# Једначине у простору дистрибуција

**Деф.** За дистрибуцију $f \in D'(\mathbb{R}^n)$ кажемо да се анулира у области $\Omega \subseteq \mathbb{R}^n$ ако важи $\langle f, \varphi \rangle = 0$, $\forall \varphi \in D(\Omega)$. Тада пишемо $f(x) = 0$ за $x \in \Omega$.

$$\varphi \in D(\Omega) \rightsquigarrow \tilde{\varphi} \in D(\mathbb{R}^n): \quad \tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \in \Omega \\ 0, & x \notin \Omega \end{cases}. \text{ Имамо } \langle f, \varphi \rangle := \langle f, \tilde{\varphi} \rangle$$

**Деф.** Нулти скуп дистрибуције $f \in D'(\mathbb{R}^n)$ у ознаци $O_f$ је унија свих области у којима се $f$ анулира.

**Деф.** Носач дистрибуције $f \in D'(\mathbb{R}^n)$ је скуп $S_f = \mathbb{R}^n \setminus O_f$.

① Одредити носач дистрибуције $\delta \in D'(\mathbb{R})$.

$$\langle \delta, \varphi \rangle = \varphi(0)$$

- $\delta$ се анулира на $(-\infty, 0)$:

$$\varphi \in D(-\infty, 0) \rightsquigarrow \tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \in (-\infty, 0) \\ 0, & x \in [0, +\infty) \end{cases}$$

$$\langle \delta, \varphi \rangle = \langle \delta, \tilde{\varphi} \rangle = \tilde{\varphi}(0) = 0$$

- $\delta$ се анулира на $(0, +\infty)$:

исто!

$$\Rightarrow (-\infty, 0) \cup (0, +\infty) \subseteq O_\delta$$

Међутим, не важи $O_\delta = \mathbb{R}$, јер би онда $\forall \varphi \in D(\mathbb{R}): \langle \delta, \varphi \rangle = \varphi(0) = 0$ ↯

$$\Rightarrow O_\delta = (-\infty, 0) \cup (0, +\infty) \Rightarrow S_\delta = \{0\}.$$

**Т** Ако је носач дистрибуције $f \in D'(\mathbb{R}^n)$ подскуп скупа $\{0\}$, онда се $f$ на јединствен начин може представити као

$$f = \sum_{|\alpha| \le m} c_\alpha D^\alpha \delta, \quad m \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n, c_\alpha \in \mathbb{R}.$$

② Решити једначину $x \cdot y(x) = 0$ у $D'(\mathbb{R})$.

$g:\mathbb{R}\to\mathbb{R}$, $g(x)=x$ $\qquad gy=0$

$y\in D'(\mathbb{R})$ - неизвестна

Намираме носач решения. Доказваме $S_y=\{0\}$.

- $y$ се анулира $(0,+\infty)$:

$\varphi\in D(0,+\infty)$

$$\tilde{\varphi}(x)=\begin{cases}\varphi(x), & x\in(0,+\infty)\\ 0, & x\in(-\infty,0]\end{cases}, \quad \psi(x)=\frac{\tilde{\varphi}(x)}{x}\ ?$$ не даже, нека: $$\psi(x)=\begin{cases}\frac{\varphi(x)}{x}, & x\in(0,+\infty)\\ 0, & x\in(-\infty,0]\end{cases}$$

$$\psi\in D(\mathbb{R})\ \left(\psi\in C^\infty(\mathbb{R}) \text{ и } supp(\psi)=supp(\varphi)\right) \quad \text{и} \quad \psi(x)\cdot x=\tilde{\varphi}(x)$$

$$\langle y,\varphi\rangle=\langle y,\tilde{\varphi}\rangle=\langle y, x\cdot\psi(x)\rangle=\langle \underbrace{xy}_{=0},\psi(x)\rangle=\langle 0,\psi(x)\rangle=0$$

- $y$ се анулира на $(-\infty,0)$:

аналогично!

$$\Rightarrow O_y\subseteq(-\infty,0)\cup(0,+\infty) \Rightarrow S_y\subseteq\{0\} \overset{T}{\Rightarrow} y=\sum_{|\alpha|\le m}c_\alpha D^\alpha\delta=\sum_{\alpha=0}^{m}c_\alpha\delta^{(\alpha)}$$

$\varphi\in D(\mathbb{R})$, $\langle xy,\varphi\rangle=\langle 0,\varphi\rangle=0$

$$\Rightarrow \langle x\cdot\sum_{\alpha=0}^{m}c_\alpha\delta^{(\alpha)},\varphi\rangle=\sum_{\alpha=0}^{m}c_\alpha\langle\delta^{(\alpha)},x\varphi\rangle=\sum_{\alpha=0}^{m}c_\alpha\cdot(-1)^\alpha\langle\delta,(x\varphi)^{(\alpha)}\rangle=$$

$$=\sum_{\alpha=0}^{m}c_\alpha(-1)^\alpha\langle\delta,x\varphi^{(\alpha)}+\alpha\varphi^{(\alpha-1)}\rangle=\sum_{\alpha=0}^{m}c_\alpha(-1)^\alpha\cdot\left(\underbrace{0\cdot\varphi^{(\alpha)}(0)}_{=0}+\alpha\cdot\varphi^{(\alpha-1)}(0)\right)=\sum_{\alpha=0}^{m}\underbrace{c_\alpha(-1)^\alpha\cdot\alpha}_{k_\alpha}\cdot\varphi^{(\alpha-1)}(0)=\sum_{\alpha=1}^{m}k_\alpha\varphi^{(\alpha-1)}(0).$$

$k_0=0$

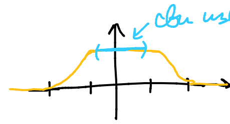
$$(x\varphi)^{(\alpha)}=\sum_{j=0}^{\alpha}\binom{\alpha}{j}(x)^{(j)}\cdot\varphi^{(\alpha-j)}=\binom{\alpha}{0}\cdot x\cdot\varphi^{(\alpha)}+\binom{\alpha}{1}\cdot 1\cdot\varphi^{(\alpha-1)}=x\varphi^{(\alpha)}+\alpha\varphi^{(\alpha-1)}$$

$x^{(0)}=x$

$x^{(1)}=x'=1$

$x^{(2)}=0$

$$\eta(x)=\begin{cases}1, & x\in(-1,1)\\ 0, & x\notin(-2,-2)\end{cases}\in D(\mathbb{R})$$



всички изводи $\eta^{(k)}(0)=0$

Вземаме ни $\varphi_k(x)=x^{k-1}\cdot\eta(x)\in D(\mathbb{R})$, $1\le k\le m$

Узмимо низ $\varphi_k(x) = x^{k-1}\cdot\eta(x) \in D(\mathbb{R})$, $1 \le k \le m$

$$\varphi_k^{(l)}(0) = (x^{k-1})^{(l)}(0)\cdot\underbrace{\eta(0)}_{1} = \begin{cases} \frac{(k-1)!}{(k-1-l)!}\cdot x^{k-1-l}\big|_{x=0}, & l \le k-1 \\ 0, & l > k-1 \end{cases} \underset{\uparrow}{=} \begin{cases} (k-1)!, & l = k-1 \\ 0, & l \ne k-1 \end{cases}$$

преживи само $l = k-1$

$$\varphi_1(0) = 0!, \quad \varphi_1'(0) = 0, \quad \varphi_1''(0) = 0, \quad \ldots, \quad \varphi_1^{(m-1)}(0) = 0$$

$$\varphi_2(0) = 0, \quad \varphi_2'(0) = 1!, \quad \varphi_2''(0) = 0, \quad \ldots, \quad \varphi_2^{(m-1)}(0) = 0$$

$$\varphi_3(0) = 0, \quad \varphi_3'(0) = 0, \quad \varphi_3''(0) = 2!, \quad \ldots, \quad \varphi_3^{(m-1)}(0) = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\varphi_m(0) = 0, \quad \varphi_m'(0) = 0, \quad \varphi_m''(0) = 0, \quad \ldots, \quad \varphi_m^{(m-1)}(0) = (m-1)!$$

убацивати редом $\varphi_1, \varphi_2, \ldots, \varphi_m$:

$$0 = \langle xy, \varphi_k \rangle = \sum_{\alpha=1}^{m} K_\alpha \varphi_k^{(\alpha-1)}(0) \underset{\uparrow}{=} K_k \cdot \varphi_k^{(k-1)}(0) = K_k \cdot (k-1)! \Rightarrow K_k = 0, \ \forall 1 \le k \le m$$

преживи само $k = \alpha$

$$\Rightarrow K_1 = K_2 = \ldots = K_m = 0 \Rightarrow c_1 = c_2 = \ldots = c_m = 0$$

$\Rightarrow y$ је облика $y = c_0\delta$, $c_0 \in \mathbb{R}$. Да ли ово задовољава? Да!

$$\langle xy, \varphi \rangle = \langle x c_0 \delta, \varphi \rangle = c_0 \langle \delta, x\varphi \rangle = c_0 (x\varphi)(0) = c_0 (0 \cdot \varphi(0)) = 0 \ \checkmark$$

домаћи: решити $x^n \cdot y(x) = 0$

Решење: $y = c_0\delta + c_1\delta' + \ldots + c_{n-1}\delta^{(n-1)}$, $c_k \in \mathbb{R}$