① $\Omega \subseteq \mathbb{R}^n$, $a \in \Omega$. Доказати да $\nexists f \in L^1_{loc}(\Omega)$ тд. $T_f = \delta_a$.

ППС $\exists f \in L^1_{loc}(\Omega)$ $\quad T_f = \delta_a$

$x = (x_1, \dots, x_n)$

$a = (a_1, \dots, a_n)$

$\forall \varphi \in D(\Omega)$: $\quad \langle T_f, \varphi \rangle = \langle \delta_a, \varphi \rangle = \varphi(a)$

↳ произвољна

$$\int_\Omega f(x)\varphi(x)\,dx = \varphi(a)$$

$\varphi_1(x) = (x_1 - a_1)\cdot \varphi(x)$: $\qquad \int_\Omega f(x)\cdot(x_1-a_1)\cdot\varphi(x)\,dx = \varphi_1(a) = 0$

$\varphi_1(a) = 0 \cdot \varphi(a) = 0$

$\boxed{T}$ (Du Bois-Reymond лема, лема Ди Буа-Рејмонда, фундаментална лема варијационог рачуна)
Нека је $\Omega \subseteq \mathbb{R}^n$ област и $f \in L^1_{loc}(\Omega)$. Ако је $\langle T_f, \varphi \rangle = 0$ $\forall \varphi \in D(\Omega)$, онда је $f = 0$ скоро свуда (с.с.) на Ω.

по T $\Rightarrow f(x)\cdot(x_1 - a_1) = 0$ с.с. $\Rightarrow f(x) = 0$ с.с. $\qquad \Rightarrow \langle T_f, \varphi \rangle = 0 = \varphi(a)$ ↯

↑ скуп $\{x_1 = a_1\} \subseteq \Omega$ мере нула

② Испитати да ли је u дистрибуција из $D'(\mathbb{R})$:

а) $\langle u, \varphi \rangle = \sum_{j=0}^{+\infty} 2^{-j}\varphi^{(r(j))}(0)$ $\qquad$ б) $\langle u, \varphi \rangle = \varphi(0)^2$ $\qquad$ в) $\langle u, \varphi \rangle = \sum_{j=0}^{+\infty} 2^{-j}\varphi^{(j)}(0)$

$r(j) = (j \bmod 3)$

а) добра дефинисаност:
$$\langle u, \varphi \rangle = \sum_{k=0}^{+\infty} 2^{-3k}\varphi^{(0)}(0) + \sum_{k=0}^{+\infty} 2^{-3k-1}\varphi^{(1)}(0) + \sum_{k=0}^{+\infty} 2^{-3k-2}\varphi^{(2)}(0) =$$

$j=3k$ $\quad j=3k+1$ $\quad j=3k+2$

$$r(j) = \begin{cases} 0, & j = 3k \\ 1, & j = 3k+1 \\ 2, & j = 3k+2 \end{cases}$$

$$= \sum_{k=0}^{+\infty} \frac{1}{8^k}\varphi(0) + \sum_{k=0}^{+\infty} \frac{1}{8^k \cdot 2}\varphi'(0) + \sum_{k=0}^{+\infty} \frac{1}{8^k\cdot 4}\varphi''(0) =$$

$$= \left(\varphi(0) + \frac{\varphi'(0)}{2} + \frac{\varphi''(0)}{4}\right) \cdot \underbrace{\sum_{k=0}^{\infty} \frac{1}{8^k}}_{= \frac{1}{1-\frac{1}{8}} = \frac{8}{7}} = \frac{8}{7}\left(\varphi(0) + \frac{\varphi'(0)}{2} + \frac{\varphi''(0)}{4}\right) \in \mathbb{R}$$

линеарност: $\langle u, \lambda_1\varphi_1 + \lambda_2\varphi_2 \rangle = \frac{8}{7}\left((\lambda_1\varphi_1+\lambda_2\varphi_2)(0) + \frac{(\lambda_1\varphi_1+\lambda_2\varphi_2)'(0)}{2} + \frac{(\lambda_1\varphi_1+\lambda_2\varphi_2)''(0)}{4}\right) = \dots \checkmark$

непрекидност: $\varphi_j \xrightarrow[j\to\infty]{\mathcal{D}(\mathbb{R})} 0 \overset{?}{\Rightarrow} \langle u, \varphi_j \rangle \xrightarrow[j\to+\infty]{} 0$

$$\left.\begin{array}{lll}
\overset{\text{ТПТ}}{\varphi_j(0) \to 0}: & \exists j_1,\ \forall j \geq j_1 & |\varphi_j(0)| < \varepsilon \\
\overset{\text{ТПТ}}{\varphi_j'(0) \to 0}: & \exists j_2,\ \forall j \geq j_2 & |\varphi_j'(0)| < \varepsilon \\
\overset{\text{ТПТ}}{\varphi_j''(0) \to 0}: & \exists j_3,\ \forall j \geq j_3 & |\varphi_j''(0)| < \varepsilon
\end{array}\right\} \Rightarrow j \geq \max\{j_1, j_2, j_3\}:$$

$$|\langle u, \varphi_j \rangle| \leq \frac{8}{7}\left(|\varphi_j(0)| + \frac{|\varphi_j'(0)|}{2} + \frac{|\varphi_j''(0)|}{4}\right) \leq$$

$$\leq \frac{8}{7}\left(\varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}\right) = \frac{8}{7} \cdot \frac{7}{4}\varepsilon = 2\varepsilon$$

б) линеарност: $\langle u, \varphi_1 + \varphi_2 \rangle \overset{?}{=} \langle u, \varphi_1 \rangle + \langle u, \varphi_2 \rangle$

$$\left.\begin{array}{ccc}
\langle u, \varphi_1+\varphi_2 \rangle & \langle u, \varphi_1 \rangle & \langle u, \varphi_2 \rangle \\
\| & \| & \| \\
((\varphi_1+\varphi_2)(0))^2 & \varphi_1(0)^2 & \varphi_2(0)^2
\end{array}\right\} \ (\varphi_1(0)+\varphi_2(0))^2 = \varphi_1(0)^2 + \varphi_2(0)^2$$

$$\varphi_1(0) \cdot \varphi_2(0) = 0 \quad \forall \varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}) \ ↯$$

$\Rightarrow u$ није дистр.

в) домаћи: одговор: НЕ

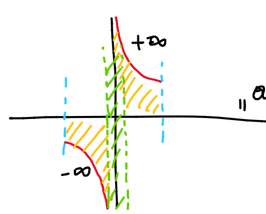
помоћ: 3. задатак са списка

$$\int_{-1}^{1} \frac{dx}{x} \text{ не постоји}$$

$$\|$$

$$\int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x}$$

не конвергирају



„али су исте"

$\int_a^b f(x)\,dx$, $c \in (a,b) \rightarrow$ имамо сингуларитет

$\int_a^b f(x)\,dx$, $c \in (a,b) \to$ имамо сингуларитет

главна вредност интеграла: $\quad \text{p.v.} \int_a^b f(x)\,dx = \lim\limits_{\varepsilon \to 0+} \left(\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^b f(x)\,dx \right)$

(principal value, value of principal) $\quad$ $\parallel$ $\quad$ $\text{v.p.} \int_a^b f(x)\,dx$

пр. $\text{p.v.} \int_{-1}^1 \frac{dx}{x} = \lim\limits_{\varepsilon \to 0+} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^1 \frac{dx}{x} \right) = \lim\limits_{\varepsilon \to 0+} \left(\ln|x| \Big|_{-1}^{-\varepsilon} + \ln|x| \Big|_{\varepsilon}^{1} \right) = \lim\limits_{\varepsilon \to 0+} \left(\ln\varepsilon - \ln 1 + \ln 1 - \ln\varepsilon \right) = \lim\limits_{\varepsilon \to 0+} 0 = 0$

нпр. $\int_{-1}^1 \frac{dx}{x} = \lim\limits_{\varepsilon \to 0+} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{2\varepsilon}^1 \frac{dx}{x} \right) = \lim\limits_{\varepsilon \to 0+} (\ln\varepsilon - \ln 2\varepsilon) = -\ln 2 \neq \text{p.v.} \int_{-1}^1 \frac{dx}{x}$

(3) Пресликавање $\mathcal{P}\frac{1}{x^2} : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ је дефинисано са $\mathcal{P}\frac{1}{x^2}(\varphi) = \text{p.v.} \int_{-\infty}^{+\infty} \frac{\varphi(x) - \varphi(0)}{x^2}\,dx$

а) Доказати да је $\mathcal{P}\frac{1}{x^2}$ добро дефинисано.

б) Доказати да је $\mathcal{P}\frac{1}{x^2}$ дистрибуција

а) $\varphi \in \mathcal{D}(\mathbb{R})$

$$\text{p.v.} \int_{-\infty}^{+\infty} \frac{\varphi(x) - \varphi(0)}{x^2}\,dx = \lim_{\varepsilon \to 0+} \left(\underbrace{\int_{-\infty}^{-\varepsilon} \frac{\varphi(x) - \varphi(0)}{x^2}\,dx} + \int_{\varepsilon}^{+\infty} \frac{\varphi(x) - \varphi(0)}{x^2}\,dx \right) = \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{+\infty} \frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^2}\,dx \overset{(*)}{=} \int_0^{+\infty} \frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^2}\,dx$$

$y = -x$

$-\varepsilon \to \varepsilon$

$-\infty \to \infty$

$dx = -dy$

$$= \int_{+\infty}^{\varepsilon} \frac{\varphi(-y) - \varphi(0)}{y^2}(-dy) = \int_{\varepsilon}^{+\infty} \frac{\varphi(-x) - \varphi(0)}{x^2}\,dx$$

$(*)$ $\quad x \to 0+:$

$$\frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^2} \sim \frac{\left(\varphi(0) + x\varphi'(0) + \frac{x^2}{2}\varphi''(0) + o(x^2)\right) + \left(\varphi(0) - x\varphi'(0) + \frac{x^2}{2}\varphi''(0) + o(x^2)\right) - 2\varphi(0)}{x^2} \sim$$

$$\sim \frac{x^2\varphi''(0) + o(x^2)}{x^2} \sim \varphi''(0) + o(1) \sim \varphi''(0)$$

$$\mathcal{P}\frac{1}{x^2}(\varphi) = \int_0^{+\infty} \frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^2}\,dx = \int_0^M \frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^2}\,dx - \underbrace{\int_M^{+\infty} \frac{2\varphi(0)}{x^2}\,dx} \quad \checkmark$$

$\varphi \in \mathcal{D}(\mathbb{R})$, $\quad \text{supp}(\varphi) \subseteq [-M, M]$

$\text{supp}(-\varphi) \subseteq [-M, M]$

$[0, M]$ коначан $\Rightarrow$ конв.

$\int_M^{+\infty} \frac{1}{x^2}\,dx$ конв. $\Rightarrow$ конв.

$\varphi \in D(\mathbb{R})$, $supp(\varphi) \subseteq [-M,M]$

$supp(-\varphi) \subseteq [-M,M]$

$[0,M]$ компакан $\Rightarrow$ конв.

$\int_M \frac{1}{x^2} dx$ конв. $\Rightarrow$ конв.

б) линеарност: ✓

непрекидност: $\varphi_j \xrightarrow[j\to\infty]{D(\mathbb{R})} 0$. Треба нам $\mathcal{P}\frac{1}{x^2}(\varphi_j) \xrightarrow[j\to\infty]{} 0$.

$\varepsilon > 0$ дато $\exists M>0$, $supp(\varphi_j) \subseteq [-M,M]$

$$\left|\mathcal{P}\frac{1}{x^2}(\varphi_j)\right| = \left|\int_0^{+\infty} \frac{\varphi_j(x)+\varphi_j(-x)-2\varphi(0)}{x^2}dx\right| = \left|\int_0^M \frac{\varphi_j(x)+\varphi_j(-x)-2\varphi_j(0)}{x^2}dx - \int_M^{+\infty}\frac{2\varphi_j(0)}{x^2}dx\right| \leq$$

$$\leq \int_0^M \left|\frac{\varphi_j(x)+\varphi_j(-x)-2\varphi_j(0)}{x^2}\right|dx + \underbrace{\int_M^{+\infty}\frac{2|\varphi_j(0)|}{x^2}dx}$$

$$= 2|\varphi_j(0)|\cdot\int_M^{+\infty}\frac{dx}{x^2} = 2|\varphi_j(0)|\left(-\frac{1}{x}\right)\Big|_M^{+\infty} = \frac{2|\varphi_j(0)|}{M}$$

$$\frac{\varphi_j(x)+\varphi_j(-x)-2\varphi_j(0)}{x^2} = \frac{(\varphi_j(0)+x\varphi_j'(0)+\frac{1}{2}\varphi_j''(\xi_1)x^2)+(\varphi_j(0)-x\varphi_j'(0)+\frac{1}{2}\varphi_j''(\xi_2)x^2)-2\varphi_j(0)}{x^2} =$$

$$= \frac{1}{2}\left(\varphi_j''(\xi_1)+\varphi_j''(\xi_2)\right)$$

$\xi_1 = \xi_1(x)$, $\xi_1 \in (0,x)$

$\xi_2 = \xi_2(x)$, $\xi_2 \in (0,x)$

$$\Rightarrow \left|\mathcal{P}\frac{1}{x^2}(\varphi_j)\right| \leq \int_0^M \frac{1}{2}|\varphi_j''(\xi_1)| + \frac{1}{2}|\varphi_j''(\xi_2)|dx + \frac{2|\varphi_j(0)|}{M}$$

$\exists j_1, \forall j \geq j_1 : |\varphi_j(x)| \leq \frac{\varepsilon}{M}$, $\forall x \in [-M,M]$ $\quad (\varphi_j'' \rightrightarrows 0)$

$\exists j_2, \forall j \geq j_2 : |\varphi_j(0)| \leq M\varepsilon$ $\quad (\varphi_j(0) \to 0$ ТТ$)$

$$\forall j \geq \max\{j_1,j_2\} : \left|\mathcal{P}\frac{1}{x^2}(\varphi_j)\right| \leq \int_0^M\left(\frac{1}{2}\cdot\frac{\varepsilon}{M}+\frac{1}{2}\cdot\frac{\varepsilon}{M}\right)dx + \frac{2\cdot M\varepsilon}{M} = \frac{\varepsilon}{M}\int_0^M dx + 2\varepsilon = 3\varepsilon$$

домаћи: $\mathcal{P}\frac{1}{x}(\varphi) = p.v.\int_{-\infty}^{+\infty}\frac{\varphi(x)}{x}dx$ – доказати да је дистрибуција

$D'(\mathbb{R})$ – векторски простор: $\langle u_1+u_2, \varphi\rangle = \langle u_1,\varphi\rangle + \langle u_2,\varphi\rangle$

$\langle \lambda u, \varphi \rangle = \langle u, \lambda \varphi \rangle$

<u>Деф.</u> За низ дистрибуција $\{f_j\}_{j\in\mathbb{N}} \in \mathcal{D}'(\Omega)$ кажемо да конвергира ка дистрибуцији $f\in\mathcal{D}'(\mathbb{R})$ ако $\forall\varphi\in\mathcal{D}(\Omega)$ важи $\lim\limits_{j\to\infty}\langle f_j,\varphi\rangle = \langle f,\varphi\rangle$.

$\boxed{T}$ Нека је $\varphi\in\mathcal{D}(\Omega)$ произвољно. Ако $\exists \lim\limits_{j\to\infty}\langle f_j,\varphi\rangle$, онда $\exists f\in\mathcal{D}'(\Omega)$ т.д. $f_j \xrightarrow[j\to\infty]{\mathcal{D}'(\Omega)} f$.

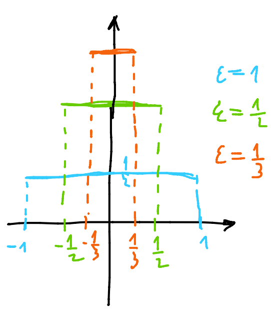
Напомена: $\{f_j\}_{j\in\mathbb{N}}$ из деф. може да буде и други индекс.

(4) Нека је $\varepsilon>0$ и $f_\varepsilon:\mathbb{R}\to\mathbb{R}$ као $f_\varepsilon(x)=\begin{cases}\frac{1}{2\varepsilon}, & |x|<\varepsilon\\ 0, & |x|\geq\varepsilon\end{cases}$. Одредити $\lim\limits_{\varepsilon\to0} f_\varepsilon$ у $\mathcal{D}'(\mathbb{R})$.

$\left(f_\varepsilon \xrightarrow[\varepsilon\to0]{\mathcal{D}'(\mathbb{R})} f = ?\right)$

правилније рећи: $\lim\limits_{\varepsilon\to0} T_{f_\varepsilon}$.

$$\|f_\varepsilon\|_{L_1} = \int_{\mathbb{R}}|f_\varepsilon(x)|\,dx = \int_{\mathbb{R}} f_\varepsilon(x)\,dx = \int_{-\varepsilon}^{\varepsilon}\frac{1}{2\varepsilon}\,dx = 1 \Rightarrow f_\varepsilon\in L^1(\mathbb{R}) \Rightarrow f_\varepsilon\in L^1_{loc}(\mathbb{R}) \Rightarrow f_\varepsilon \text{ деф. дистрибуцију}$$



$\varphi\in\mathcal{D}(\mathbb{R})$ произвољно

$$\lim_{\varepsilon\to0}\langle f_\varepsilon,\varphi\rangle = \lim_{\varepsilon\to0}\int_{\mathbb{R}} f_\varepsilon(x)\varphi(x)\,dx = \lim_{\varepsilon\to0}\int_{-\varepsilon}^{\varepsilon}\frac{\varphi(x)}{2\varepsilon}\,dx = \lim_{\varepsilon\to0}\int_{-1}^{1}\frac{\varphi(\varepsilon y)}{2\varepsilon}\cdot\varepsilon\,dy$$

смена: $\frac{x}{\varepsilon}=y$, $dx=\varepsilon\,dy$

$\pm\varepsilon\to\pm1$

$$= \lim_{\varepsilon\to0}\int_{-1}^{1}\frac{\varphi(\varepsilon y)}{2}\,dy = \int_{-1}^{1}\lim_{\varepsilon\to0}\frac{\varphi(\varepsilon y)}{2}\,dy =$$

ТДК

$M=\frac{1}{2}\max\limits_{x\in\mathbb{R}}|\varphi(x)|$ ($\exists$ јер је $supp(\varphi)$ компактан)

$$= \int_{-1}^{1}\frac{\varphi(\lim_{\varepsilon\to0}\varepsilon y)}{2}\,dy = \int_{-1}^{1}\frac{\varphi(0)}{2}\,dy = \varphi(0)$$

φ непр.

$T \Rightarrow \exists f,\ f_\varepsilon \xrightarrow{\mathcal{D}'(\mathbb{R})} f \qquad \langle f,\varphi\rangle = \varphi(0) \Rightarrow f=\delta$