$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + a_{n-2}(t)x^{(n-2)} + \dots + a_1(t)x' + a_0(t)x = f(t) \quad \to \text{лин јна ВР}$$

$a_{n-1}(t), \dots, a_0(t)$ – константе: ЛЈВРКК

$f \neq 0$: нехомогена

$f = 0$ хомогена

$$x^{(n)} + a_{n-1}x^{(n-1)} + \dots + a_1x' + a_0x = f(t)$$

ОР: $x(t) = \underbrace{x_H(t)}_{\text{ОР хомогене}} + \underbrace{x_p(t)}_{\text{ПР нехомогене}}$,

$(f=0,\ x_p=0)$

$\downarrow$ за хомогену

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0 \quad (*) \quad \text{(карактеристична јна)}$$

$\Rightarrow$ n нула (над $\mathbb{C}$)

$\to$ како изгледа база ВП решења x_H? (ВП дим $= n$)

1) $\mu \in \mathbb{R}$ нула $(*)$ реда k.

базни ел су $e^{\mu t}, te^{\mu t}, \dots, t^{k-1}e^{\mu t}$ (има их k)

2) $\alpha \pm i\beta \in \mathbb{C} \setminus \mathbb{R}$ нула $(*)$ реда k:

базни ел су $e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t,\ te^{\alpha t}\cos\beta t,\ te^{\alpha t}\sin\beta t, \dots, t^{k-1}e^{\alpha t}\cos\beta t,\ t^{k-1}e^{\alpha t}\sin\beta t$

(2k их је)

① (хомогене)

а) $x''' - 13x' - 12x = 0$

$\downarrow$ кар јна.

$\lambda^3 - 13\lambda - 12 = 0$

$\lambda = 1$ ×

$\lambda = -1$ ✓

$$\lambda^3 - 13\lambda - 12 = (\lambda+1)(\lambda^2 - \lambda - 12) = (\lambda+1)(\lambda-4)(\lambda+3)$$

$\lambda_1 = -1 \rightsquigarrow e^{-t}$

$\lambda_2 = -3 \rightsquigarrow e^{-3t}$

$\lambda_3 = 4 \rightsquigarrow e^{4t}$

$$\text{ОР}: x(t) = c_1 e^{-t} + c_2 e^{-3t} + c_3 e^{4t}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

б) $x''' - 3x'' + 9x' + 13x = 0$

$$\lambda^3 - 3\lambda^2 + 9\lambda + 13 = 0$$

$\lambda_1 = -1 \rightsquigarrow e^{-t}$

$$\lambda^3 - 3\lambda^2 + 9\lambda + 13 = (\lambda+1)\underbrace{(\lambda^2 - 4\lambda + 13)}$$

$$\lambda^2 - 4\lambda + 13 = 0$$

$$D = (-4)^2 - 4 \cdot 13 = 16 - 52 = -36$$

$$\lambda_{2/3} = \frac{4 \pm i \cdot 6}{2} = 2 \pm 3i$$

$$\rightsquigarrow e^{2t}\cos 3t, \; e^{2t}\sin 3t$$

$$\text{ОР}: x(t) = c_1 e^{-t} + c_2 e^{2t}\cos 3t + c_3 e^{2t}\sin 3t, \quad c_i \in \mathbb{R}$$

в) $x''' - 7x'' + 16x' - 12x = 0$

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

$\lambda = 1$ x

$\lambda = -1$ x

$\underline{\underline{\lambda = 2}}$

$$(\lambda - 2)(\lambda^2 - 5\lambda + 6) = (\lambda-2)(\lambda-3)$$

$\lambda_1 = \lambda_2 = 2 \rightsquigarrow e^{2t}, te^{2t}$

$\lambda_3 = 3 \rightsquigarrow e^{3t}$

$$\text{ОР}: x(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 e^{3t}, \quad c_i \in \mathbb{R}$$

г) $x^{(6)} - 4x^{(5)} + 8x^{(4)} - 8x''' + 4x'' = 0$

$$\lambda^6 - 4\lambda^5 + 8\lambda^4 - 8\lambda^3 + 4\lambda^2 = 0$$

$$\lambda^2(\lambda^4 - 4\lambda^3 + 8\lambda^2 - 8\lambda + 4) = 0$$

$$\lambda^4 - 4\lambda^3 + 8\lambda^2 - 8\lambda + 4 = (\lambda^2 + a\lambda + b)(\lambda^2 + c\lambda + d)$$

± 1, ± 2, ± 4 X

↳ делује да нема $\mathbb{R}$ нула

$\lambda^3: \quad a + c = -4$

$\lambda^2: \quad b + d + ac = 8$

$\lambda: \quad ad + bc = -8$

$1: \quad bd = 4 \rightsquigarrow (1,4), (2,2), (-2,-2), (-1,-4) \ldots$

$b=d=2,\ a=c=-2 \quad \Rightarrow \quad (\lambda^2-2\lambda+2)^2$

$\lambda_1=\lambda_2=0 \rightsquigarrow \underbrace{e^{0t}}_{=1},\ \underbrace{te^{0t}}_{=t}$

$\lambda_{3/4}=\lambda_{5/6}=1\pm i \rightsquigarrow e^t\cos t,\ e^t\sin t,\ te^t\cos t,\ te^t\sin t$

ОР: $x(t)=c_1+c_2t+c_3e^t\cos t+c_4e^t\sin t+c_5te^t\cos t+c_6te^t\sin t \quad , c_i\in\mathbb{R}$

Кошијев проблем: $x^{(n)}+a_{n-1}(t)x^{(n-1)}+\dots+a_0(t)x=f(t)$

услови
$$\left.\begin{aligned} x(t_0)&=x_0\\ x'(t_0)&=x_1\\ &\vdots\\ x^{(n-1)}(t_0)&=x_{n-1}\end{aligned}\right\}\ n \text{ јна}$$

↳ у истој тачки

није КП $x(0)=x(1)=0$ X

$x(0)=0,\ x'(1)=2$ X

(2) Реши Кошијев проблем: $x''' + x'' = 0$ (3. реда)

$x(0)=1$

$x'(0)=0$

$x''(0)=1$

$\lambda^3+\lambda^2=0$

$\lambda^2(\lambda+1)=0$

$\lambda_1=\lambda_2=0 \rightsquigarrow 1, t$

$\lambda_3=-1 \rightsquigarrow e^{-t}$

ОР $x(t)=c_1+c_2t+c_3e^{-t} \quad , c_i\in\mathbb{R}$

$x'(t)=c_2-c_3e^{-t}$

$x''(t)=c_3e^{-t}$

$$\left.\begin{aligned} x(0)=1 &\Rightarrow c_1+c_3=1\\ x'(0)=0 &\Rightarrow c_2-c_3=0\\ x''(0)=1 &\Rightarrow c_3=1\end{aligned}\right\}$$

$c_1=0$

$c_2=c_3=1$

$x_K(t)=t+e^{-t}$

нехомогени случај: $f\neq 0$, $\quad x^{(n)}+a_{n-1}x^{(n-1)}+\dots+a_1x'+a_0x=f(t)$

у посебним случајевима: $f(t)=e^{\alpha t}(P_n(t)\cdot\cos\beta t+Q_m(t)\sin\beta t)$

P_n - пол ст n

Q_m - пол ст m

λ - вишеструкост броја $\alpha\pm i\beta$ као решења каракт. јне

$x_p(t)=t^\lambda e^{\alpha t}(R_k(t)\cos\beta t+T_k(t)\sin\beta t)$ $\quad R_k, T_k$ - пол

$$\Downarrow$$

$$x_p(t) = t^{s}\, e^{\alpha t}\left(R_k(t)\cos\beta t + T_k(t)\sin\beta t\right)$$

R_k, T_k - пол ст. k

$k = \max\{m, n\}$

③ а) $x''' - x'' + x' - x = t^2 + t$

$\lambda^3 - \lambda^2 + \lambda - 1 = 0$

$(\lambda - 1)(\lambda^2 + 1) = 0$

$1, \pm i$

OP. $x(t) = x_H(t) + x_p(t)$

$x_H(t) = C_1 e^t + C_2\cos t + C_3 \sin t$

$f(t) = t^2 + t$

$\left.\begin{array}{l} \alpha = 0 \quad (e^{\alpha t} = 1) \\ \beta = 0 \quad (\cos\beta t = 1,\ \sin\beta t = 0) \end{array}\right\} \quad \alpha \pm i\beta = 0 \pm i0 = \underline{\underline{0}}$

$P_n(t) = t^2 + t \quad \left.\begin{array}{l} , n = 2 \\ m = 0 \text{ (небитно)} \end{array}\right\} \quad k = \max\{0, 2\} = 2$

$s = ?$ 0 није у скупу $\{1, \pm i\}$ $\Rightarrow s = 0$

$x_p(t) = R_2(t) = at^2 + bt + c \quad \rightsquigarrow \quad x_p''' - x_p'' + x_p' - x_p = t^2 + t$

$\left.\begin{array}{l} x_p' = 2at + b \\ x_p'' = 2a \\ x_p''' = 0 \end{array}\right\} \longrightarrow \quad 0 - 2a + 2at + b - at^2 - bt - c = t^2 + t$

$\left.\begin{array}{l} -2a + b - c = 0 \\ 2a - b = 1 \\ -a = 1 \end{array}\right\}$

$a = -1,\ b = -3,\ c = -1$

$x_p(t) = -t^2 - 3t - 1$

б) $x''' - x'' + x' - x = \cos t + 2e^t$ → није у одг. облику

$1, \pm i$

$x(t) = C_1 + C_2\cos t + C_3 \sin t + x_{p_1}(t) + x_{p_2}(t)$

$f_1(t) = \cos t$

$f_2(t) = 2e^t$

$L(x) = f_1(t) + f_2(t)$ → лин

$L(x_{p_1}) = f_1(t)$

$L(x_{p_2}) = f_2(t)$

$L(x_{p_1} + x_{p_2}) = L(x_{p_1}) + L(x_{p_2}) = f_1(t) + f_2(t)$

$f_1.$ $\alpha=0$, $\beta=1$

$\left.\begin{array}{ll} P_n\equiv 1 & ,n=0 \\ Q_m\equiv 0 & ,m=-\infty \end{array}\right\} k=0 \Rightarrow R_0=\underline{c_1},\ T_0=\underline{c_2}$

$\alpha\pm i\beta = 0\pm i\cdot 1 = \pm i \Rightarrow \underline{\underline{s=1}}$ $\qquad x_{p_1}(t)=t\left(c_1\cdot\cos t + c_2\cdot\sin t\right)$ $\ldots$ $c_1=c_2=-\frac{1}{4}$

$f_2:$ $\alpha=1$

$\beta=0$

$\left.\begin{array}{l} P_n(t)=2 \Rightarrow n=0 \\ Q_m(t)=\text{ништавог}\ (m=0) \end{array}\right\} k=0$, $R_0=c_1,\ T_0=c_2$

$\alpha\pm i\beta = 1\pm i\,0 = 1 \Rightarrow s=1$ $\qquad x_{p_2}(t)=t\cdot e^{t}\cdot(c_1)$ $\ldots$ $c_1=1$

В) $x''-x=\sin^2 t$

$$f(t)=\sin^2 t=\frac{1-\cos 2t}{2}=\underbrace{\frac{1}{2}}_{x_{p_1}}-\underbrace{\frac{\cos 2t}{2}}_{x_{p_2}}$$

Г) $\underbrace{x''-4x'+5x}_{2\pm i}=(\sin t+2\cos t)\cdot e^{2t}$

$\left.\begin{array}{l} \alpha=2 \\ \beta=1 \\ \left.\begin{array}{l} P_n(t)\equiv 2 \Rightarrow n=0 \\ Q_m(t)\equiv 1 \Rightarrow m=0 \end{array}\right\} k=0 \Rightarrow R_0(t)\equiv c_1, T_0(t)\equiv c_2 \\ \alpha\pm i\beta = \underline{2\pm i} \Rightarrow s=1 \end{array}\right\} \Rightarrow x_p(t)=t\cdot e^{2t}\cdot(c_1\cos t+c_2\sin t)$ $\vdots$

Д) $x''-2x'+x=\dfrac{e^t}{t}$ → није у овој облику

$(\lambda-1)^2=0 \rightsquigarrow 1,1 \rightsquigarrow e^t,\ te^t$

идеја: $x_p(t)=e^t\ g(t)$

$x_p'(t)=e^t\cdot g(t)+e^t\ g'(t)=e^t\cdot(g(t)+g'(t))$

$x_p''(t)=e^t\cdot(g(t)+g'(t))+e^t\ (g'(t)+g''(t))=e^t\ (\ g(t)+2g'(t)+g''(t))$

$$\dots g(t) = e^t \cdot (g(t)+g'(t))$$

$$x_p''(t) = e^t\cdot(g(t)+g'(t)) + e^t(g'(t)+g''(t)) = e^t(g(t)+2g'(t)+g''(t))$$

$$x_p'' - 2x_p' + x_p = \frac{e^t}{t}$$

$$e^t(g+2g'+g'') - 2e^t(g+g') + e^t g = \frac{e^t}{t} \quad / \, e^t$$

$$g'' = \frac{1}{t} \; / \int \Rightarrow g' = \ln|t| + c_1 \Rightarrow g(t) = t\cdot\ln|t| - t + c_1 t + c_2 = t\cdot\ln|t|$$

$$\left.\begin{array}{l} c_1 = 1 \\ c_2 = 0 \end{array}\right\}$$

$$x_p(t) = e^t\cdot t\cdot\ln|t|$$

OP: $x(t) = c_1e^t + c_2e^t\cdot t + e^t\cdot t\cdot\ln|t|$, $c_1, c_2 \in \mathbb{R}$