Примитивна дистрибуција

Деф. Нека је $f \in D'(\Omega)$, $\Omega \subseteq \mathbb{R}$. За $g \in D'(\Omega)$ кажемо да је примитивна дистрибуција од f ако важи $g' = f$. Често се пише $g = f^{(-1)}$.

$\boxed{T}$ Свака дистрибуција има примитивну дистрибуцију.

Ако су $g_1, g_2 \in D'(\Omega)$ примитивне дистрибуције од $f \in D'(\Omega)$ $(\Omega \subseteq \mathbb{R})$. Тада важи $\exists c \in \mathbb{R}$ тд. $g_1 - g_2 = c$. Тј, $\forall \varphi \in D(\Omega)$ је $\langle g_1, \varphi\rangle - \langle g_2, \varphi\rangle = \langle c, \varphi\rangle = c\int_{-\infty}^{+\infty} \varphi(x)\,dx$.

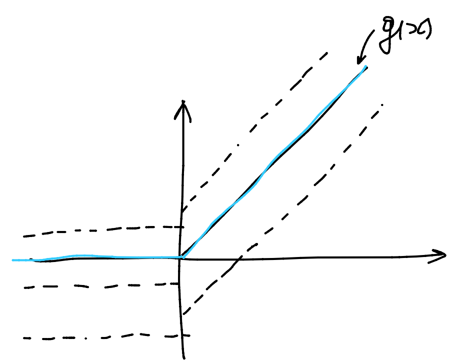
Последица: Опште решење $y' = 0$ у $D'(\Omega)$, $\Omega \subseteq \mathbb{R}$ је $y = c$, $c \in \mathbb{R}$.

Важи: $f \in D'(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, $\nabla f = 0 \Rightarrow f = c$.

Последица: Опште решење $y' = f$ у $D'(\Omega)$, $\Omega \subseteq \mathbb{R}$ је $y = f^{(-1)} + c$, $c \in \mathbb{R}$.

① Одредити примитивну дистрибуцију од Θ.

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$



$g = ?$, $g' = \Theta$

$$g(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

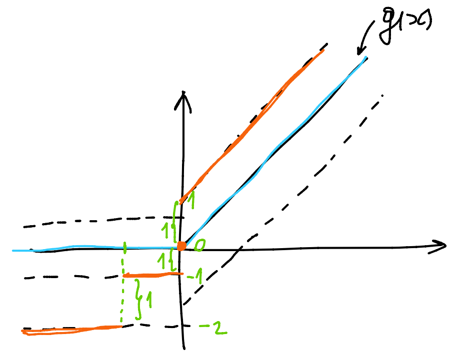
Докажимо $g' = \Theta$. $\varphi \in D(\mathbb{R})$, $\operatorname{supp}\varphi \subseteq [-M, M]$, $M > 0$.

$$\langle g', \varphi\rangle = -\langle g, \varphi'\rangle = -\int_{-\infty}^{+\infty} g(x)\varphi'(x)\,dx = -\int_0^{+\infty} x\varphi'(x)\,dx = -\int_0^M x\varphi'(x)\,dx = -\left(\underbrace{x\cdot\varphi(x)\Big|_0^M}_{0} - \int_0^M 1\cdot\varphi(x)\,dx\right) =$$

$$= \int_0^M \varphi(x)\,dx = \int_0^{+\infty} \varphi(x)\,dx = \int_{-\infty}^{+\infty} \Theta(x)\cdot\varphi(x)\,dx = \langle \Theta, \varphi\rangle$$

$\Rightarrow g' = \Theta$, све прим. су $\{g + c \mid c \in \mathbb{R}\}$

② Доказати да $h(x)=\begin{cases}-2, & x<-1\\ -1, & -1\le x<0\\ 0, & x=0\\ x+1, & x>0\end{cases}$ није примитивна дистрибуција од θ.



Као фје $h'(x)=\theta(x)$ с.с. $\to$ скоро свуда

али као дистрибуције НЕ!

$\varphi\in\mathcal{D}(\mathbb{R})$, $\operatorname{supp}\varphi\subseteq[-M,M]$, $M>1$

$$\langle h',\varphi\rangle=-\langle h,\varphi'\rangle=-\int_{-M}^{-1}(-2)\cdot\varphi'(x)\,dx-\int_{-1}^{0}(-1)\cdot\varphi'(x)\,dx-\int_{0}^{M}(x+1)\varphi'(x)\,dx=$$

$$=2\int_{-M}^{-1}\varphi'(x)\,dx+\int_{-1}^{0}\varphi'(x)\,dx-\Big(\underbrace{(x+1)\cdot\varphi(x)\Big|_0^M}_{\varphi(M)=0}-\int_0^M 1\cdot\varphi(x)\,dx\Big)=$$

$$=2\varphi(x)\Big|_{-M}^{-1}+\varphi(x)\Big|_{-1}^{0}-\Big(-1\cdot\varphi(0)-\int_0^M\varphi(x)\,dx\Big)=$$

$$=2\varphi(-1)-2\underbrace{\varphi(-M)}_{0}+\varphi(0)-\varphi(-1)+\varphi(0)+\int_{-\infty}^{+\infty}\theta(x)\cdot\varphi(x)\,dx=$$

$$=\varphi(-1)+2\varphi(0)+\langle\theta,\varphi\rangle=\langle\delta_{-1},\varphi\rangle+2\langle\delta_0,\varphi\rangle+\langle\theta,\varphi\rangle=$$

$$=\langle\delta_{-1}+2\delta_0+\theta,\varphi\rangle$$

$$\Rightarrow h'=\delta_{-1}+2\delta_0+\theta$$

Диференцијалне једначине

③ Решити $x^3\cdot y''=0$ у $\mathcal{D}'(\mathbb{R})$.

смена: $z=y''$, $x^3\cdot z=0 \overset{\text{зад.}}{\Longrightarrow} z=c_0\delta+c_1\delta'+c_2\delta''$, $c_0,c_1,c_2\in\mathbb{R}$

$$y''=c_0\delta+c_1\delta'+c_2\delta''$$

$y'=(y'')^{(-1)}$

$$\Rightarrow y'=c_0\theta+c_1\delta+c_2\delta'+c_3$$

знамо $\theta'=\delta\Rightarrow\delta^{(-1)}=\theta$

$$\Rightarrow y=c_0 g+c_1\theta+c_2\delta+c_3x+c_4 \quad , c_0,c_1,c_2,c_3,c_4\in\mathbb{R}$$

$g'=\theta\Rightarrow\theta^{(-1)}=g$

$$\Rightarrow y = c_0 g + c_1\theta + c_2\delta + c_3 x + c_4, \quad c_0, c_1, c_2, c_3, c_4 \in \mathbb{R}$$

$$g' = \theta \Rightarrow \theta^{(-1)} = g$$

$$x' = 1 \Rightarrow 1^{(-1)} = x$$

④ Решити $xy' = 1$ у $\mathcal{D}'(\mathbb{R})$.

дистрибуција

идеја: 1 пребацити у 0

$$\langle P\tfrac{1}{x}, \varphi\rangle = p.v.\int_{-\infty}^{+\infty} \frac{\varphi(x)}{x}\,dx$$

$$\langle xP\tfrac{1}{x}, \varphi\rangle = \langle P\tfrac{1}{x}, x\varphi\rangle = p.v.\int_{-\infty}^{+\infty} \frac{x\varphi(x)}{x}\,dx = \int_{-\infty}^{+\infty}\varphi(x) = \langle 1, \varphi\rangle$$

$$\Rightarrow x\cdot P\tfrac{1}{x} = 1$$

$$xy' - 1 = 0$$

$$x(y' - ?) = 0 \quad \to \frac{1}{x}?$$

$$x\cdot\left(y' - P\tfrac{1}{x}\right) = 0$$

знамо:

$$(\ln|x|)' = P\tfrac{1}{x}$$

$$\left(P\tfrac{1}{x}\right)^{(-1)} = \ln|x|$$

зад.

$$\Rightarrow y' - P\tfrac{1}{x} = c_0\delta$$

$$y' = c_0\delta + P\tfrac{1}{x}$$

$$y = c_0\theta + \ln|x| + c_1, \quad c_0, c_1 \in \mathbb{R}$$

⑤ Решити $x^2\cdot(y' - \cos x) = 0$ у $\mathcal{D}'(\mathbb{R})$. Наћи и решење које задовољава услове

$$y(-\pi) = 1, \quad y(\pi) = 0, \quad \int_{-\pi}^{\pi} y(x)\,dx = 2\pi.$$

ОР тражимо:

$$y' - \cos x = c_0\delta + c_1\delta'$$

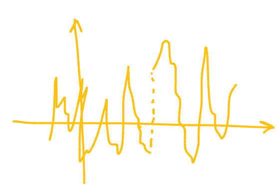
$$y' = c_0\delta + c_1\delta' + \cos x$$

$$y = c_0\theta + c_1\delta + \sin x + c_2, \quad c_0, c_1, c_2 \in \mathbb{R}$$

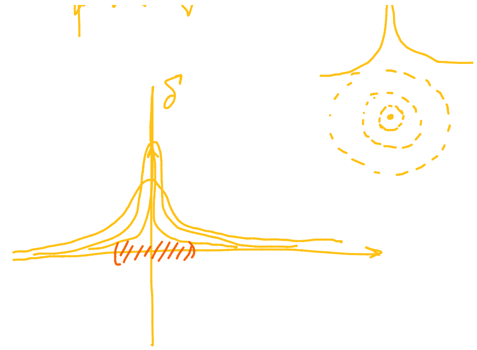
→ шта значе ове ствари? Конвенција из примена:

- $f \in L^1_{loc}(\mathbb{R}) \rightsquigarrow T_f(x) = f(x)$

$$\int_a^b T_f(x)\,dx = \int_a^b f(x)\,dx$$



$$\int_a T_f(x)\,dx = \int_a f(x)\,dx$$



- $\delta(x) = \begin{cases} \infty, & x=0 \\ 0, & x \neq 0 \end{cases}$ "δ функција" у $\mathbb{R}$

$$\int_a^b \delta(x)\,dx = \begin{cases} 1, & 0 \in (a,b) \\ 0, & 0 \notin (a,b) \end{cases}$$

- $\Omega \subseteq \mathbb{R}^n$, $a \in \Omega$, $\delta_a(x) = \begin{cases} \infty, & x=a \\ 0, & x \neq a \end{cases}$

$$\int_K \delta_a(x)\,dx = \begin{cases} 1, & a \in K \\ 0, & a \notin K \end{cases}$$

$$y = c_0\theta + c_1\delta + \sin x + c_2, \quad c_0, c_1, c_2 \in \mathbb{R}$$

$$y(-\pi) = 1, \quad y(\pi) = 0, \quad \int_{-\pi}^{\pi} y(x)\,dx = 2\pi.$$

$$y(-\pi) = c_0\underbrace{\theta(-\pi)}_{0} + c_1\underbrace{\delta(-\pi)}_{0} + \underbrace{\sin(-\pi)}_{0} + c_2 = c_2 = 1$$

$$y(\pi) = c_0\underbrace{\theta(\pi)}_{1} + c_1\underbrace{\delta(\pi)}_{0} + \underbrace{\sin(\pi)}_{0} + c_2 = c_0 + c_2 = 0$$

$$\int_{-\pi}^{\pi} y(x)\,dx = c_0\underbrace{\int_{-\pi}^{\pi}\theta(x)\,dx}_{\pi} + c_1\underbrace{\int_{-\pi}^{\pi}\delta(x)\,dx}_{1} + \underbrace{\int_{-\pi}^{\pi}\sin x\,dx}_{0} + c_2\underbrace{\int_{-\pi}^{\pi}dx}_{2\pi} = c_0\pi + c_1 + c_2\cdot 2\pi = 2\pi$$

$$\left.\begin{aligned} &c_2 = 1 \\ &c_0 = -1 \\ &-\pi + c_1 + 2\pi = 2\pi \implies c_1 = \pi \end{aligned}\right\} \quad y(x) = -\theta(x) + \pi\delta(x) + \sin x + 1$$

Линеарна ДЈ 1. реда

$$y' + py = q, \quad y \in D'(\mathbb{R})$$

$$p \in C^\infty(\mathbb{R}), \quad q \in D'(\mathbb{R})$$

$$y' + py = q \,/\cdot \underbrace{e^{\int p(x)dx}}_{} \in C^\infty(\mathbb{R})$$

$$e^{\int p(x)dx}\cdot y' + e^{\int p(x)dx}\cdot p(x)\cdot y = e^{\int p(x)dx}\cdot q$$

$a \in C^\infty(\mathbb{R})$
$f \in \mathcal{D}'(\mathbb{R})$
$(af)' = a'f + af'$

$$\left(e^{\int p(x)dx} \cdot y\right)' = e^{\int p(x)dx} \cdot g$$

$$e^{\int p(x)dx} \cdot y = \left(e^{\int p(x)dx} \cdot g\right)^{(-1)} + c \quad \in \mathcal{D}'(\mathbb{R})$$

$$y = e^{-\int p(x)dx} \cdot \left(\left(e^{\int p(x)dx} \cdot g\right)^{(-1)} + c\right), \quad c \in \mathbb{R}$$

(6) Решити $y' - 2y = 5\delta$ у $\mathcal{D}'(\mathbb{R})$.

$p(x) = -2 \rightsquigarrow \int p(x)dx = -2x$

$$y' - 2y = 5\delta \,/\cdot e^{-2x}$$

$$e^{-2x}y' - 2e^{-2x}y = 5e^{-2x}\delta$$

$$\left(e^{-2x}y\right)' = 5e^{-2x}\delta \quad , a(x) = 5e^{-2x}$$

$$a(0) = 5 \cdot e^{-2\cdot 0} = 5$$

$\circledast$ $a \in C^\infty(\mathbb{R})$, онда $a\delta = a(0)\cdot\delta$

$\langle a\delta, \varphi\rangle = \langle \delta, a\varphi\rangle = (a\varphi)(0) = a(0)\cdot\varphi(0) =$
$= a(0)\cdot\langle\delta,\varphi\rangle \Rightarrow a\delta = a(0)\delta$

$$\circledast \Rightarrow \quad \left(e^{-2x}y\right)' = 5\delta$$

$$e^{-2x}y = 5\delta^{(-1)} + c = 5\theta + c \quad /\cdot e^{2x}$$

$$y = 5\theta e^{2x} + c\cdot e^{2x}, \quad c \in \mathbb{R}$$

(7) Решити $u'' - 3u' + 2u = \delta$ у $\mathcal{D}'(\mathbb{R})$.

$\lambda^2 - 3\lambda + 2 = 0 \rightsquigarrow \lambda_1 = 1, \lambda_2 = 2 \qquad e^x, e^{2x}$?
$(\lambda-1)(\lambda-2)$

$$u'' - 3u' + 2u = \left(\frac{d^2}{dx^2} - 3\frac{d}{dx} + 2id\right)(u) = \left(\frac{d}{dx} - id\right)\underline{\left(\frac{d}{dx} - 2id\right)(u)}$$

смена: $\underline{u' - 2u = v}$

$$u'' - 3u' + 2u = (u' - 2u)' - (u' - 2u) = v' - v = \delta$$

$$v' - v = \delta \,/\cdot e^{-x}$$

$$e^{-x}v' - e^{-x}v = e^{-x}\delta$$

$$(e^{-x}v)' = e^{-x}\delta = e^{-0}\cdot\delta = 1\cdot\delta = \delta$$

$$e^{-x}v = \theta + c_1 \qquad , c_1 \in \mathbb{R}$$

$$v = e^{x}\theta + c_1 e^{x}$$

$$u' - 2u = e^{x}\theta + c_1 e^{x} \quad /\cdot e^{-2x}$$

$$e^{-2x}u' - 2e^{-2x}u = e^{-x}\theta + c_1 e^{-x}$$

$$(e^{-2x}u)' = e^{-x}\theta + c_1 e^{-x}$$

$$e^{-2x}u = (e^{-x}\theta + c_1 e^{-x})^{(-1)} + c_2 \quad , c_2 \in \mathbb{R}$$

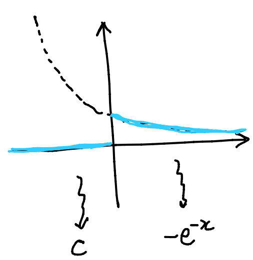
$$e^{-2x}u = (e^{-x}\theta)^{(-1)} + c_1 \underbrace{(e^{-x})^{(-1)}}_{-e^{-x}} + c_2$$

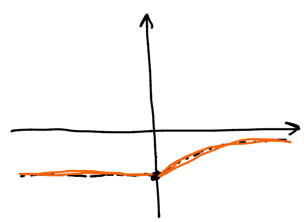
$$e^{-2x}u = (e^{-x}\theta)^{(-1)} - c_1 e^{-x} + c_2 \quad /\cdot e^{2x}$$

$$u = e^{2x}\cdot(e^{-x}\cdot\theta)^{(-1)} - c_1 e^{x} + c_2 e^{2x}$$

Треба још наћи $(e^{-x}\theta)^{(-1)}$.

$$e^{-x}\theta(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$





$$g(x) = \begin{cases} -1, & x < 0 \\ -e^{-x}, & x \geq 0 \end{cases}$$

Докажимо $(e^{-x}\theta)^{(-1)} = g(x)$.

$\varphi \in \mathcal{D}(\mathbb{R})$, $\operatorname{supp}\varphi \subseteq [-M, M]$

$$\langle g', \varphi \rangle = -\langle g, \varphi' \rangle = -\int_{-\infty}^{+\infty} g(x)\varphi'(x)\,dx = -\int_{-M}^{0}(-1)\varphi'(x)\,dx + \int_{0}^{M}(+e^{-x})\cdot\varphi'(x)\,dx =$$

$$= \varphi(x)\Big|_{-M}^{0} + \left(e^{-x}\cdot\varphi(x)\Big|_{0}^{M} - \int_{0}^{M}(-e^{-x})\varphi(x)\,dx\right) =$$

$$= \varphi(0) - \underbrace{\varphi(-M)}_{0} + e^{-M}\underbrace{\varphi(M)}_{0} - \varphi(0) + \int_{0}^{M} e^{-x}\varphi(x)\,dx =$$

$+\infty$ $+\infty$

$$= \int_0^{+\infty} e^{-x}\varphi(x)\,dx = \int_{-\infty}^{+\infty} \theta(x)\cdot e^{-x}\cdot\varphi(x)\,dx = \langle e^{-x}\theta, \varphi\rangle$$

$$\Rightarrow u = e^{2x}\cdot g(x) - c_1 e^{x} + c_2 e^{2x} \quad , c_1, c_2 \in \mathbb{R}$$