Eksponent matrice

$X' = AX$ линеаран систем дј са константним коефицијентима

$X : \mathbb{R} \to \mathbb{R}^n$

$t \mapsto X(t)$

$X'(t) = A\,X(t)$, $X', X \in \mathbb{R}^n$

$A \in M_n(\mathbb{R})$, $A = [a_{ij}]_{i,j=1}^{n}$

1 једн 1 реда у $\mathbb{R}^n$

$\wr$

n једн 1. реда у $\mathbb{R}$

$$X = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \Rightarrow \begin{matrix} x_1'(t) = a_{11}x_1(t) + \dots + a_{1n}x_n(t) \\ \vdots \qquad \ddots \qquad \vdots \\ x_n'(t) = a_{n1}x_1(t) + \dots + a_{nn}x_n(t) \end{matrix}$$

ОР. $X(t) = e^{tA}c$, $c \in \mathbb{R}^n$, $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, $c_1, \dots, c_n \in \mathbb{R}$

нема смисла:

$X(t) = c\,e^{tA}$, $c \in \mathbb{R}^2$

$2\times1 \to 1\times1 \,,\, 2\times2$

$\exp(A) = e^A$

$\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, $\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

(1) Решити систем дј $X' = AX$, одређивањем e^{tA} у облику реда, ако је:

а) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ б) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ в) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ г) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a, b \in \mathbb{R}$

$X(t) = e^{tA} \cdot c$, $c \in \mathbb{R}^2$

$e^{tA} = ?$

а) $e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$

$A^k = ?$

$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$\Bigg\}$ индукцијом: $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $k \geq 1$

Б: $k=1$, $A^1 = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ✓

Х: $A^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

К: $k+1$, $A^{k+1} = A^k A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+k \\ 0 & 1 \end{bmatrix}$ ✓

$$e^{tA}=\sum_{k=0}^{\infty}\frac{t^kA^k}{k!}=\sum_{k=0}^{\infty}\frac{t^k\begin{bmatrix}1&k\\0&1\end{bmatrix}}{k!}=\begin{bmatrix}\sum_{k=0}^{\infty}\frac{t^k}{k!} & \sum_{k=0}^{\infty}\frac{t^k}{k!}k\\ 0 & \sum_{k=0}^{\infty}\frac{t^k}{k!}\end{bmatrix}=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}$$

$$e^t=\sum_{k=0}^{\infty}\frac{t^k}{k!}$$

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}\cdot k=\sum_{k=1}^{\infty}\frac{t^k}{(k-1)!}=t\cdot\sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}=t\sum_{k=0}^{\infty}\frac{t^k}{k!}=t\cdot e^t$$

ОР. $X(t)=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}c=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}c_1e^t+c_2te^t\\ c_2e^t\end{bmatrix}$

$c\in\mathbb{R}^2$

$x_1(t)=c_1e^t+c_2te^t$

$x_2(t)=c_2e^t$

б) домаћи

$\boxed{\text{Тврђење 52.}}$ (Својства експонента.)

(1) $e^{\mathbf{0}}=\mathrm{Id}$; 0-нула матрица , $Id = I = E = \begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix}$

(2) $AB=BA\Rightarrow e^{A+B}=e^Ae^B$;

(3) $AB=BA\Rightarrow Be^A=e^AB$; спец: $A=B$, $Ae^A=e^AA$

(4) $e^A=\lim\limits_{n\to\infty}(\mathrm{Id}+\frac{A}{n})^n$;

(5) за $U=\mathbb{R}^n$, тј. $A\in M_n(\mathbb{R})$ важи $\frac{d}{dt}e^{tA}=e^{tA}A=Ae^{tA}$;

(6) за $U=\mathbb{R}^n$ важи $\det e^A=e^{\operatorname{tr}A}$;

(7) за $U=\mathbb{R}^n$ важи $e^{P^{-1}AP}=P^{-1}e^AP$.

Домаћи: (2) не важи за $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $B=\begin{bmatrix}0&0\\0&1\end{bmatrix}$.

в) $A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ $(\lambda_{1/2}=\pm i)$

$e^{tA}=?$ $A^k=?$

$$A^2=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=-E$$

$$A^3=A^2A=-E\,A=-A$$

$$A^4=A^3\cdot A=(-A)\,A=-A^2=-(-E)=E=A^0$$

$$A^k=A^{4t+r}=(A^4)^tA^r=E^t\cdot A^r=A^r=\begin{cases}E, & r=0\\ A, & r=1\\ -E, & r=2\\ -A, & r=3\end{cases}=\begin{cases}(-1)^lE, & k=2l\\ (-1)^lA, & k=2l+1\end{cases}$$

$k=4t+r$

$r\in\{0,1,2,3\}$

$$e^{tA}=\sum^{\infty}\frac{t^kA^k}{k!}=\sum^{\infty}\frac{t^{2l}\cdot A^{2l}}{}+\sum^{\infty}\frac{t^{2l+1}\cdot A^{2l+1}}{}=\sum^{\infty}\frac{t^{2l}(-1)^l}{}\cdot E+\sum^{\infty}\frac{t^{2l+1}\cdot(-1)^l}{}\cdot A=\cos t\;E+\sin t\cdot A$$

$$e^{tA}=\sum_{k=0}^{\infty}\frac{t^kA^k}{k!}=\sum_{l=0}^{\infty}\frac{t^{2l}\cdot A^{2l}}{(2l)!}+\sum_{l=0}^{\infty}\frac{t^{2l+1}\cdot A^{2l+1}}{(2l+1)!}=\underbrace{\sum_{l=0}^{\infty}\frac{t^{2l}(-1)^l}{(2l)!}}_{\substack{1-\frac{t^2}{2!}+\frac{t^4}{4!}-\dots\\=\cos t}}\cdot E+\underbrace{\sum_{l=0}^{\infty}\frac{t^{2l+1}\cdot(-1)^l}{(2l+1)!}}_{\substack{t-\frac{t^3}{3!}+\frac{t^5}{5!}-\dots\\=\sin t}}\cdot A=\cos t\; E+\sin t\cdot A$$

$$=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}=R_t$$

ОР: $X(t)=R_t\cdot c,\ c\in\mathbb{R}^2$

г) $A=\begin{bmatrix}a & b\\ -b & a\end{bmatrix}$

предобавка: $A^k \leftrightarrow (a+ib)^k$

II начин: $A=\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}+\begin{bmatrix}0 & b\\ -b & 0\end{bmatrix}=aE+b\underset{}{B}$, $\quad B=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

$$e^{tA}=e^{t(aE+bB)}=e^{taE+tbB}\overset{(2)}{=}e^{taE}\cdot e^{tbB}\underset{\text{домашно}}{=\dots=}e^{ta}\cdot R_{tb}$$

$$(taE)\cdot(tbB)=t^2ab\overbrace{E\,B}^{B}=t^2ab\,BE=(tbB)(taE)\Rightarrow \text{комут.}$$

(1) Изследвайте дали $\exists A\in M_2(\mathbb{R})$ т.ч.

а) $e^A=\begin{bmatrix}1 & 0\\ 0 & -4\end{bmatrix}$ $\qquad$ б) $e^A=\begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix}$

в $\mathbb{R}$

$e^a=b$

$b>0$ ✓

$b\le 0$ ✗

а) $(6)\Rightarrow \det(e^A)=e^{\overset{\in\mathbb{R}}{trA}}>0$

$\det\left(\begin{bmatrix}1 & 0\\ 0 & -4\end{bmatrix}\right)=-4<0$ $\Big\}$ ↯

б) $\det\left(\begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix}\right)=4>0$

$4=e^{trA}\Rightarrow trA=\ln 4$

$(3)\Rightarrow Ae^A=e^AA$ $\qquad$ $A=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix}-1 & \\ & -4\end{bmatrix}=\begin{bmatrix}-1 & \\ & -4\end{bmatrix}\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$$

$-\alpha=-\alpha$ ✗

$\left.\begin{aligned}-4\beta&=-\beta\\-\gamma&=-4\gamma\end{aligned}\right\}\ \beta=\gamma=0\ \Rightarrow\ A=\begin{bmatrix}\alpha & 0\\ 0 & \delta\end{bmatrix}$

$-4\delta=-4\delta$ ✗

(*) $A = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\} \Rightarrow e^A = \mathrm{diag}\{e^{\lambda_1}, \dots, e^{\lambda_n}\}$

[illegible] $e^A = \begin{bmatrix} e^\alpha & 0 \\ 0 & e^\delta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \Rightarrow \left.\begin{matrix} e^\alpha = -1 \\ e^\delta = -4 \end{matrix}\right\} \lightning$

(2) $\lambda \in \mathbb{C}$ сопс вр $A \Rightarrow e^\lambda$ сопс вр e^A

I начин: $\exists v \neq 0: \ \lambda v = Av$

$? \exists v_1 \neq 0: \ e^\lambda v_1 = e^A v_1$ $\rightsquigarrow$ узећемо баш $v_1 = v$

$$e^A v = \left[\sum_{k=0}^{\infty} \frac{A^k}{k!}\right] \cdot v = \sum_{k=0}^{\infty} \frac{A^k v}{k!} = \sum_{k=0}^{\infty} \frac{\lambda^k v}{k!} = \left[\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\right] \cdot v = e^\lambda \cdot v$$

$$A^k v = A^{k-1}(Av) = A^{k-1}\lambda v = \lambda \cdot A^{k-1} v = \lambda^2 A^{k-2} v = \dots = \lambda^k v$$

II начин: $\det(A - \lambda E) = 0$

$? \det(e^A - e^\lambda E) = 0$

$$\det(e^A - e^\lambda E) = \det\left(\sum_{k=0}^{\infty} \frac{A^k}{k!} - \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} E\right) = \det\left(\sum_{k=0}^{\infty} \frac{A^k - (\lambda E)^k}{k!}\right) = \dots$$

доказати

(3) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ наћи $\det(e^{e^A})$.

$A, e^A, e^{e^A} \in M_3(\mathbb{R})$

(6) $\Rightarrow \det(e^{e^A}) = e^{tr(e^A)}$

$e^A = ?$

$$A = \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{E} + \underbrace{\begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix}}_{B}$$

$EB = B = BE \overset{(2)}{\Rightarrow} e^A = e^{B+E} = e^B e^E$

$E = \mathrm{diag}\{1,1,1\} \Rightarrow e^E = \mathrm{diag}\{e, e, e\} = e \cdot E$

$$B^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = B$$

$$B^4 = (B^2)^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left.\begin{array}{l} B^0 = E \\ B^1 = B^3 = \quad = B \\ B^2 = B^4 = \dots = B^2 \end{array}\right\} \text{индукција}$$

$$e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!} = \underset{\substack{\uparrow \\ k=0}}{E} + \sum_{l=1}^{\infty} \frac{B^{2l}}{(2l)!} + \sum_{l=0}^{\infty} \frac{B^{2l+1}}{(2l+1)!} =$$

$$= E + B^2 \cdot \Big(\underbrace{\sum_{l=0}^{\infty} \frac{1}{(2l)!}}_{ch 1} \underset{\substack{\nearrow \\ l=0}}{-1} \Big) + B \underbrace{\sum_{l=0}^{\infty} \frac{1}{(2l+1)!}}_{sh 1}$$

$$\left.\begin{array}{l} e = \sum_{k=0}^{\infty} \frac{1}{k!} = П + Н \\ e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = П - Н \end{array}\right\} \quad П = \frac{e+e^{-1}}{2} = ch 1$$

$$Н = \frac{e-e^{-1}}{2} = sh 1$$

$$= E + B^2(ch 1 - 1) + B\, sh 1 =$$

$$= \begin{bmatrix} ch 1 & 0 & sh 1 \\ 0 & 1 & 0 \\ sh 1 & 0 & ch 1 \end{bmatrix}$$

$$\Rightarrow e^A = e E \cdot e^B = e \cdot e^B \quad \Rightarrow tr\, A = e\, tr B = e\,(ch 1 + 1 + ch 1) = e\,(e + e^{-1} + 1) = e^2 + e + 1$$

$$\det(e^{e^A}) = e^{e^2+e+1}$$

(4) Нека је $A = [a_{ij}]_{i,j=1}^{n} \in M_n(\mathbb{R})$, $a_{ij} \geq 0, \forall i \neq j$. Нека је $B = e^A = [b_{ij}]_{i,j=1}^{n} \in M_n(\mathbb{R})$

Доказати $b_{ij} \geq 0$, $\forall i,j$

$$A = \begin{bmatrix} \ddots & \geq 0 \\ \geq 0 & \ddots \end{bmatrix} \rightsquigarrow B = e^A = \begin{bmatrix} \geq 0 \end{bmatrix}.$$

(*) Ако $a_{ij} \geq 0, \forall i,j \Rightarrow b_{ij} \geq 0, \forall i,j$

$$\text{нпр } A = \begin{bmatrix} 7 & & \\ & -8 & \\ & & -2 \end{bmatrix} = \overset{X}{\begin{bmatrix} -8 & & \\ & -8 & \\ & & -8 \end{bmatrix}} + \overset{Y}{\begin{bmatrix} 15 & & \\ & 0 & \\ & & 6 \end{bmatrix}}, \quad XY = YX \Rightarrow e^A = e^X \cdot e^Y$$

$A = X + Y$

$X = M \cdot E \quad , Y = A - ME$

$$M = \min\{a_{ii} \mid 1 \le i \le n\}$$

$$Y = \begin{cases} \underbrace{a_{ij}}_{} \geq 0, & i \neq j \\ a_{ii} - M & \end{cases} = [\geq 0] \Rightarrow e^Y = [\geq 0]$$

≥ 0 (jer je $M = \min$)

$$XY = YX? \qquad X = ME, \quad ME \cdot Y = M \cdot Y = MYE = YME = YX \ \checkmark \overset{(2)}{\Rightarrow} e^A = e^X \cdot e^Y$$

$$X = \operatorname{diag}\{M, \dots, M\} \Rightarrow e^X = \operatorname{diag}\{e^M, \dots, e^M\} = e^M \cdot E$$

$$B = e^A = e^M E \cdot e^Y = \underbrace{e^M}_{\geq 0} \cdot \underbrace{e^Y}_{= [\geq 0]} = [\geq 0] \Rightarrow \forall i,j \cdot b_{ij} \geq 0.$$