = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

$y' = -\frac{1}{x^2} \cdot x' = -x^{-2} x'$

$y(t) = e^{-\int p\,dt} \left(c + \int q \cdot e^{\int p\,dt}\, dt \right)$

$\int p\,dt = -\int \frac{dt}{t} = -\ln t$

$e^{-\ln t} = (e^{\ln t})^{-1} = t^{-1}$

$\int q \cdot e^{\int p\,dt}\, dt = -\int \frac{\ln t}{t} \cdot t^{-1}\, dt = -\int \frac{\ln t}{t^2}\, dt$

$= -\left(-\frac{1}{t} \cdot \ln t - \int \left(-\frac{1}{t}\right) \cdot \frac{1}{t}\, dt \right)$

$= \frac{\ln t}{t} - \int \frac{dt}{t^2} = \frac{\ln t}{t} + \frac{1}{t}$

$y(t) = t \cdot \left(c + \frac{\ln t + 1}{t^2} \right)$

$$x(t) = \frac{1}{t \cdot \left(c + \frac{\ln t + 1}{t^2} \right)}, \quad c \in \mathbb{R}$$

$\left(\frac{\ln t + 1}{t} \right)' = \frac{\frac{1}{t} \cdot t - 1 \cdot (\ln t + 1)}{t^2} = \frac{-\ln t}{t^2}$ ✓

в) $t x' - 2t\sqrt{x} = 4x \quad / : t$

$x' - 2\sqrt{x} = \frac{4x}{t}$

$x' - \frac{1}{t} \cdot 4x = 2\underbrace{\sqrt{x}}_{x^\alpha}$ } $\alpha = \frac{1}{2}$ ...

5 Рикатијева ДЈ

$x' = p(t) x^2 + q(t) x + r(t)$ $\quad p, q, r : (a, b) \to \mathbb{R}$ непр.

СМЕНА: $x_p(t)$ – једно партикуларно решење

$p \equiv 0$: лин

$x(t) = x_p(t) + \frac{1}{y(t)}$ $\quad x(t) \rightsquigarrow y(t)$

$r \equiv 0$ · бер ($\alpha = 2$)

$\rightsquigarrow$ лин

$x' = x_p' + \left(\frac{1}{y}\right)' = x_p' - \frac{1}{y^2} y'$

x_p: $x_p' = p x_p^2 + q x_p + r$

$x_p' - \frac{y'}{y^2} = p\left(x_p + \frac{1}{y}\right)^2 + q\left(x_p + \frac{1}{y}\right) + r$

$x_p' - \frac{y'}{y^2} = \left(p x_p^2 + q x_p + r \right) + \frac{2 p x_p}{y} + \frac{p}{y^2} + \frac{q}{y} \quad / \cdot y^2$

$$x_p' - \frac{y'}{y^2} = \left(px_p^2 + qx_p + r\right) + \frac{2px_p}{y} + \frac{p}{y^2} + \frac{q}{y} \;/\cdot y^2$$

$$-y' = 2px_p y + p + qy \Rightarrow y' + y(2px_p + q) = -p$$

$$\text{op: } \begin{cases} x(t) \\ x_p(t) \end{cases}$$

(4) a) $t(2t-1)x' + x^2 - (4t+1)x + 4t = 0$

б) $x' = \dfrac{2\sin t - x^2 \sin t \cos^2 t}{\cos^2 t}$

в) $x' + x^2 + \dfrac{4x}{t} + \dfrac{2}{t^2} = 0$

а) $x_p = ?$

$t(2t-1)x' + x^2 - (4t+1)x + 4t = 0$ — 0 cim., 2 cim., 1 cim.

$\Rightarrow x_p = at + b$, $a, b \in \mathbb{R}$

$x_p' = a$

$t(2t-1)\cdot a + (at+b)^2 - (4t+1)(at+b) + 4t = 0$

$2at^2 - at + a^2t^2 + 2abt + b^2 - 4at^2 - 4bt - at - b + t = 0$

$t^2(2a + a^2 - 4a) + t(-a + 2ab - 4b - a + 4) + (b^2 - b) = 0$

$a^2 - 2a = 0 \;,\; a(a-2) = 0$

$b^2 - b = 0 \;,\; b(b-1) = 0$

$=0$?

$(a,b) \in \{(0,0), (0,1), (2,0), (2,1)\}$

$(0,0)$ ✗ $(0,1)$ ✓ $(2,0)$ ✓ $(2,1)$ ✓

$x_p(t) = 2t$

$x(t) \rightsquigarrow y(t)$

$$x(t) = x_p(t) + \frac{1}{y(t)}$$

$$x = 2t + \frac{1}{y} \Rightarrow x' = 2 + \left(-\frac{1}{y^2}\right)\cdot y' = 2 - \frac{y'}{y^2}$$

$$t(2t-1)x' + x^2 - (4t+1)x + 4t = 0 \longrightarrow t(2t-1)\cdot\left(2 - \frac{y'}{y^2}\right) + \left(2t + \frac{1}{y}\right)^2 - (4t+1)\left(2t + \frac{1}{y}\right) + 4t = 0 \;/\cdot y^2$$

$$t(2t-1)(2y^2-y') + (2ty+1)^2 - (4t+1)(2ty^2+y) + 4ty^2 = 0$$

$$4t^2y^2 - 2t^2y' - 2ty^2 + ty' + 4t^2y^2 + 4ty + 1 - 8t^2y^2 - 4ty - 2ty^2 - y + 4ty^2 = 0$$

$$-2t^2y' + ty' + 1 - y = 0$$

$$y'(t-2t^2) - y + 1 = 0$$

$$y' - \underbrace{\frac{1}{t-2t^2}}_{\text{ANH.}} y = -\frac{1}{t-2t^2} \quad \therefore \quad \underbrace{y(t)}_{x(t)}$$

6) $x' = \dfrac{2\sin t - x^2 \sin t \cos^2 t}{\cos^2 t}$

$x' = 2\dfrac{\sin t}{\cos^2 t} - \boxed{x^2 \cdot \sin t} \rightarrow \dfrac{\sin t}{\cos^2 t} \cdot a$

$\pm \sin = (\cos t)'$

$\boxed{x_p(t) = \dfrac{a}{\cos t}}$, $x_p' = -\dfrac{a \cdot (-\sin t)}{\cos^2 t} = a\dfrac{\sin t}{\cos^2 t}$

B) $x' + x^2 + \dfrac{4x}{t} + \dfrac{2}{t^2} = 0$

$\boxed{x_p = \dfrac{a}{t}}$

$x_p^2 = \dfrac{a^2}{t^2}$

$\dfrac{4x_p}{t} = \dfrac{4a}{t^2}$

$\dfrac{2}{t^2}$