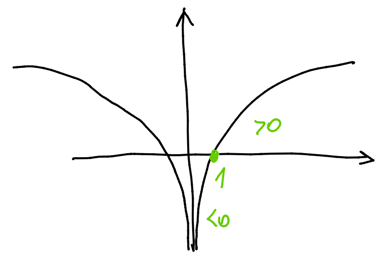
(1) Нека је $f(x)=\ln|x|$.

а) Доказати да $f\in D'(\mathbb{R})$.

б) Одредити f' у $D'(\mathbb{R})$.



а) $K\subseteq\mathbb{R}$ компакт, $\exists M>0$, $K\subseteq[-M,M]$ (узмемо $M>1$)

$$\int \ln x\,dx = x\cdot\ln x - x + c$$

$$\int_K |\ln|x||\,dx \le \int_{-M}^{M} |\ln|x||\,dx = 2\int_0^M |\ln x|\,dx = 2\cdot\left(\int_0^1 -\ln x\,dx + \int_1^M \ln x\,dx\right) = 2\cdot(-x\ln x + x)\Big|_0^1 + 2\cdot(x\ln x - x)\Big|_1^M =$$

$$= 2\cdot(1) + 2\cdot(M\ln M - M + 1) = 4 + 2M\ln M - 2M < +\infty \Rightarrow f\in L^1_{loc}(\mathbb{R}) \Rightarrow f\in D'(\mathbb{R})$$

б) $\varphi\in D(\mathbb{R})$, $\exists M>0$, $\operatorname{supp}(\varphi)\subseteq[-M,M]$

$$\langle f',\varphi\rangle = -\langle f,\varphi'\rangle = -\int_{\mathbb{R}} f(x)\cdot\varphi'(x)\,dx = -\int_{-M}^{M}\ln|x|\cdot\varphi'(x)\,dx = -\int_0^M \ln x\cdot(\varphi'(x)+\varphi'(-x))\,dx =$$

$$\int_{-M}^{0}\ln|x|\cdot\varphi'(x)\,dx = \int_{M}^{0}\ln|y|\cdot\varphi'(-y)\cdot(-dy) = \int_0^M \ln|y|\cdot\varphi'(-y)\cdot dy$$

$y=-x$
$dx=-dy$

одвајамо $\varepsilon\to0+$, због парцијалне

$$= -\lim_{\varepsilon\to0+}\int_\varepsilon^M \ln x\cdot(\varphi'(x)+\varphi'(-x)) = -\lim_{\varepsilon\to0+}\left(\ln x\cdot(\varphi(x)-\varphi(-x))\Big|_\varepsilon^M - \int_\varepsilon^M \frac{1}{x}(\varphi(x)-\varphi(-x))\,dx\right) =$$

$$= -\lim_{\varepsilon\to0+}\left(0-\ln\varepsilon\cdot(\varphi(\varepsilon)-\varphi(-\varepsilon))\right) + \lim_{\varepsilon\to0+}\int_\varepsilon^M \frac{\varphi(x)-\varphi(-x)}{x}\,dx = \int_0^M \frac{\varphi(x)-\varphi(-x)}{x}\,dx \overset{*}{=} \langle P\tfrac{1}{x},\varphi\rangle$$

$$\ln\varepsilon\cdot(\varphi(\varepsilon)-\varphi(-\varepsilon)) = \ln\varepsilon\cdot\left(\varphi(0)+\varepsilon\cdot\varphi'(0)+o(\varepsilon^2)-\varphi(0)+\varepsilon\varphi'(0)+o(\varepsilon^2)\right) =$$

$$= \ln\varepsilon\cdot\left(\varepsilon\cdot\varphi'(0)+o(\varepsilon^2)\right)\to 0,\quad \varepsilon\to0+$$

$$\frac{\varphi(x)-\varphi(-x)}{x} \sim \frac{\varphi(0)+x\varphi'(0)-\varphi(0)+x\varphi'(0)+o(x^2)}{x} \sim 2\varphi'(0),\quad x\to0+ \Rightarrow \exists\lim_{\varepsilon\to0+}\int_\varepsilon^M$$

$$* \quad \langle P\tfrac{1}{x}, \ldots\rangle = \ldots \int^{+\infty} \frac{\varphi(x)}{\ldots} \ldots \left[\ldots -\varepsilon \ldots +\infty\right] \ldots \infty \ldots \infty$$

(*) $\langle P\frac{1}{x},\varphi\rangle = p.v.\int\limits_{-\infty}^{+\infty}\frac{\varphi(x)}{x}\,dx = \lim\limits_{\varepsilon\to0+}\left[\int\limits_{-\infty}^{-\varepsilon}+\int\limits_{\varepsilon}^{+\infty}\right] = \lim\limits_{\varepsilon\to0+}\int\limits_{\varepsilon}^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}\,dx = \int\limits_{0}^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}\,dx$

смена: $y=-x$

$$\int\limits_{\infty}^{\varepsilon}\frac{\varphi(-y)}{-y}(-dy) = \int\limits_{\varepsilon}^{\infty}\frac{-\varphi(-y)}{y}\,dy$$

$\Rightarrow \quad (\ln|x|)' = P\frac{1}{x}$

② Доказати да у $\mathcal{D}'(\mathbb{R})$ важи $\quad |\sin x|''+|\sin x| = 2\cdot\sum\limits_{k=-\infty}^{+\infty}\delta_{k\pi}.$

Шта је $\sum\limits_{k=-\infty}^{+\infty}\delta_{k\pi}$?

(1) $\sum\limits_{k=-\infty}^{+\infty}\delta_{k\pi} = \lim\limits_{N\to+\infty}\sum\limits_{k=-N}^{N}\delta_{k\pi}$

(2) $\Rightarrow$ (1)

(1) $\not\Rightarrow$ (2)

(2) $\sum\limits_{k=-\infty}^{+\infty}\delta_{k\pi} = \sum\limits_{k=0}^{+\infty}\delta_{k\pi} + \sum\limits_{k=-\infty}^{-1}\delta_{k\pi} = \lim\limits_{N\to+\infty}\sum\limits_{k=0}^{N}\delta_{k\pi} + \lim\limits_{N\to+\infty}\sum\limits_{k=-N}^{-1}\delta_{k\pi}$

Лимес $\lim\limits_{N\to+\infty}$ је на оба места у простору $\mathcal{D}'(\mathbb{R})$.

Код нас није суштински битно, јер ће за $\varphi\in\mathcal{D}(\mathbb{R})$, $\langle$десно$,\varphi\rangle$ бити коначна сума.

$\varphi\in\mathcal{D}(\mathbb{R})$, $\exists N$ тд. $\operatorname{supp}(\varphi)\subseteq[-\pi N,\pi N]$

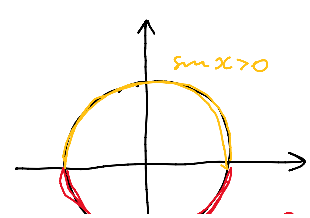
$$\langle 2\cdot\sum_{k=-\infty}^{+\infty}\delta_{k\pi},\varphi\rangle = \langle 2\cdot\sum_{k=-N}^{N}\delta_{k\pi},\varphi\rangle = 2\cdot\sum_{k=-N}^{N}\langle\delta_{k\pi},\varphi\rangle = 2\sum_{k=-N}^{N}\varphi(k\pi) = 2\cdot\sum_{k=-N+1}^{N-1}\varphi(k\pi)$$

$g(x)=|\sin x|''+|\sin x|$, $|\sin x|$ – непрекидна $\Rightarrow$ $|\sin x|\in\mathcal{D}'(\mathbb{R})$, $\exists |\sin x|''\in\mathcal{D}'(\mathbb{R})$ $\Big\}\ g\in\mathcal{D}'(\mathbb{R})$

$$\langle g,\varphi\rangle = \langle|\sin x|'',\varphi\rangle+\langle|\sin x|,\varphi\rangle = \langle|\sin x|,\varphi''\rangle+\langle|\sin x|,\varphi\rangle = \langle|\sin x|,\varphi+\varphi''\rangle =$$

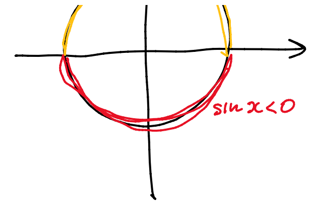
$$= \int\limits_{\mathbb{R}}|\sin x|\cdot(\varphi(x)+\varphi''(x))\,dx = \int\limits_{-\pi N}^{\pi N}|\sin x|\cdot(\varphi(x)+\varphi''(x))\,dx =$$

$$= \sum^{N-1}\int\limits^{(k+1)\pi}|\sin x|\cdot(\varphi(x)+\varphi''(x))\,dx =$$



$$= \sum_{k=-N}^{N-1} \int_{k\pi}^{(k+1)\pi} |\sin x| \cdot (\varphi(x) + \varphi''(x))\,dx =$$

$$= \sum_{k=-N}^{N-1} \int_{k\pi}^{(k+1)\pi} (-1)^k \sin x \cdot (\varphi(x) + \varphi''(x))\,dx = (*)$$



$\sin x > 0 \iff \sin x \in (2k\pi, \pi + 2k\pi)$

$\sin x < 0 \iff \sin x \in (\pi + 2k\pi, 2k\pi + 2\pi)$

$|\sin x| = (-1)^k \cdot \sin x,$

$x \in (k\pi, (k+1)\pi)$

$$\int_{k\pi}^{(k+1)\pi} \sin x \cdot \varphi''(x)\,dx = \varphi'(x)\cdot \sin x \Big|_{k\pi}^{(k+1)\pi} - \int_{k\pi} \varphi'(x)\cdot \cos x\,dx =$$

$$= -\int_{k\pi}^{(k+1)\pi} \varphi'(x)\cdot\cos x\,dx = -\varphi(x)\cdot\cos x\Big|_{k\pi}^{(k+1)\pi} + \int_{k\pi}^{(k+1)\pi} \varphi(x)\cdot(-\sin x)\,dx =$$

$$= -\varphi((k+1)\pi)\cdot(-1)^{k+1} + \varphi(k\pi)\cdot(-1)^k - \int_{k\pi}^{(k+1)\pi} \varphi(x)\cdot\sin x\,dx =$$

$$= (-1)^k\left(\varphi((k+1)\pi) + \varphi(k\pi)\right) - \int_{k\pi}^{(k+1)\pi} \varphi(x)\sin x\,dx$$

$$(*) = \sum_{k=-N}^{N-1} (-1)^k \cdot \left[\int_{k\pi}^{(k+1)\pi} (\sin x\cdot\varphi(x))\,dx + \int_{k\pi}^{(k+1)\pi} (\sin x\cdot\varphi''(x))\,dx\right] =$$

$$= \sum_{k=-N}^{N-1} (-1)^k\cdot(-1)^k\cdot\left(\varphi((k+1)\pi + \varphi(k\pi)\right) =$$

$$= \sum_{k=-N}^{N-1}\left(\varphi((k+1)\pi) + \varphi(k\pi)\right) = \left(\varphi((-N+1)\pi) + \underbrace{\varphi(-N\pi)}_{=0}\right) + \left(\varphi((-N+2)\pi) + \varphi((-N+1)\cdot\pi)\right) + \ldots$$

$$\ldots + \left(\varphi((N-1)\pi) + \varphi((N-2)\pi)\right) + \left(\underbrace{\varphi(N\pi)}_{=0} + \varphi((N-1)\pi)\right) =$$

$$= 2\cdot\sum_{k=-N+1}^{N-1} \varphi(k\pi)$$

③ а) Доказати да $\left((x,y)\mapsto \log(x^2+y^2)\right) \in D'(\mathbb{R}^2)$.

б) Наћи $\Delta \log(x^2+y^2)$ у $D'(\mathbb{R}^2)$.

а) Хоћемо $\log(x^2+y^2) \in L^1_{loc}(\mathbb{R}^2)$. $K \subseteq \mathbb{R}^2$ компакт, $\exists R>1$, $K \subseteq B[0,R]$

$$\int_K |\log(x^2+y^2)|\,dx\,dy \le \int |\log(x^2+y^2)|\,dx\,dy = \int |\log(x^2+y^2)|\,dx\,dy + \int |\log(x^2+y^2)|\,dx\,dy =$$

$$\int_K |\log(x^2+y^2)|\,dx\,dy \le \int_{B[0;R]} |\log(x^2+y^2)|\,dx\,dy = \int_{B[0;1]} |\log(x^2+y^2)|\,dx\,dy + \int_{B[0;R]\setminus B(0;1)} |\log(x^2+y^2)|\,dx\,dy =$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\Rightarrow dx\,dy = r\,dr\,d\theta$$

$$\int \log r^2 \cdot r\,dr = \frac{r^2}{2}\log r^2 - \int \frac{r^2}{2}\cdot\frac{1}{r^2}\cdot 2r\,dr = r^2\log r - \frac{r^2}{2} + c$$

$$= -\int_0^1\int_0^{2\pi} \log r^2 \cdot r\,d\theta\,dr + \int_1^R\int_0^{2\pi} \log r^2\cdot r\,d\theta\,dr =$$

$$= -2\pi\left(r^2\log r - \frac{r^2}{2}\right)\Big|_0^1 + 2\pi\left(r^2\log r - \frac{r^2}{2}\right)\Big|_1^R =$$

$$= -2\pi\left(-\frac{1}{2}\right) + 2\pi\left(R^2\log R - \frac{R^2}{2} + \frac{1}{2}\right) =$$

$$= \left(R^2\log R - \frac{R^2}{2} + 1\right)\cdot 2\pi < +\infty \Rightarrow \log(x^2+y^2) \in D'(\mathbb{R}^2)$$

б) $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\varphi \in D(\mathbb{R}^2)$

$$\langle \Delta u, \varphi\rangle = \langle D^{(2,0)}u + D^{(0,2)}u, \varphi\rangle = \langle D^{(2,0)}u, \varphi\rangle + \langle D^{(0,2)}u, \varphi\rangle =$$

$$= (-1)^2\langle u, D^{(2,0)}\varphi\rangle + (-1)^2\langle u, D^{(0,2)}\varphi\rangle = \langle u, D^{(2,0)}\varphi + D^{(0,2)}\varphi\rangle = \langle u, \Delta\varphi\rangle$$

Подсећање: Лапласијан у поларним координатама

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\cdot\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\cdot\frac{\partial}{\partial y} = \cos\theta\cdot\frac{\partial}{\partial x} + \sin\theta\cdot\frac{\partial}{\partial y}$$

$$\frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r}\left(\cos\theta\cdot\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial r}\left(\sin\theta\cdot\frac{\partial}{\partial y}\right) =$$

$$= \cos\theta\cdot\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}\right) + \sin\theta\cdot\frac{\partial}{\partial r}\left(\frac{\partial}{\partial y}\right) =$$

$$= \cos\theta\cdot\left(\cos\theta\cdot\frac{\partial^2}{\partial x^2} + \sin\theta\cdot\frac{\partial^2}{\partial y\partial x}\right) + \sin\theta\cdot\left(\cos\theta\cdot\frac{\partial^2}{\partial x\partial y} + \sin\theta\cdot\frac{\partial^2}{\partial y^2}\right) =$$

$$= \cos^2\theta\,\frac{\partial^2}{\partial x^2} + \sin^2\theta\cdot\frac{\partial^2}{\partial y^2} + 2\sin\theta\cos\theta\,\frac{\partial^2}{\partial x\partial y}$$

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\,\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\,\frac{\partial}{\partial y} = (-r\sin\theta)\frac{\partial}{\partial x} + (r\cos\theta)\frac{\partial}{\partial y}$$

$$\frac{\partial^2}{\partial\theta^2} = \frac{\partial}{\partial\theta}(-r\sin\theta)\cdot\frac{\partial}{\partial x} + (-r\sin\theta)\cdot\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial\theta}(r\cos\theta)\frac{\partial}{\partial y} + (r\cos\theta)\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial y}\right) =$$

$$= (-r\cos\theta)\frac{\partial}{\partial x} + (-r\sin\theta)\frac{\partial}{\partial y} + (-r\sin\theta)^2\frac{\partial^2}{\partial x^2} - r^2\sin\theta\cos\theta\,\frac{\partial^2}{\partial y\partial x} - r^2\sin\theta\cos\theta\,\frac{\partial^2}{\partial x\partial y} + (r\cos\theta)^2\frac{\partial^2}{\partial y^2} =$$

$$= (-r\cos\theta)\frac{\partial}{\partial x} + (-r\sin\theta)\frac{\partial}{\partial y} + (-r\sin\theta)^2\frac{\partial^2}{\partial x^2} - r^2\sin\theta\cos\theta\frac{\partial^2}{\partial y\partial x} - r^2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y} + (r\cos\theta)^2\frac{\partial^2}{\partial y^2} =$$

$$= \left(-r\cos\theta\frac{\partial}{\partial x} - r\sin\theta\frac{\partial}{\partial y}\right) + r^2\sin^2\theta\frac{\partial^2}{\partial x^2} + r^2\cos^2\theta\frac{\partial^2}{\partial y^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{r}\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{r}\frac{\partial}{\partial r}$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}}$$

$dx\,dy = r\,dr\,d\theta$
$x^2+y^2=r^2$

$$\langle \Delta\log(x^2+y^2), \varphi\rangle = \langle \log(x^2+y^2), \Delta\varphi\rangle = \int_{\mathbb{R}^2}\log(x^2+y^2)\cdot\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi(x,y)\,dx\,dy =$$

$$= \int_0^{+\infty}\int_0^{2\pi}\log(r^2)\cdot\left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\varphi(r,\theta)\cdot r\,d\theta\,dr =$$

supp φ компакт.
$\exists R>0$ тд
supp $\varphi \subseteq B[0;R]$

$$= \lim_{\varepsilon\to 0+}\int_\varepsilon^R\int_0^{2\pi}\log(r^2)\cdot\left(\frac{1}{r}\frac{\partial^2}{\partial\theta^2} + r\cdot\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\right)\varphi(r,\theta)\,d\theta\,dr$$

$$= \lim_{\varepsilon\to 0+}\int_\varepsilon^R\left(\int_0^{2\pi}\log(r^2)\cdot\frac{1}{r}\frac{\partial^2\varphi}{\partial\theta^2}d\theta\right)dr + \lim_{\varepsilon\to 0+}\int_0^{2\pi}\left(\int_\varepsilon^R\log(r^2)\cdot\left(r\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\right)\varphi(r,\theta)\,dr\right)d\theta$$

$[\varepsilon,R]\times[0,2\pi]$ компактан,
подинтегр. фја
непр. $\Rightarrow$ Фубини важи
$\Rightarrow \int_\varepsilon^R\int_0^{2\pi} = \int_0^{2\pi}\int_\varepsilon^R$

$$\int_0^{2\pi}\log(r^2)\cdot\frac{1}{r}\frac{\partial^2\varphi}{\partial\theta^2}d\theta = \log(r^2)\cdot\frac{1}{r}\int_0^{2\pi}\frac{\partial^2\varphi(r,\theta)}{\partial\theta^2}d\theta = \frac{\log(r^2)}{r}\cdot\left.\frac{\partial\varphi(r,\theta)}{\partial\theta}\right|_0^{2\pi} = 0$$

$\varphi(\cdot,\theta)$ је 2π-период. $\Rightarrow \dfrac{\partial\varphi(\cdot,\theta)}{\partial\theta}$ је 2π-пер.

$$\int_\varepsilon^R\log(r^2)\cdot\left(r\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\right)\varphi(r,\theta)\,dr = \int_\varepsilon^R\log(r^2)\,r\,\frac{\partial^2\varphi(r,\theta)}{\partial r^2}dr + \int_\varepsilon^R\log(r^2)\frac{\partial\varphi(r,\theta)}{\partial r}dr =$$

$$= \left.\left(\log(r^2)\cdot r\cdot\frac{\partial\varphi(r,\theta)}{\partial r}\right)\right|_\varepsilon^R - \int_\varepsilon^R(2+\log(r^2))\cdot\frac{\partial\varphi(r,\theta)}{\partial r}dr + \left.\left(\log(r^2)\cdot\varphi(r,\theta)\right)\right|_\varepsilon^R - \int_\varepsilon^R\frac{2}{r}\cdot\varphi(r,\theta)\,dr =$$

$$= \left(\log(r^2)\cdot r\cdot \frac{\partial \varphi(r,\theta)}{\partial r}\right)\Big|_\varepsilon^R - \int_\varepsilon^R (2+\log(r^2))\cdot \frac{\partial\varphi(r,\theta)}{\partial r}dr + \left(\log(r^2)\cdot\varphi(r,\theta)\right)\Big|_\varepsilon^R - \int_\varepsilon^R \frac{2}{r}\cdot\varphi(r,\theta)dr =$$

$$= -\log(\varepsilon^2)\cdot\varepsilon\cdot\frac{\partial\varphi(\varepsilon,\theta)}{\partial r} - \left((2+\log(r^2))\cdot\varphi(r,\theta)\Big|_\varepsilon^R - \int_\varepsilon^R\frac{2}{r}\varphi(r,\theta)dr\right) - \log(\varepsilon^2)\varphi(\varepsilon,\theta) - \int_\varepsilon^R\frac{2}{r}\varphi(r,\theta)dr =$$

$(\log(r^2)\cdot r)' =$

$= \frac{1}{r^2}\cdot 2r\cdot r + \log(r^2) =$

$= 2+\log(r^2)$

$$= -\log(\varepsilon^2)\cdot\varepsilon\cdot\frac{\partial\varphi(\varepsilon,\theta)}{\partial r} - \log(\varepsilon^2)\varphi(\varepsilon,\theta) + (2+\log(\varepsilon^2))\cdot\varphi(\varepsilon,\theta) =$$

$$= -\log(\varepsilon^2)\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial r}(\varepsilon,\theta) + 2\varphi(\varepsilon,\theta)$$

$$\Rightarrow \langle \Delta\log(x^2+y^2),\varphi\rangle = \lim_{\varepsilon\to0+}\int_0^{2\pi}\left(-\log(\varepsilon^2)\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial r}(\varepsilon,\theta) + 2\varphi(\varepsilon,\theta)\right)d\theta = \int_0^{2\pi}\lim_{\varepsilon\to0+}2\varphi(\varepsilon,\theta)\,d\theta = 2\varphi(0,0)\cdot\int_0^{2\pi}d\theta$$

φ непр.

$\lim_{\varepsilon\to0+}\varphi(\varepsilon,\theta) = \varphi(0,0)$

у Декартовим

$\frac{\partial\varphi}{\partial r}$ у околини $(0,0)$ је ограничено,

а $\log(\varepsilon^2)\cdot\varepsilon \xrightarrow[\varepsilon\to0+]{} 0$ $\Rightarrow \lim_{\varepsilon\to0+}\left(-\log(\varepsilon^2)\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial r}(\varepsilon,\theta)\right) = 0$

ТДК овде пролази, јер су φ, $\frac{\partial\varphi}{\partial r}$ ограничене

$\int\lim = \lim\int$

$$\Rightarrow \langle\Delta\log(x^2+y^2),\varphi\rangle = 2\varphi(0,0)\cdot 2\pi = 4\pi\cdot\langle\delta,\varphi\rangle \Rightarrow \Delta\log(x^2+y^2) = 4\pi\delta.$$

Напомена: функција $\log(x^2+y^2)$ на $\mathbb{R}^2\setminus\{(0,0)\}$ је хармонијска $\Delta\log(x^2+y^2)=0$