$$X(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad X'(t)=\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

$X' = A \cdot X$ → хомогена линеарна дј (систем реда 2)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

→ динамички систем

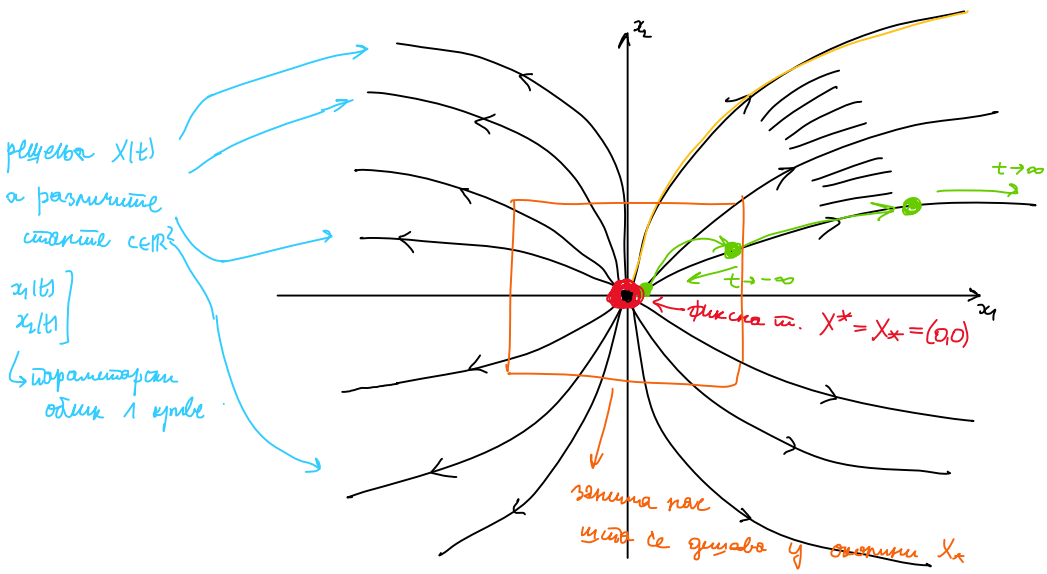
може $A(t)$

$A \in M_2(\mathbb{R})$ ⇒ са константним коеф (ЛСДЈКК)

решења $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ (зависи од $c \in \mathbb{R}^2$)

фазни портрет

фазна трајекторија



Еквилибријуми. $X' = 0$ $(X^* = X_e)$
(те равнотеже, фикс. тј.)

ЛСДЈКК: $A \cdot X = 0$ ⇒ решење је век потпростор од $\mathbb{R}^2$ ⇒ 1) тачка $(0,0)$
2) права (права кроз $(0,0)$)
3) раван ($\mathbb{R}^2$)

$X' = AX$, $A = PDP^{-1}$, $A \sim D$, $P \in GL_2(\mathbb{R})$

$\cdot P^{-1}$ \ $X' = PDP^{-1}X$

$P^{-1}X' = DP^{-1}X$)

$P^{-1}X' = D P^{-1} X$

смена: $P^{-1}X = Y$, $Y' = P^{-1}X$ $\Rightarrow$ $Y' = D \cdot Y$

D — линеарна трансф $\rightarrow A$

Шта је D? 1) дијагонална $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

2) Жорданов блок $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

3) комплексно-конјуговани сопствени вредности $\alpha \pm i\beta$, $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

1. Скицирати фазни портрет динамичког система $X' = AX$, ако је:

(1) $A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$; (2) $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$; (3) $A = \begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}$.

(1.1) $X_* = ?$ $AX = 0$, $\begin{matrix} -x_1 + 2x_2 = 0 \\ -3x_2 = 0 \end{matrix}$ $\Rightarrow x_1 = x_2 = 0 \Rightarrow X_* = (0,0)$

сопс. вр $\det(A - \lambda E) = \det\left(\begin{bmatrix} -1-\lambda & 2 \\ 0 & -3-\lambda \end{bmatrix}\right) = (-1-\lambda)(-3-\lambda) - 0 = (\lambda+1)(\lambda+3)$

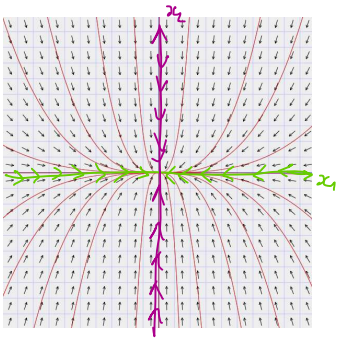
$\lambda_1 = -1$, $\lambda_2 = -3$ $\quad (\lambda_1, \lambda_2 < 0)$

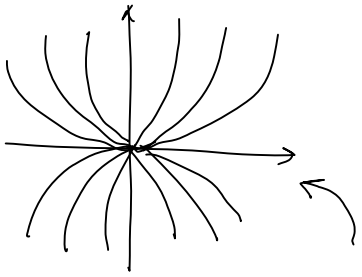
$A \sim D = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \rightarrow$ стабилан чвор

$Y' = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \cdot Y$ (x_1, x_2)

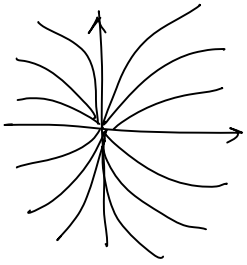
$-1 < 0 \rightarrow$ ка еквилибријуму

$-3 < 0 \rightarrow$ ка еквилибријуму





V



$|\lambda_1| < |\lambda_2| \rightarrow$ брже уз x_1-осу

$(A - \lambda_1 E)v_1 = 0$

$\begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} v_1 = 0$

$v_1 = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$(A - \lambda_2 E)v_2 = 0$

$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} v_2 = 0$

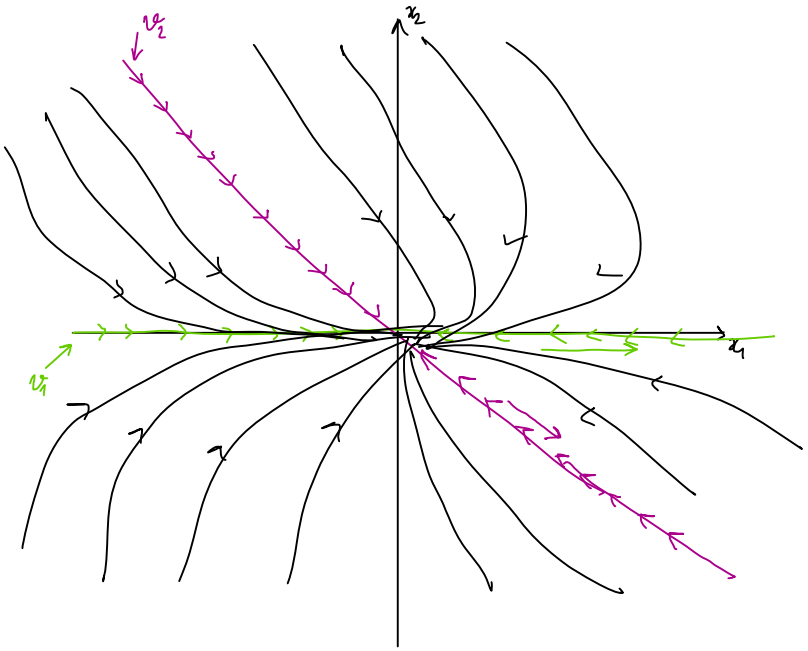
$v_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $2\alpha + 2\beta = 0$

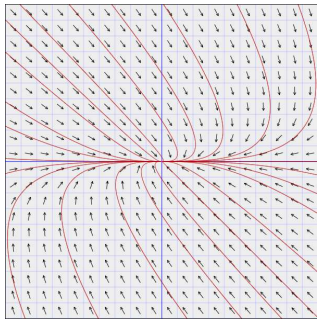
$v_1 = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$v_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $2\alpha + 2\beta = 0$

$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P = [v_1 \downarrow \; v_2 \downarrow] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$





да ли је чвор оријентисан? $\det P = -1 < 0 \rightarrow$ има рефлексије

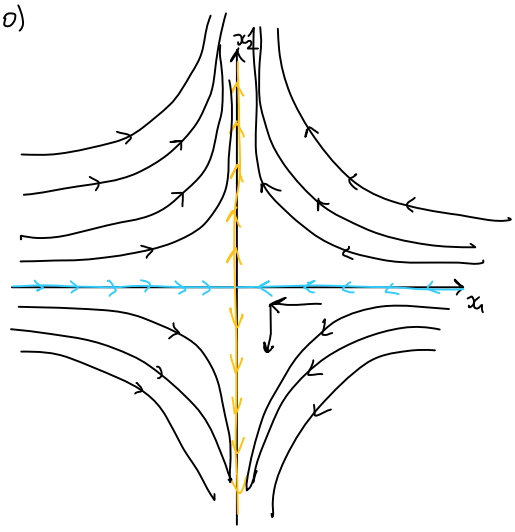
(1.2) $A = \begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}$ $\quad \left.\begin{matrix} -x_1 + 3x_2 = 0 \\ 5x_1 - 3x_2 = 0 \end{matrix}\right\} \; X^* = (0,0)$

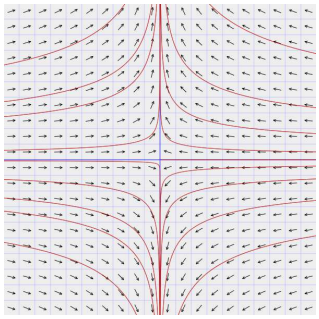
$\lambda_1 = -6, \; \lambda_2 = 2 \rightarrow$ седло

$A \sim D = \begin{bmatrix} -6 & \\ & 2 \end{bmatrix}$ $\quad -6 < 0$, $2 > 0$

$|\lambda_2| < |\lambda_1|$

↳ уз x_2-осу

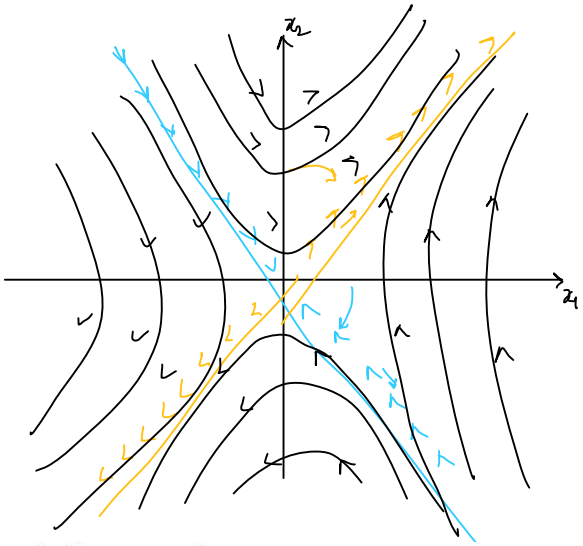


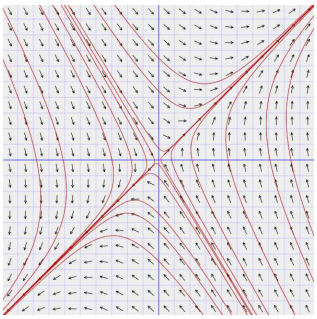


сопс век. $(A - \lambda_j E) v_j = 0$

$v_1 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 1 \\ -5 & 1 \end{bmatrix}$, $\det P = 8 > 0$





2. Скицирати фазни портрет динамичког система $X' = AX$, ако је:

(1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$;

(2) $A = \begin{bmatrix} -2 & -5 \\ 2 & 2 \end{bmatrix}$;

(3) $A = \begin{bmatrix} -1 & -1 \\ 2 & -3 \end{bmatrix}$.

(4) $A = \begin{bmatrix} 7 & -10 \\ 4 & -5 \end{bmatrix}$.

(2.2) $\lambda_{1/2} = \pm i\sqrt{6}$ ⟶ центар → елипсе око коорд почетка (слеш у (2.1) су кругови са центром у (0,0))

$(\alpha \pm i\beta \wedge \alpha = 0 \Rightarrow \pm i\beta)$

$A = PDP^{-1}$

$D = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{6} \\ -\sqrt{6} & 0 \end{bmatrix}$, $P = \begin{bmatrix} Re(v)\downarrow & Im(v)\downarrow \end{bmatrix}$

v – комплексан сопствени вектор за $\alpha + i\beta$

$\lambda_1 = \overline{\lambda_2} = i\sqrt{6}$

$(A - \lambda_1 E)v_1 = 0$

$\lambda_1 \rightsquigarrow v_1$

$\lambda_2 \rightsquigarrow v_2$

$\lambda_2 = \overline{\lambda_1} \Rightarrow v_2 = \overline{v_1}$

$\begin{bmatrix} -2-i\sqrt{6} & -5 \\ 2 & 2-i\sqrt{6} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$, $a, b \in \mathbb{C}$

$(-2-i\sqrt{6})a - 5b = 0$

~~$2a + (2-i\sqrt{6})b = 0$~~

$b = \frac{-2-i\sqrt{6}}{5} a$

$a = -5$, $b = 2+i\sqrt{6} \Rightarrow v_1 = \begin{bmatrix} -5 \\ 2+i\sqrt{6} \end{bmatrix}$

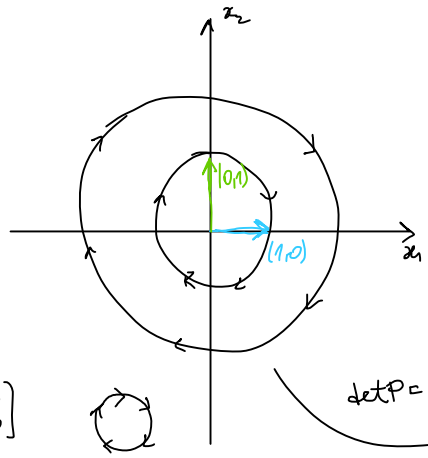
$Re(v) = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$, $Im(v) = \begin{bmatrix} 0 \\ \sqrt{6} \end{bmatrix}$

$P = \begin{bmatrix} -5 & 0 \\ 2 & \sqrt{6} \end{bmatrix}$

$X' = DX \Rightarrow \begin{cases} x_1' = x_2\sqrt{6} \\ x_2' = -x_1\sqrt{6} \end{cases} \rightsquigarrow \begin{cases} x_1 = c_1\cos(\sqrt{6}t) + c_2\sin(\sqrt{6}t) \\ x_2 = -c_1\sin(\sqrt{6}t) + c_2\cos(\sqrt{6}t) \end{cases}$

$\|X\|^2 = x_1^2 + x_2^2 = c_1^2\cos^2(\sqrt{6}t) + c_2^2\sin^2(\sqrt{6}t) + c_1^2\sin^2(\sqrt{6}t) + c_2^2\cos^2(\sqrt{6}t) = c_1^2 + c_2^2 \equiv \text{const}$

$\Rightarrow$ решења су на кругу

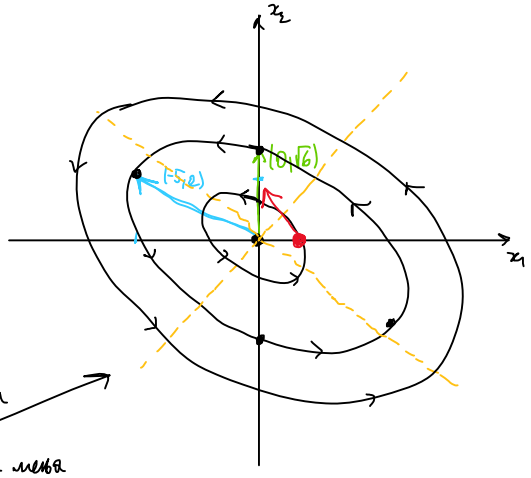


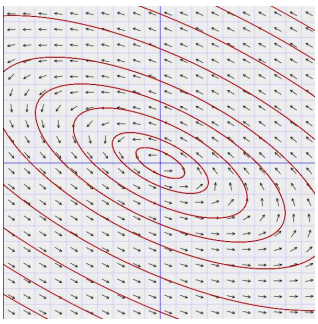
$D = \begin{bmatrix} 0 & \sqrt{6} \\ -\sqrt{6} & 0 \end{bmatrix}$

$D = \begin{bmatrix} 0 & -\sqrt{6} \\ \sqrt{6} & 0 \end{bmatrix}$

$\det P = -\sqrt{6} < 0 \rightarrow$ седло

$\rightarrow$ смер се мења





$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$X' = AX = \begin{bmatrix} -2 & -5 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

(2.3) $A = \begin{bmatrix} -1 & -1 \\ 2 & -3 \end{bmatrix}$

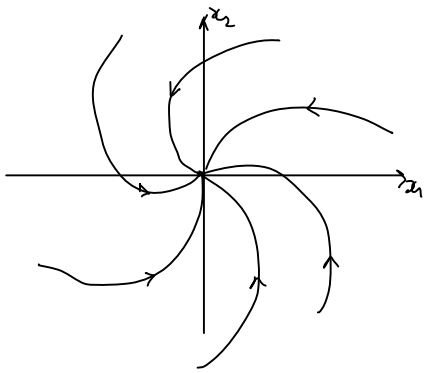
$\lambda_{1/2} = -2 \pm i \quad (\alpha \neq 0)$

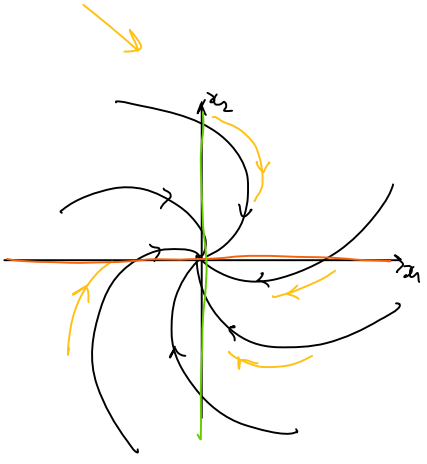
$(-2 < 0)$

стабилна спирала

$D = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$ $1 > 0$

нестабилан смер





$\lambda = \alpha + i\beta = -2 + i$

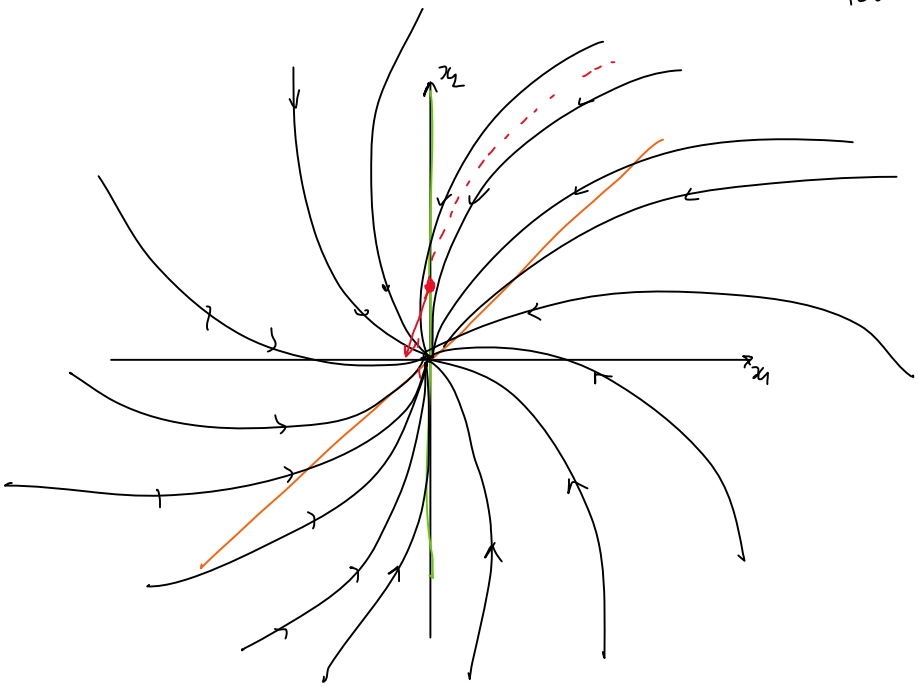
$(A - \lambda E)v = 0$

$v = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \rightsquigarrow P = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

$\det P = -1 < 0$

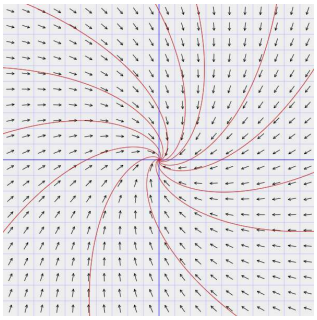
$v = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightsquigarrow P = \begin{bmatrix} 1 & -1 \\ \uparrow & \uparrow \\ Re\,v & Im\,v \end{bmatrix}$

$\det P = -1 < 0$ ↓ имена оригинальну

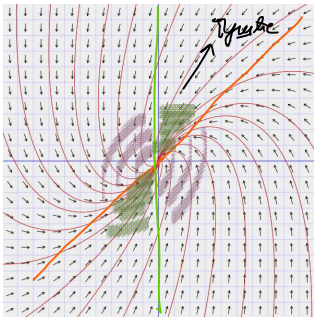


$X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $X' = AX = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$

D ↴



A ↴



3. Скицирати фазни портрет динамичког система $X' = AX$, ако је:

(1) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$;

(2) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$;

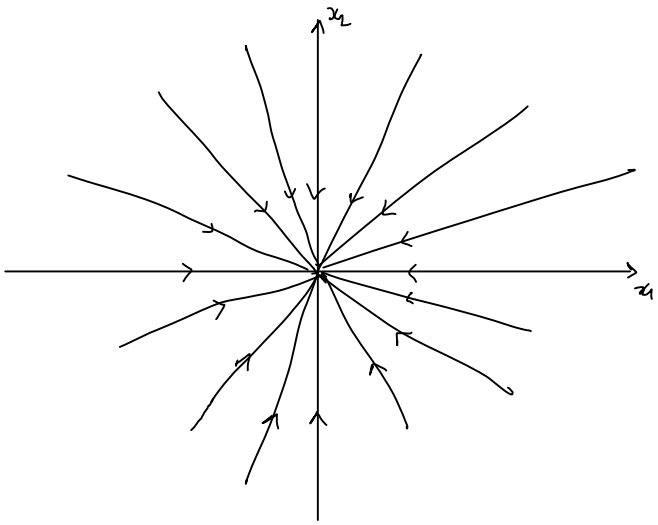
(3) $A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$;

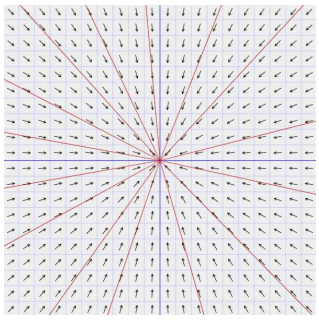
(4) $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$;

(3.1) $\lambda_1 = \lambda_2 = -1 < 0$

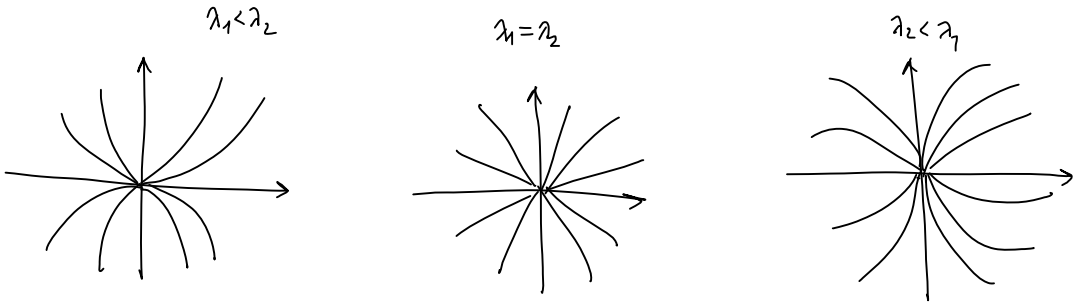
$A \sim \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$? ✓ $A \sim \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

$-1 < 0$ → стабилна звезда





$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$



(3.4) $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$ $\quad X^* = (0,0)$ $(\det A \neq 0)$

$\lambda = \lambda_1 = \lambda_2 = -2$

$$A \sim \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad V \quad \underbrace{A \sim \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}}$$

ако је димензија сопственог потпростора од λ мања од 2 (једнака 1)

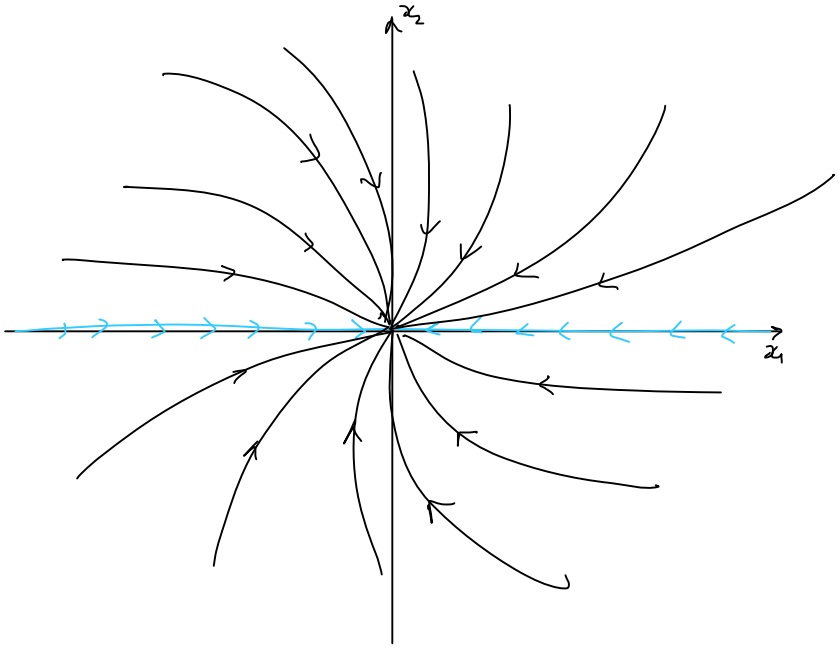
$(A - \lambda E)v = 0$

$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \Rightarrow -\alpha + \beta = 0 \Rightarrow \alpha = \beta \rightarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

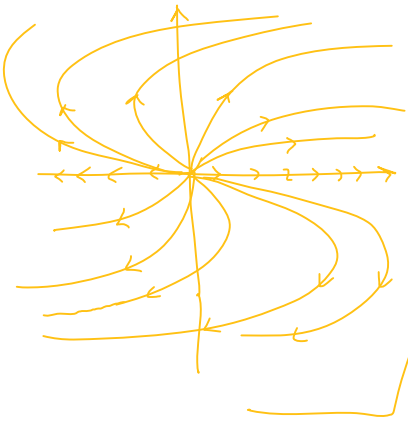
$\dim\left(\mathrm{Lin}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right) = 1 \Rightarrow$ један Жорданов блок

$A \sim \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} = D \rightarrow$ стабилан дегенерисан чвор

$(-2 < 0)$



У случаю: $D = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ исти.



$P = ?$ → правимо v_2 ← генерисани сопствени вектор за $v_1 = v$

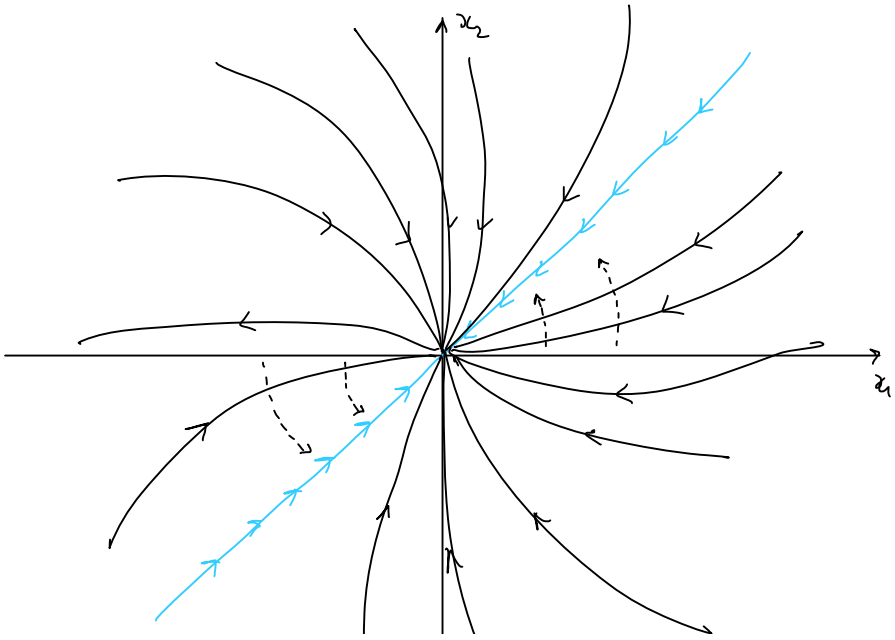
$$(A - \lambda E) v_2 = v_1$$

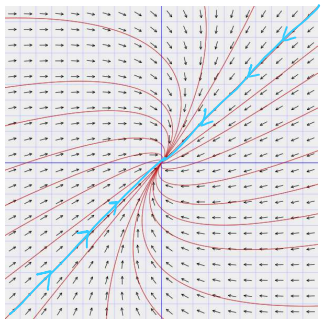
$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

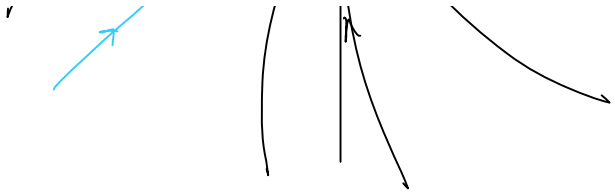
$-\gamma + \delta = 1 \Rightarrow \delta = 1 + \gamma$

нпр $\gamma = 0$, $\delta = 1$ $\Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} v_1 \downarrow & v_2 \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \det P = 1 > 0$$ нема рефл







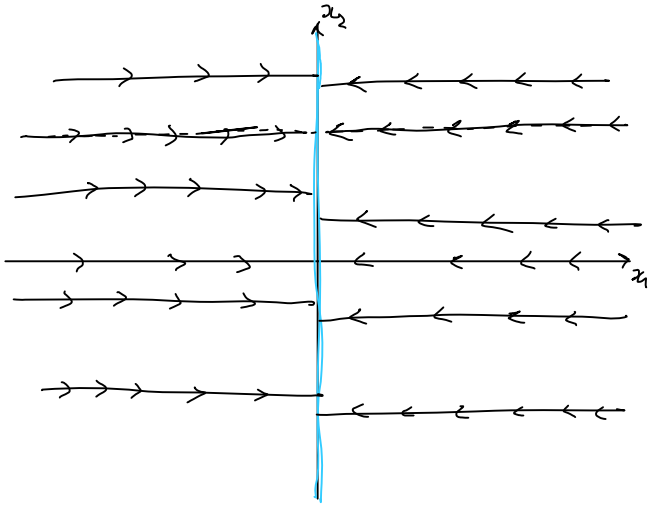
4. Скицирати фазни портрет динамичког система $X' = AX$, ако је:

(1) $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$; (2) $A = \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}$; (3) $A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$.

(4.2) $X^* = ?$

$\left.\begin{matrix} -5x_1 = 0 \\ 0 = 0 \end{matrix}\right\} \; x_1 = 0$, $X^* \in \{(0,s) \mid s \in \mathbb{R}\}$

↳ неизоловани еквилибријуми



$x_1' = -5x_1$

$x_2' = 0$

$x_2 = c_2$

$x_1 = c_1 e^{-5t}$, $c_1, c_2 \in \mathbb{R}$

$c_1 > 0$: $c_1 e^{-5t} > 0$

$t \to -\infty$ $\quad t = 0$ $\quad t \to \infty$

$+\infty \longrightarrow c_1 \longrightarrow 0$

$c_1 < 0$: $c_1 e^{-5t} < 0$

$t \to -\infty$ $\quad t = 0$ $\quad t \to +\infty$

$-\infty \longrightarrow c_1 \longrightarrow 0$

(5) Нека су $a, b \in \mathbb{R}$ и нека је $a \neq \pm b$. Свести диференцијалну једначину $y'' + 2ay' + b^2 y = 0$ на систем диференцијалних једначина. У зависности од параметара a и b испитати тип еквилибријума и скицирати фазне портрете.

$y'' = -2ay' - b^2 y$

$\left.\begin{matrix} x_1 = y \\ x_2 = y' \end{matrix}\right\} \quad \begin{matrix} x_1' = y' = x_2 \\ x_2' = -2ay' - b^2 y = -2ax_2 - b^2 x_1 \end{matrix}$, $X' = AX$, $A = \begin{bmatrix} 0 & 1 \\ -b^2 & -2a \end{bmatrix}$

$\det(A - \lambda E) = 0$

$(0 - \lambda)(-2a - \lambda) - (-b^2) \cdot 1 = 0$

$\lambda^2 + 2a\lambda + b^2 = 0$

$D = (2a)^2 - 4b^2 = 4(a^2 - b^2) \neq 0 \quad (a \neq \pm b)$

$1^\circ\ a^2-b^2>0$, $D>0$, $\lambda_{1/2}=\frac{-2a\pm\sqrt{D}}{2}=-a\pm\sqrt{a^2-b^2}$

$2^\circ\ b^2-a^2>0$, $D<0$, $\lambda_{1/2}=\frac{-2a\pm i\sqrt{-D}}{2}=-a\pm i\sqrt{b^2-a^2}$ $\quad(\lambda_1\neq\lambda_2)$

$X^*=?$ $\quad x_2=0$, $-b^2x_1-2ax_2=0$ $\Big\}$ $-b^2x_1=0$ $\begin{cases}\to b=0\\ \to x_1=0\end{cases}$

$b=0$: $(s,0)$, $s\in\mathbb{R}$

$b\neq 0$ $(0,0)$

$1^\circ\ a^2-b^2>0$ $\quad(a=0$ nemoguće)

$1.1^\circ\ b=0$ $\quad X^*\in\{(s,0)\mid s\in\mathbb{R}\}$ – neizolovani ekvilibrijumi

(u svim ostalim slučajevima je $X^*=(0,0)$)

$1.2^\circ\ b\neq 0$

$\lambda_1=-a-\sqrt{a^2-b^2}<-a+\sqrt{a^2-b^2}=\lambda_2$

$1.2.1^\circ\ a>0$

$\lambda_1<0$

$\lambda_2<0\iff -a+\sqrt{a^2-b^2}<0$

$\sqrt{a^2-b^2}<a\ \Big/^2$ $\quad(a>0)$

$a^2-b^2<a^2$

$-b^2<0$ ✓

$\lambda_1<\lambda_2<0$ → stabilan čvor

$1.2.2^\circ\ a<0$

$\lambda_2>0$

$\lambda_1>0\iff -a-\sqrt{a^2-b^2}>0\iff -a>\sqrt{a^2-b^2}\ \Big/^2\ (a<0)$

$a^2>a^2-b^2$ ✓

$\lambda_2>\lambda_1>0$ → nestabilan čvor

$2^\circ\ b^2-a^2>0$ $\quad(b=0$ nemoguće)

$\lambda_{1/2}=-a\pm i\sqrt{b^2-a^2}$

$2.1^\circ\ a=0$

$\lambda_{1/2}=\pm i\sqrt{b^2-a^2}$ → centar

$2.2^\circ\ a>0$ $\quad Re(\lambda_1)=Re(\lambda_2)<0$ → stabilan fokus

$2.3^\circ\ a<0$ $\quad Re(\lambda_1)=Re(\lambda_2)>0$ → nestabilan fokus

⑥ Скицирати фазни портрети дадената $x' = x(1-x)$.

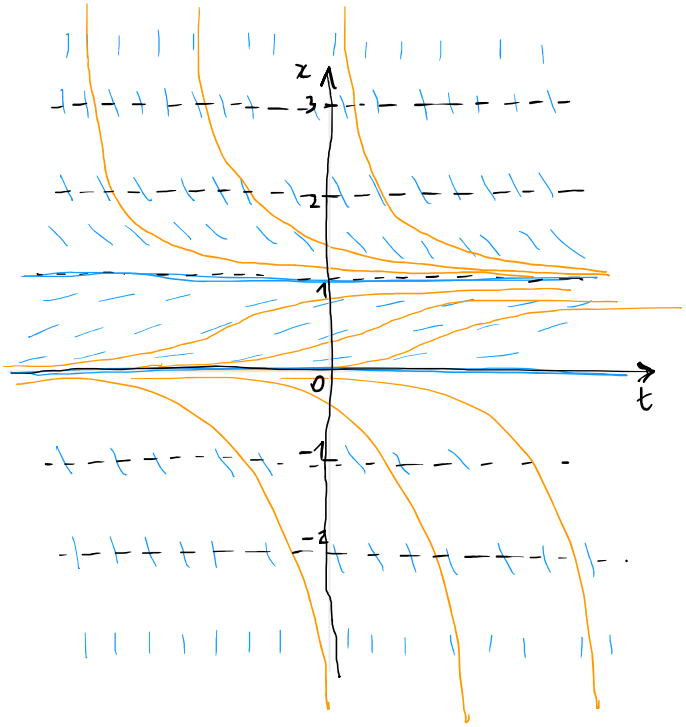
Ова је 1д, па је ФП на правој $(\mathbb{R}^1)$.

$x \in (-\infty, 0)$: $x(1-x) < 0$ ↓

$x \in (0,1)$: $x(1-x) > 0$ ↑

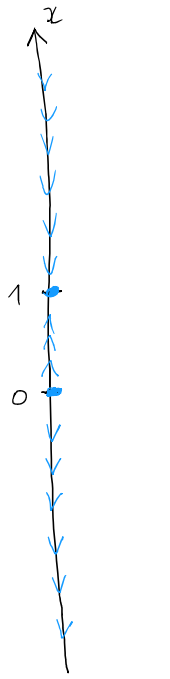
$x \in (1, +\infty)$: $x(1-x) < 0$ ↓

Интегралне криве



проекција

Фазни портрети



Еквилибриуми су $X_* \in \{0,1\}$, што одговара константним решењима.