10 - уторак 24.05.2022.

① $u:(-1,1)\to\mathbb{R}$, $u(x)=\begin{cases}\frac{x^2}{2}, & x\ge 0\\ -\frac{x^2}{2}, & x<0\end{cases}$. Одредити највеће $k\in\mathbb{N}_0$ т.д.:

а) $u\in C^k(-1,1)$

б) $u\in H^k(-1,1)$

а) $u'(x)=\begin{cases} x, & x\ge 0\\ -x, & x<0\end{cases} = |x| \in C(-1,1) \Rightarrow u\in C^1(-1,1)$

$$u'(0)=\lim_{h\to 0}\frac{u(h)}{h}=\lim_{h\to 0}\frac{\operatorname{sgn}(h)\cdot\frac{h^2}{2}}{h}=0 \qquad \Rightarrow \boxed{k=1}$$

$|x|$ није диф. $\Rightarrow |x|\notin C^1(-1,1) \Rightarrow u\notin C^2(-1,1)$

б) $f\in C(a,b) \Rightarrow f\in L^2(a,b)$

$u'(x)=|x| \Rightarrow u'\in L^2(-1,1)$

проверити у $D'(-1,1)$

$u''(x)=\operatorname{sgn} x$

$u'''(x)=2\delta\notin L^2(-1,1)$

$$\|\operatorname{sgn} x\|_{L^2}=\left(\int_{-1}^{1}|\operatorname{sgn} x|^2\,dx\right)^{1/2}=\sqrt{2} \Rightarrow u''\in L^2(-1,1)$$

$\Rightarrow \boxed{k=2}$

② Нека је $1\le p<+\infty$ и $u\in W^{1,p}(0,1)$. Доказати $|u(x)-u(y)|\le|x-y|^{1-\frac{1}{p}}\cdot\left(\int_0^1|u'(t)|^p\,dt\right)^{1/p}$ за скоро свако $x,y\in[0,1]$.

$u,u'\in L^p(0,1)$

Из зад. са прошлог часа: $u\in AC[0,1]$ и $u'\in L^1(0,1) \xRightarrow{T}$ важи НЛ формула

$$u(x)-u(0)=\int_0^x u'(t)\,dt$$

БУО $y\ge x$:

$$|u(y)-u(x)|=\left|u(0)+\int_0^y u'(t)dt-\left(u(0)+\int_0^x u'(t)dt\right)\right|=\left|\int_x^y u'(t)dt\right|\le\int_x^y|u'(t)|dt$$

1° $p=1$ $\quad |u(y)-u(x)|\le\int_x^y \underbrace{|u'(t)|}_{\ge 0}dt\le\int_0^1|u'(t)|dt$ ✓

2° $p>1$ ⇝ Хелдер

2° $p>1 \rightsquigarrow$ Хелдер

$$|u(y)-u(x)| \le \int_x^y |u'(t)|\cdot 1\,dt \underset{u'\in L^p}{\le} \left(\int_x^y |u'(t)|^p\,dt\right)^{1/p}\cdot\left(\int_x^y 1^q\,dt\right)^{\frac{1}{q}} \le \left(\int_0^1 |u'(t)|^p\,dt\right)^{1/p}\cdot (y-x)^{1-\frac{1}{p}}$$

$$\frac{1}{q} = 1-\frac{1}{p}$$

(3) $\Omega=(0,1)^2$, $f:\Omega\to\mathbb{R}$, $f(x,y)=\sqrt{x}$. Испитати да ли f припада $W^{0,2}(\Omega)$, $W^{1,1}(\Omega)$, $W^{1,2}(\Omega)$.

$f\in L^2$ $\qquad$ $f, f'_x, f'_y\in L^1$ $\qquad$ $f, f'_x, f'_y\in L^2$

$f\in L^2$? $\quad \|f\|_{L^2} = \left(\int_\Omega |f(x,y)|^2\,dx\,dy\right)^{1/2} = \left(\int_\Omega x\,dx\,dy\right)^{1/2} =$

$$= \left(\int_0^1 dy\cdot\int_0^1 x\,dx\right)^{1/2} = \frac{1}{\sqrt{2}} < +\infty \Rightarrow f\in L^2(\Omega) \Rightarrow f\in W^{0,2}(\Omega)$$

$$\frac{\partial f}{\partial y}(x,y) = 0$$

$\frac{\partial f}{\partial x}(x,y) = \frac{1}{2\sqrt{x}}$ (проверити у $\mathcal{D}'(\Omega)$ да важи)

$f'_y\in L^1(\Omega)$, $f'_y\in L^2(\Omega)$

$$\|f\|_{L^1} = \int_\Omega |f(x,y)|\,dx\,dy = \int_0^1 dy\cdot\int_0^1 \sqrt{x}\,dx = 1\cdot\frac{2}{3}1 = \frac{2}{3} < +\infty \Rightarrow f\in L^1(\Omega)$$

$$\|f'_x\|_{L^1} = \int_\Omega \frac{1}{2\sqrt{x}}\,dx\,dy = \int_0^1 dy\cdot\int_0^1\frac{dx}{2\sqrt{x}} = 1\cdot\frac{2}{2}\cdot 1 = 1 < +\infty \Rightarrow f'_x\in L^1(\Omega) \Rightarrow f\in W^{1,1}(\Omega)$$

$$\|f'_x\|^2_{L^2} = \int_\Omega |f'_x(x,y)|^2\,dx\,dy = \int_\Omega \frac{1}{4x}\,dx\,dy = +\infty \Rightarrow f'_x\notin L^2(\Omega) \Rightarrow f\notin W^{1,2}(\Omega)$$

$$\int_0^1 \frac{1}{x}\,dx = +\infty$$

(4) Нека је $1\le p<+\infty$ и $\alpha\in\mathbb{R}$ и $u(x)=|x|^\alpha$, $x\in\mathbb{R}$. Одредити под којим условима за p и α важи $u\in W^{1,p}(-1,1)$.

$u\in W^{1,p}(-1,1) \iff u\in L^p(-1,1)$ и $u'\in L^p(-1,1)$

$$u\in L^1(-1,1) \iff \int_0^1 |x|^\alpha dx < +\infty \iff -\alpha<1 \iff \alpha>-1.$$

$$u\in L^p(-1,1) \iff \int_{-1}^1 \left(|x|^\alpha\right)^p dx < +\infty \iff \int_0^1 |x|^{\alpha p} dx < +\infty \iff -p\alpha<1 \iff \alpha > -\tfrac{1}{p}. \qquad \left(-\tfrac{1}{p}>-1\right)$$

За $\alpha>-1$ тражимо $u'\in D'(-1,1)$.

$\alpha>0$: $u(x)=|x|^\alpha$, $\varphi\in D(-1,1)$

$$\langle u',\varphi\rangle = -\langle u,\varphi'\rangle = -\int_{-1}^1 |x|^\alpha \varphi'(x)dx = -\int_{-1}^0 (-x)^\alpha\varphi'(x)dx - \int_0^1 x^\alpha\varphi'(x)dx =$$

$$= -\int_0^1 x^\alpha\varphi'(-x)dx - \int_0^1 x^\alpha \varphi'(x)dx =$$

$$= -\int_0^1 x^\alpha\left(\varphi'(x)+\varphi'(-x)\right)dx = -x^\alpha\cdot\left(\varphi(x)-\varphi(-x)\right)\Big|_0^1 + \int_0^1 \alpha x^{\alpha-1}\left(\varphi(x)-\varphi(-x)\right)dx =$$

$x^\alpha|_{x=0}=0$

$\left(\varphi(x)-\varphi(-x)\right)\big|_{x=1}=0$

$$= \int_0^1 \alpha\cdot x^{\alpha-1}\left(\varphi(x)-\varphi(-x)\right)dx = \int_0^1 \alpha x^{\alpha-1}\varphi(x)dx - \int_0^1 \alpha x^{\alpha-1}\varphi(-x)dx =$$

овај $\int$ конв. јер $\alpha-1>-1$

$$= \int_0^1 \alpha x^{\alpha-1}\varphi(x)dx - \int_{-1}^0 \alpha(-x)^{\alpha-1}\varphi(x)dx =$$

$$= \int_0^1 \alpha|x|^{\alpha-1}\varphi(x)dx + \int_{-1}^0 \alpha|x|^{\alpha-1}\varphi(x)\operatorname{sgn} x\, dx =$$

$$= \int_{-1}^1 \boxed{\alpha|x|^{\alpha-1}\cdot\operatorname{sgn}(x)}\,\varphi(x)dx \quad\Rightarrow\quad u'=\alpha|x|^{\alpha-1}\operatorname{sgn}(x)$$

$$u'\in L^p \iff \int_0^1\left(|x|^{\alpha-1}\right)^p dx<+\infty \iff -p(\alpha-1)<1 \iff \alpha-1>-\tfrac{1}{p} \iff \alpha>1-\tfrac{1}{p}$$

$\alpha=0$: $u(x)=1$, $u'(x)=0 \Rightarrow p\geq 1$ произвољно

$-1<\alpha<0$: радимо исто као и у случају $\alpha>0$, само треба оправдати парц. интегр.

$$\int_0^1 \alpha\cdot x^{\alpha-1}\cdot\left(\varphi(x)-\varphi(-x)\right)dx \text{ зашто конв?}$$

$$\int_0^1 x^{\alpha-1}dx=+\infty, \text{ јер } -(\alpha-1)=1-\alpha\in(1,2)$$

$$\int_0^1 \alpha \cdot x^{\alpha-1} \cdot \left(\varphi(x)-\varphi(-x)\right)dx = \int_0^1 \alpha \cdot x^{\alpha} \cdot \frac{\varphi(x)-\varphi(-x)}{x}dx$$

$$\varphi \in C^{\infty}(-1,1) \Rightarrow \frac{\varphi(x)-\varphi(-x)}{x} = \frac{\varphi(0)+x\varphi'(0)+o(x)-\left(\varphi(0)-x\varphi'(0)+o(x)\right)}{x} = \frac{2x\varphi'(0)+o(x)}{x} =$$

$$= 2\varphi'(0)+o(1) \to 2\varphi'(0)\ ,\ x\to 0$$

Када $x\to 0$: $\alpha x^{\alpha-1}\left(\varphi(x)-\varphi(-x)\right) \sim 2\alpha\varphi'(0)\cdot x^{\alpha}$ $\Rightarrow$ конвергира

$\Rightarrow$ и овде важи $u' = \alpha|x|^{\alpha-1}\cdot \mathrm{sgn}(x)$.

$$u' \in L^p \iff \int_0^1 |x|^{(\alpha-1)p}dx < +\infty \iff \left.\begin{matrix}\alpha > 1-\frac{1}{p} \\ -1<\alpha<0\end{matrix}\right\} \quad 1-\frac{1}{p}<\alpha<0 \iff 1<\frac{1}{p} \iff p<1 \ \lightning$$

Одговор: $\alpha = 0 \ \vee \ \alpha > 1-\frac{1}{p}$.