6 DJ у тотални диференцијал

$$M(t,x)\,dt + N(t,x)\,dx = 0$$

$$x' = \frac{f(t,x)}{g(t,x)} \iff \frac{dx}{dt} = \frac{f(t,x)}{g(t,x)} \;/\; \text{„}dt\text{"} \iff$$

$$g(t,x)\,dx = f(t,x)\,dt \iff f(t,x)\,dt - g(t,x)\,dx = 0$$

↑ dj у диференцијалном облику

$\exists F(t,x)$ ↙ претпоставка , $dF(t,x) = M(t,x)\,dt + N(t,x)\,dx$

$$\frac{\partial F}{\partial t}(t,x)\,dt + \frac{\partial F}{\partial x}(t,x)\,dx$$

$$\Rightarrow \begin{cases} M(t,x) = \frac{\partial F}{\partial t}(t,x) \Big/ {}^{1}_{x} \\ N(t,x) = \frac{\partial F}{\partial x}(t,x) \Big/ {}^{1}_{t} \end{cases} \Rightarrow M'_x = \frac{\partial^2 F}{\partial t\,\partial x} = \frac{\partial^2 F}{\partial x\,\partial t} = N'_t$$

тот $\Rightarrow M'_x = N'_t$

важи и $\Leftarrow$ ако радимо у прост-повезаној области

↑ само где је дна деф, где је тополатрамо, --

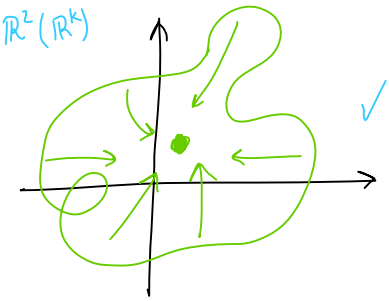
ОР: $0 = M\,dt + N\,dx = dF \Rightarrow F(t,x) = c,\ c \in \mathbb{R}$

↑ имплицитно задато

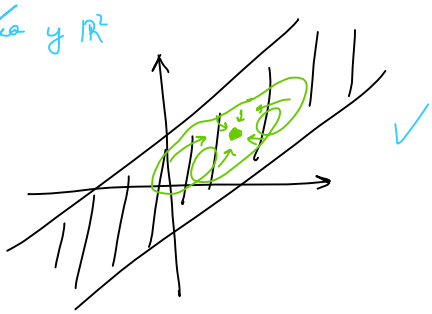
прост-повезане области (ПП)

$D \subseteq \mathbb{R}^k$ је ПП $\iff$ свака непрекидна затворена крива у D се може скупити у тачку кроз D и D је повезана

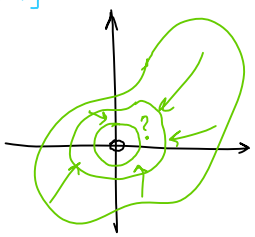
пр. 1) $\mathbb{R}^2 (\mathbb{R}^k)$ ✓



2) трака у $\mathbb{R}^2$ ✓



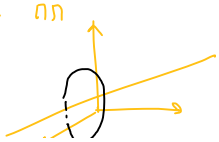
3) $\mathbb{R}^2 \setminus \{(0,0)\}$ ✗

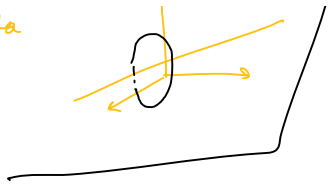


„ПП у $\mathbb{R}^2$ ако нема рупа"

↓ не важи у $\mathbb{R}^k$!

- $\mathbb{R}^3 \setminus \{(0,0,0)\}$ јесте ПП
- $\mathbb{R}^3 \setminus \ell$ није ПП (↓ права)





(1) а) $2t(1+\sqrt{t^2-x})dt - \sqrt{t^2-x}\,dx = 0$

б) $(1+x^2\sin t)\,dt - 2x\cos^2 t\,dx = 0$

в) $(tx^2+3t^2x)\,dt + (t^3+t^2x)\,dx = 0$

а) $M(t,x) = 2t(1+\sqrt{t^2-x})$

$N(t,x) = -\sqrt{t^2-x}$

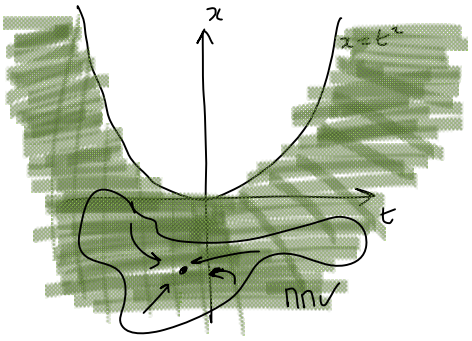
$M'_x = 2t\cdot\frac{1}{2\sqrt{t^2-x}}(-1) = \frac{-t}{\sqrt{t^2-x}}$

$N'_t = -\frac{1}{2\sqrt{t^2-x}}\cdot 2t = \frac{-t}{\sqrt{t^2-x}}$))

област?

$t^2-x\geq 0$

$x\leq t^2$



⇓

тот. д

$\Rightarrow \exists F,\ dF = M\,dt + N\,dx$, $F=?$

$\frac{\partial F}{\partial t} = M = 2t + 2t\sqrt{t^2-x}$

$\frac{\partial F}{\partial x} = N = -\sqrt{t^2-x}$ $\Big/\int dx$

$F = -\int\sqrt{t^2-x}\,dx = \frac{2}{3}(t^2-x)^{3/2} + \underbrace{c(t)}_{(c\in\mathbb{R}\times)}$, $c:\mathbb{R}\to\mathbb{R}$

$\Rightarrow \frac{\partial F}{\partial t} = \frac{2}{3}\cdot\frac{3}{2}(t^2-x)^{1/2}\cdot 2t + c'(t) = 2t\sqrt{t^2-x} + c'(t)$ (= $2t+2t\sqrt{t^2-x}$) $\Rightarrow c'(t) = 2t$

↙ ако не дозвољено да зависи само од t

⇒ ГРЕШКА (у рачуну)

$c(t) = t^2$

ОР: $\frac{2}{3}(t^2-x)^{3/2} + t^2 = C$, $C\in\mathbb{R}$

7 Интегрирајући фактор

$M(t,x)\,dt + N(t,x)\,dx = 0$

$M'_x \neq N'_t \Rightarrow$ није тот. д

Идеја: $\underbrace{\mu(t,x)\cdot\big(M(t,x)\,dt + N(t,x)\,dx\big) = 0}_{\text{услов тот. д}}$ ⇝ $\mu = ?$ (интегрирајући фактор)

$$\left.\begin{aligned}\widetilde{M} &= \mu \cdot M \\ \widetilde{N} &= \mu N\end{aligned}\right\} \quad \widetilde{M}'_x = \widetilde{N}'_t$$

$\mu(t,x)$ је често у облику $\mu = \mu(w) = \mu(w(t,x))$

$\mu: \mathbb{R}^2 \to \mathbb{R}$, $\mu: \mathbb{R} \to \mathbb{R}$, $w: \mathbb{R}^2 \to \mathbb{R}$

$$\widetilde{M}'_x = \frac{\partial}{\partial x}(\mu M) = \mu'_x \cdot M + \mu M'_x = \mu'(w)\, w'_x \cdot M + \mu M'_x$$
$$\widetilde{N}'_t = \frac{\partial}{\partial t}(\mu N) = \mu'_t \cdot N + \mu N'_t = \mu'(w)\, w'_t N + \mu \cdot N'_t$$

$$\Rightarrow \quad \mu'(w)\left(w'_x \cdot M - w'_t N\right) = \mu\left(N'_t - M'_x\right)$$

$\mu'(w) = \frac{d\mu}{dw}$

$$\frac{\mu'(w)}{\mu(w)} = \frac{N'_t - M'_x}{w'_x \cdot M - w'_t \cdot N}$$

$$\frac{d\mu}{\mu} = \frac{N'_t - M'_x}{w'_x \cdot M - w'_t N}\, dw$$

→ мора да зависи само од w

Циљ: наћи w!

Често:
- $w(t,x) = at + bx$ ($w = x$, $w = t$, ...)
- $w(t,x) = a\ln|t| + b\ln|x|$
- $w(t,x) = f(t) + g(x)$
- $w(t,x) = f(t)\, g(x)$

Проверити $\frac{1}{\mu} = 0$ за једноставно решење

(2) а) $(q(t) - p(t)x)\,dt - dx = 0$, p, q познате ← ($w = t$), утврдити да ли линеарне

б) $2tx\ln x\,dt + (t^2 + x^2\sqrt{x^2+1})\,dx = 0$

в) $x(2 - 3tx^2)\,dt - t(1 + tx^2)\,dx = 0$, на области $G = \{x > 0, t > 0\}$, наћи решење кроз $(2,1)$

г) $\left(\sqrt{t^2 - x} + 2t\right)dt - dx = 0$, μ у облику $\mu(t^2 - x)$ ($w = t^2 - x$)

д) $(t+2)\sin x\,dt + 2t\cos x\,dx = 0$ ($w = x$)

$\left(\searrow\ x' = \frac{dx}{dt} = -\frac{t+2}{t} \cdot \operatorname{tg} x \rightsquigarrow \text{РП}\right)$

б) $M'_x = (2tx\ln x)'_x = 2t(1 + \ln x)$

$N'_t = \left(t^2 + x^2\sqrt{4+x^2}\right)'_t = 2t$ ##

б) $M'_x = (2tx\ln x)'_x = 2t(1+\ln x)$

$N'_t = (t^2 + x^2\sqrt{1+x^2})'_t = 2t$ ##

$$\frac{d\mu}{\mu} = \frac{2t-(2t+2t\ln x)}{w'_x \cdot M - w'_t \cdot N}\,dw = \frac{-2t\ln x}{w'_x(2tx\ln x) - w'_t\underbrace{(t^2+x^2\sqrt{x^2+1})}_{=0}}\,dw = \frac{-2t\ln x}{\frac{1}{x}\,2tx\ln x}\,dw = -dw \;/\int$$

$w'_t = 0 \Rightarrow w = w(x)$

$w = x$ нарушает

$w = \ln x$

$w = x^2$

Нар $w = x^2$

$$\frac{d\mu}{\mu} = \frac{-2t\ln x}{2x\,2tx\ln x}\,dw = -\frac{1}{2x^2}\,dw = -\frac{1}{2w}\,dw = -\frac{dw}{2w} \;/\int$$

$\Rightarrow \ln|\mu| = -w + \tilde{c} \;/e^{\cup}$

$\mu = c\cdot e^{-w}$ ($c=$

$\mu(t,x) = \mu(w(t,w)) = c\cdot e^{-w(t,x)} = c\,e^{-\ln x} = \frac{c}{x}$ $(c=1)$

$\mu = \frac{1}{x}$

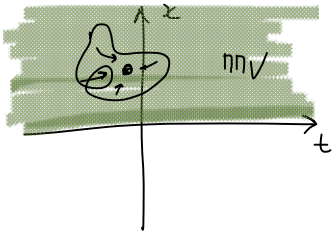
$\frac{1}{\mu} = 0 \Leftrightarrow x = 0$ ✗

($\ln x$, $x>0$)

$/\cdot\frac{1}{x} \Rightarrow \underbrace{2t\ln x\,dt}_{M} + \underbrace{\left(\frac{t^2}{x} + x\sqrt{x^2+1}\right)}_{N}dx = 0$ ТОТ.Д ?

$M'_x = N'_t$ ✓

область: $x>0$, $x\neq 0$



$\Rightarrow \exists F = ?$

$$\left.\begin{aligned}\frac{\partial F}{\partial t} &= 2t\ln x\\ \frac{\partial F}{\partial x} &= \frac{t^2}{x} + x\sqrt{x^2+1}\end{aligned}\right\} \therefore \text{ОР: } F(t,x) = t^2\ln x + \frac{1}{3}(x^2+1)^{3/2} = C,\ C\in\mathbb{R}$$

в) $x(2-3tx^2)\,dt - t(1+tx^2)\,dx = 0$, на области $G = \{x>0, t>0\}$, найти решение кроз (2,1)

$M'_x = 2 - 9tx^2$

$N'_t = -1 - 2tx^2$

$$\frac{d\mu}{\mu} = \frac{(-1-2tx^2)-(2-9tx^2)}{w'_x\cdot x(2-3tx^2) + w'_t\,t(1+tx^2)}\,dw = \frac{-3+7tx^2}{\cdots}\,dw$$

$$\frac{d\mu}{\mu} = \frac{(-1-2tx^2)-(2-3tx^2)}{w'_x \cdot x(2-3tx^2)+w'_t\, t(1+tx^2)}\,dw = \frac{-3+7tx^2}{w'_x \cdot x(2-3tx^2)+w'_t\, t(1+tx^2)}\,dw$$

идея: w'_x спрямо $x \to w'_x = \frac{1}{x}$?

w'_t спрямо $t \to w'_t = \frac{1}{t}$?

$w = a\ln|x| + b\ln|t| = a\ln x + b\ln t \Rightarrow w'_x = \frac{a}{x}$

$(x,t>0)$

$w'_t = \frac{b}{t}$

$$\frac{d\mu}{\mu} = \frac{-3+7tx^2}{a(2-3tx^2)+b(1+tx^2)}\,dw = \frac{-3+7tx^2}{(2a+b)+tx^2(-3a+b)}\,dw = \frac{-3+7tx^2}{-3+7tx^2}\,dw = 1\,dw \;/\int$$

$2a+b=-3$

$-3a+b=7$

$5a=-10 \Rightarrow a=-2$

$b=1$

$\Rightarrow w = -2\ln x + \ln t$

$\Rightarrow \mu = e^w = e^{-2\ln x+\ln t} = \frac{t}{x^2}$

$/\ \frac{t}{x^2} \Rightarrow \underbrace{\left(2\frac{t}{x}-3t^2x\right)}_{M}dt \underbrace{-\left(\frac{t^2}{x^2}+t^3\right)}_{N}dx = 0$

$M'_x = N'_t$

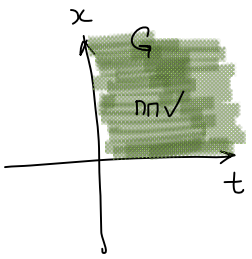
$\Rightarrow$ ТОТ. Д

$\frac{\partial F}{\partial t} = 2\frac{t}{x} - 3t^2x$

$\frac{\partial F}{\partial x} = -\left(\frac{t^2}{x^2}+t^3\right)$

$\Big\} \therefore F(t,x) = \frac{t^2}{x} - t^3x = c \in \mathbb{R}$ (ОР)

обласи



$\frac{1}{\mu} = 0 \Leftrightarrow \frac{x^2}{t} = 0 \Leftrightarrow x = 0$?

$(0\cdot dt - \ldots - 0 = 0$ ✓$)$

$\{x=0\} \cap G = \phi$

$\Rightarrow$ няма решение

крив $(2,1)$: $t=1,\ x=1 \Big\} \ \frac{4}{1} - 8 \cdot 1 = c \Rightarrow c = -4$

пр: $\frac{t^2}{x} - t^3x = -4$