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CHAOTIC AND HYPERCHAOTIC MOTION OF A CHARGED PARTICLE IN A MAGNETIC DIPOLE FIELD

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The motion of a charged particle in the field of a magnetic dipole is studied by numerically integrating the equations of motion. The widely believed picture in which a bound particle corkscrews about a line of magnetic flux, bouncing back along the same line as it nears the poles, is shown to be a substantial over-simplification. The nature of the trajectory depends on the energy of the particle, but whatever the energy this picture is not observed. For low energies the particle will corkscrew towards the poles, while at the same time drifting laterally with a variable speed in a quasiperiodic fashion. For intermediate energies the motion is found to be chaotic, and for higher energies it becomes hyperchaotic. In the equatorial plane only quasiperiodic orbits can occur. If the magnetic dipole moment is slowly varying, the particle undergoes chaotic motion even in the equatorial plane, but only for high energies.

1. Introduction

In 1896 Poincaré [1896] published a work explaining a phenomenon that was puzzling contemporary physicists. The experiment involved the effect of a magnetic field on cathode rays. A long thin magnet was placed in front of the screen, parallel to the oncoming beam. The beam converged, and if the magnet was suitably placed, was brought to a sharp focus, causing the glass to melt in some cases. What also surprised these observers was that the effect is the same when the polarity of the magnet is reversed.

Poincaré explained the phenomenon by analyzing the motion of a charged particle in the field of a magnetic monopole (the far end of the magnet being, he realized, irrelevant to the problem). Solving the equations of motion, he found that the particle followed a geodesic path on a circular cone with its apex at the monopole. As the particle approaches the monopole, it spirals around, in a direction that depends on the sign of the electric charge and the direction of the monopole field. After reaching a point of closest approach, it returns along a trajectory that is the mirror image of the incoming one. If the screen is at (or near) this point, a broad beam will be brought to a focus. (Incidentally, this is very easy to demonstrate in the laboratory.)

Poincaré's result inspired Störmer [1955] to examine the motion of a charged particle in the field of a magnetic dipole. This is a much more difficult problem, and an analytic solution, for arbitrary initial conditions, is unknown. Since the magnetic field of the Earth resembles a dipole field, Störmer's problem sheds light on the behavior of cosmic rays, the dynamics of ions in the upper atmosphere, the structure of radiation belts, and the polar aurora [Störmer, 1955; Rossi & Olbert, 1970].

If the initial position and velocity of the charge lie in the equatorial plane, the subsequent motion will remain in that plane, and we represent the trajectory using the natural coordinates r (the distance

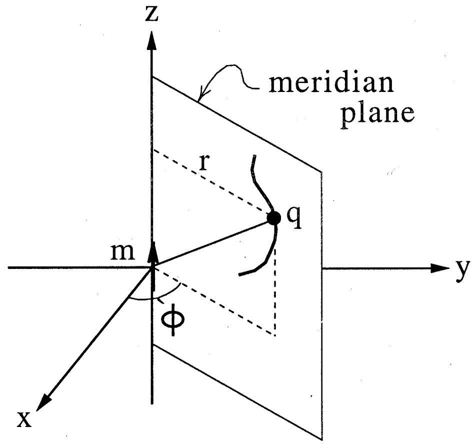

Fig. 1. The meridian plane and definition of coordinates. The magnetic dipole **m** is at the origin.

from the axis) and ϕ (the azimuthal angle). If the motion is *not* confined to the equatorial plane, it is customary to suppress the ϕ dependence and describe the trajectory in terms of r and z (see Fig. 1). We shall call this "motion in the meridian plane", but it is important to remember that the meridian plane itself rotates as the particle moves, so that its azimuthal angle matches the particle's instantaneous value.

One of the important results obtained by Störmer, was that for trajectories confined to the equatorial plane the radius of curvature at any point is proportional to the cube of the distance from the dipole. Recently, Willis and coworkers [Willis *et al.*, 1997] extended Störmer's result by deriving a general equation for the curvature of an arbitrary trajectory in any multipole field. But the complete characterization of particle trajectories in Störmer's problem remains a significant challenge. Many interesting properties of periodic orbits have been identified [Avrett, 1962; Mavraganis, 1975; Markellos *et al.*, 1978; Markellos & Klimopoulos, 1977; Markellos & Halioulias, 1977; Bayrov & Ogorodnikov, 1977; Willis *et al.*, 1997]. Markellos and coworkers [Markellos *et al.*, 1978] found a number of families of periodic orbits in the meridian plane (some symmetric and others asymmetric), and proved the existence of an infinite number of families with simple-periodic oscillations (i.e. orbits that cross the equatorial plane twice). Also, a general method has been developed [Bayrov & Ogorodnikov, 1977] for determining the boundaries of the forbidden zones for motion starting in the equatorial plane. A detailed description of bounded and unbounded motion can be found in [Rossi & Olbert, 1970].

The purpose of this paper is to characterize the bounded nonperiodic trajectories in Störmer's problem. Specifically, we would like to know whether the nonperiodic motion is chaotic, and for what conditions the chaotic behavior is present in both the meridian and equatorial planes. We also show that the familiar picture used for explaning the polar auroras, based on the trajectory of a charged particle in a dipole magnetic field, is incorrect.

2. Motion in the Meridian Plane

Consider a particle of electric charge q and mass M in the presence of a dipole magnetic field $\mathbf{B}$. The force on the electric charge, moving with velocity $\mathbf{v}$, is given by the Lorentz law:

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})\,. \tag{1}$$

The dipole magnetic field can be written as:

$$\mathbf{B} = \frac{\mu_0}{4\pi|\mathbf{x}|^3}[3(\mathbf{m}\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} - m]\,, \tag{2}$$

where $\mathbf{m} = m\hat{\mathbf{z}}$ is the dipole moment (chosen to be at the origin and pointing in the positive z direction), and $\mathbf{R}$ is the (vector) position of the particle.

There are two constants of the motion. Since magnetic forces do no work, the particle's speed never changes, so the kinetic energy is constant. And because of the rotational symmetry about the axis, the z-component of the total angular momentum is constant:

$$L_z = Mr^2\frac{d\phi}{dt} + \frac{Ar^2}{R^3}\,, \tag{3}$$

where $R \equiv \sqrt{z^2+r^2}$, and $A \equiv \mu_0 mq/4\pi$. In Eq. (3), the first term represents the mechanical angular momentum of the particle and the second is the angular momentum of the fields [Rossi & Olbert, 1970; Griffiths, 1992]. (Conservation of L_z is easily proved by examining the $\hat{\phi}$ component of Newton's second law, using (1) and (2).)

It is convenient to define a characteristic length $r_0 \equiv A/L_z$; the kinetic energy of the particle is then:

$$\begin{aligned} E \equiv \frac{1}{2}Mv^2 &= \frac{1}{2}M(\dot{r}^2 + \dot{z}^2 + r^2\dot{\phi}^2) \\ &= \frac{1}{2}M(\dot{r}^2 + \dot{z}^2) + U_{\text{eff}}\,, \end{aligned} \tag{4}$$

where the "effective potential" is:

$$U_{\text{eff}}(r,\,z) \equiv \frac{A^2}{2M}\left(\frac{1}{rr_0} - \frac{r}{R^3}\right)^2\,. \tag{5}$$

The $\dot{\phi}$ dependence has been eliminated, leaving us with the *two*-dimensional problem of finding the motion of a particle in the meridian plane (r, z), subject to the effective potential U_{eff}. The remaining equations of motion are:

$$\begin{aligned}\frac{d^2r}{dt^2} &= -\frac{1}{M}\frac{\partial U_{\text{eff}}}{\partial r} \\ &= \frac{A^2}{M^2}\frac{1}{r}\left(\frac{1}{rr_0} - \frac{r}{R^3}\right)\left(\frac{1}{rr_0} + \frac{r}{R^3} - \frac{3r^3}{R^5}\right), \quad (6)\end{aligned}$$

$$\frac{d^2z}{dt^2} = -\frac{1}{M}\frac{\partial U_{\text{eff}}}{\partial z} = -\frac{3A^2}{M^2}\frac{rz}{R^5}\left(\frac{1}{rr_0} - \frac{r}{R^3}\right). \quad (7)$$

The motion is limited by boundaries where the speed in the meridian plane vanishes ($\dot{r} = \dot{z} = 0$), so that $U_{\text{eff}} = (1/2)Mv^2$, or

$$\frac{1}{rr_0} - \frac{r}{R^3} = \pm\frac{Mv}{A}. \quad (8)$$

At the boundaries the motion is purely azimuthal.

For numerical studies it is convenient to measure length in units of r_0 and time in units of $t_0 \equiv Mr_0^3/A$ — or, what amounts to the same thing, to set r_0 and t_0 equal to 1. (Both r_0 and t_0 can in principle be negative, but we shall restrict our attention to the positive regime.) In this notation, the equations of motion assume the dimensionless form

$$\frac{d^2r}{dt^2} = \frac{1}{r}\left(\frac{1}{r} - \frac{r}{R^3}\right)\left(\frac{1}{r} + \frac{r}{R^3} - 3\frac{r^3}{R^5}\right), \quad (9)$$

$$\frac{d^2z}{dt^2} = -3\frac{rz}{R^5}\left(\frac{1}{r} - \frac{r}{R^3}\right), \quad (10)$$

and the boundary is defined by

$$\frac{1}{r} - \frac{r}{R^3} = \pm v. \quad (11)$$

The trajectories were found by numerically integrating Eqs. (9) and (10), using a fourth-order Runge–Kutta routine, with step lengths δt ranging from 10^{-4} to 10^{-2}. For initial conditions $(r_i, \dot{r}_i, z_i, \dot{z}_i)$ the speed is given by

$$v^2 = \dot{r}_i^2 + \dot{z}_i^2 + \left(\frac{1}{r_i} - \frac{r_i}{R_i^3}\right)^2. \quad (12)$$

In the study of the phenomenon of polar auroras, it is often asserted that electrons moving towards the Earth are captured by the Earth's magnetic field and tend to become attached to the field lines, spiraling about them as they move towards the poles. These electrons eventually produce the aurora, by collisions with the molecules of air in the neigborhood of the poles. However, our numerical calculations revealed a quite different picture. First, the trajectories of the electrons are strongly dependent on their energies. Second, the trajectories of the electrons never track a single field line. Finally, we argue that only electrons with relatively high energies are able to produce the auroras.

Let us first consider the situation for low energies. In Fig. 2 we depict the trajectory of a particle under the initial conditions $(1, v\cos(\pi/6), 0, v\sin(\pi/6))$, with $v = 0.045$. Here the initial velocity makes a 30° angle with the equatorial plane. The particle does not track a single line of the magnetic field; instead, it spirals towards one of the poles while drifting laterally. (Other mechanisms leading to lateral drift have been proposed, including gravitational perturbations [Chandrasekhar, 1960].) The drift velocity of the particle is not constant, since it depends on its radial position as prescribed by Eq. (3). In the regime of low energies, the particle never reaches the poles; it bounces back and forth in the neighborhood of the equator. Moreover, most of the particles coming from the Sun arrive at Earth in a direction nearly parallel to the equator, with a small velocity perpendicular to the equator. Also, from the boundary equations [Eq. (11)], the region for the allowed bound state trajectories is very small, for low energy particles, leaving them with small probability of being captured by the magnetic field. Finally, the energies of the captured particles may not be high enough to produce molecular excitation of the air molecules in the visible region of the spectrum. Therefore, one might conclude that the low energy particles are not responsible for the appearance of the aurora.

In the case of higher energies the picture is entirely different. Consider the situation shown in Fig. 3, where the initial conditions are $(1, v\cos(\pi/6), 0, v\sin(\pi/6))$, the same as Fig. 2, except that now the velocity is five times larger (that is, $v = 0.225$). First, the trajectory of the particle no longer spirals when it moves towards the poles. In fact, the motion is rather irregular, with the particle moving about the dipole in an apparently random manner. This irregular motion is found to be insensitive to the direction of the initial velocity. It turns out that as the energy of the particle increases we observe a transition from a quasiperiodic to a

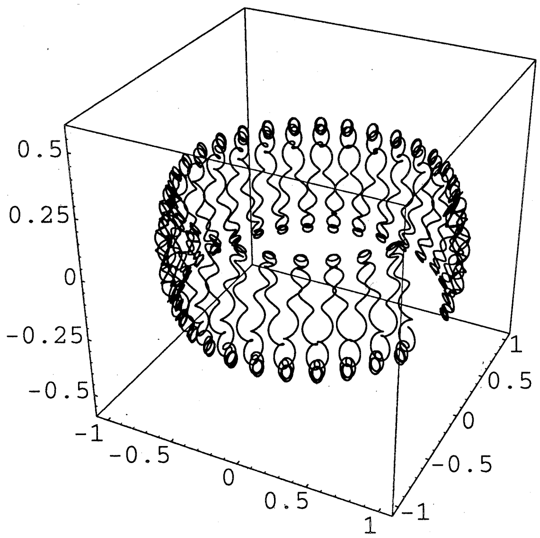

Fig. 2. Actual trajectory of a charged particle moving in the field of a magnetic dipole. The initial conditions are $(r,\, \dot{r},\, z,\, \dot{z}) = (1,\, v\cos(\pi/6),\, 0,\, v\sin(\pi/6))$, where $v = 0.045$. The particle remains in a region around the equator and never reaches the poles.

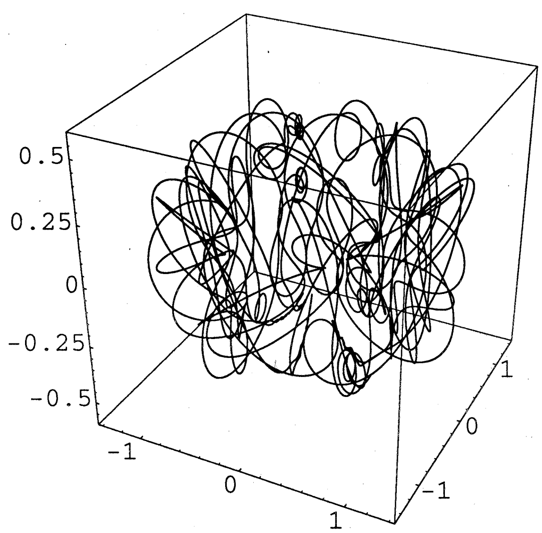

Fig. 3. Actual trajectory of a charged particle trapped in the field of a magnetic dipole. The initial conditions are $(r,\, \dot{r},\, z,\, \dot{z}) = (1,\, v\cos(\pi/6),\, 0,\, v\sin(\pi/6))$, where $v = 0.225$. The particle visits the poles in a chaotic manner.

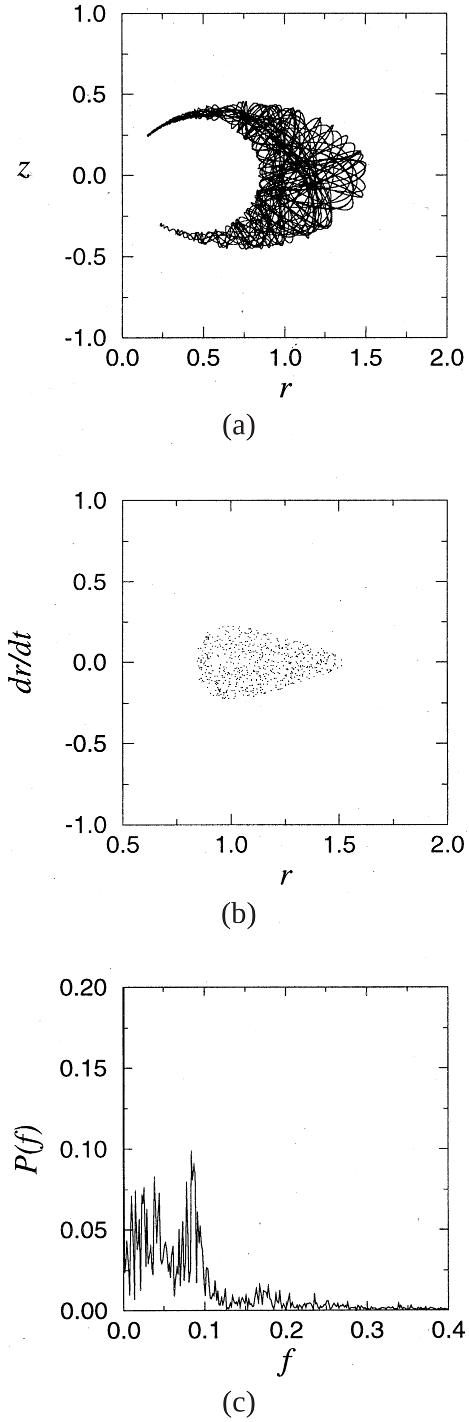

Fig. 4. Charged particle trapped in the field of a magnetic dipole. Typical motion in the meridian plane. The initial conditions are $(r,\, \dot{r},\, z,\, \dot{z}) = (1,\, 0,\, 0,\, 0.225)$. (a) Trajectory in the $(r,\, z)$-plane. (b) Poincaré section in the $(r,\, \dot{r})$ plane for $z = 0$ and $\dot{z} > 0$. (c) Power spectrum (arbitrary units) obtained from the time series of $z(t)$.

chaotic regime, and as the energy increases further we encounter a transition from a chaotic to a hyperchaotic regime. Particles with high energies are responsible for the appearance of the polar auroras, since they are easily able to reach the poles, have greater probability of capture by the Earth's magnetic field, and have enough energy to produce molecular excitation, through collisions, in the visible region of the spectrum.

We now show how the motion is characterized for the case of very high energies. In Fig. 4(a) we used initial conditions (1, 0, 0, 0.225) (in this case $v = \dot{z}_i = 0.225$). The motion is apparently ergodic; that is, the trajectory of the particle eventually fills in the bounded region. The particle tends to get "trapped" in the tips of the crescent, before it reverses its direction and returns to the opposite pole (these regions are responsible for the polar aurora). The motion can be characterized by plotting (for example) the intersection of the phase space trajectory with the plane $z = 0$, at points where the z-component of the velocity is positive. Such a "Poincaré section" is shown in Fig. 4(b). The scattered distribution of the points suggests that the motion is chaotic. This is confirmed by the broad-band power spectrum shown in Fig. 4(c). The Fourier analysis was done using the time series for $r(t)$, sampled at equal time intervals $\Delta t = 0.01$. In fact, using the method described in [Rangarajan *et al.*, 1998], we find the Lyapunov exponents to be $\lambda = (0.33, 0.12, -0.10, -0.35)$. The existence of *two* positive Lyapunov exponents indicates that the motion is hyperchaotic.

The sum of the Lyapunov exponents measures the fractional rate at which a volume V in phase space expands or contracts under the action of the flow [Bergé *et al.*, 1986]:

$$\frac{1}{V}\frac{dV}{dt} = \sum_i \lambda_i\,.$$

In conservative systems, such as this one, the volume remains constant (the divergence of the flow is zero), so the sum of the Lyapunov exponents must vanish, as indeed our numerical results confirm.

3. Motion in the Equatorial Plane

For motion in the equatorial plane it is simpler to use Cartesian coordinates; the equations of motion are:

$$\frac{d^2x}{dt^2} = -\frac{A}{M}\frac{\dot{y}}{r^3}, \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{A}{M}\frac{\dot{x}}{r^3}, \tag{13}$$

or, in dimensionless form,

$$\frac{d^2x}{dt^2} = -\frac{1}{r^3}\frac{dy}{dt}, \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{1}{r^3}\frac{dx}{dt}. \tag{14}$$

where $r \equiv \sqrt{x^2+y^2}$. In this case conservation of energy (4) says

$$\frac{1}{2}Mv^2 = \frac{1}{2}M\dot{r}^2 + U_{\text{eff}}\,, \tag{15}$$

with the effective potential

$$U_{\text{eff}}(r) = \frac{A^2}{2Mr^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)^2. \tag{16}$$

The motion is bounded at $\dot{r} = 0$, which is to say (with $r_0 = t_0 = 1$)

$$\frac{1}{r}\left(1 - \frac{1}{r}\right) = \pm v\,. \tag{17}$$

Evidently the range of r is

$$\frac{1}{2v}[\sqrt{1+4v} - 1] \le r \le \frac{1}{2v}[1 - \sqrt{1-4v}] \tag{18}$$

(there is a second allowed range, $r \ge (1/2v)[1 + \sqrt{1-4v}]$, but this corresponds to unbounded trajectories).

For initial conditions $(x_i,\, \dot{x}_i,\, y_i,\, \dot{y}_i)$, the speed is (of course) $v = \sqrt{\dot{x}_i^2 + \dot{y}_i^2}$. Figure 5(a) shows the trajectory for initial conditions (0, 0, 1, 0.225); in this case $v = 0.225$ and the motion is confined to the range $0.841 \le r \le 1.519$. The trajectory eventually fills in this region, but unlike before there is an obvious structure to the orbit; the Poincaré section [Fig. 5(b)] yields points that lie on a distinct curve, and the power spectrum [Fig. 5(c)] obtained from the time series of $x(t)$ reveals a sequence of well-defined peaks, indicating that the motion is quasiperiodic. Indeed, the motion defined by Eqs. (13) and (14) can never be chaotic. Although the phase space is four-dimensional, the existence of two constants of the motion (kinetic energy and the z-component of angular momentum) effectively reduces the dimensionality to two, hence the system cannot be chaotic. A dynamical system must be described by at least three autonomous first-order differential equations in order to support bounded chaotic orbits [Hilborn, 1994; Drazin, 1993]. However, the introduction of a time-dependent perturbation in the magnetic dipole moment *can* induce chaotic behavior. Specifically, let $m \to m(1 + \varepsilon\cos(\omega t))$. For low energies, even in the presence of an oscillating magnetic dipole, the motion of the particle remains quasiperiodic. However, as the energy of the particle is increased, we observe a transition from quasiperiodic to chaotic behavior. The inclusion of a time-dependent magnetic field increases the dimensionality of the flow by one, thus fulfilling the requirement for a system to be chaotic. This is a necessary condition, but by no means implies that the system will behave chaotically in the presence of a time-dependent magnetic dipole field. In fact the system remains quasiperiodic for low energies whether a fluctuating magnetic field is present or not. In Fig. 6(a) we show the trajectory when $\varepsilon = 0.2$ and $f = \omega/2\pi = 0.01$

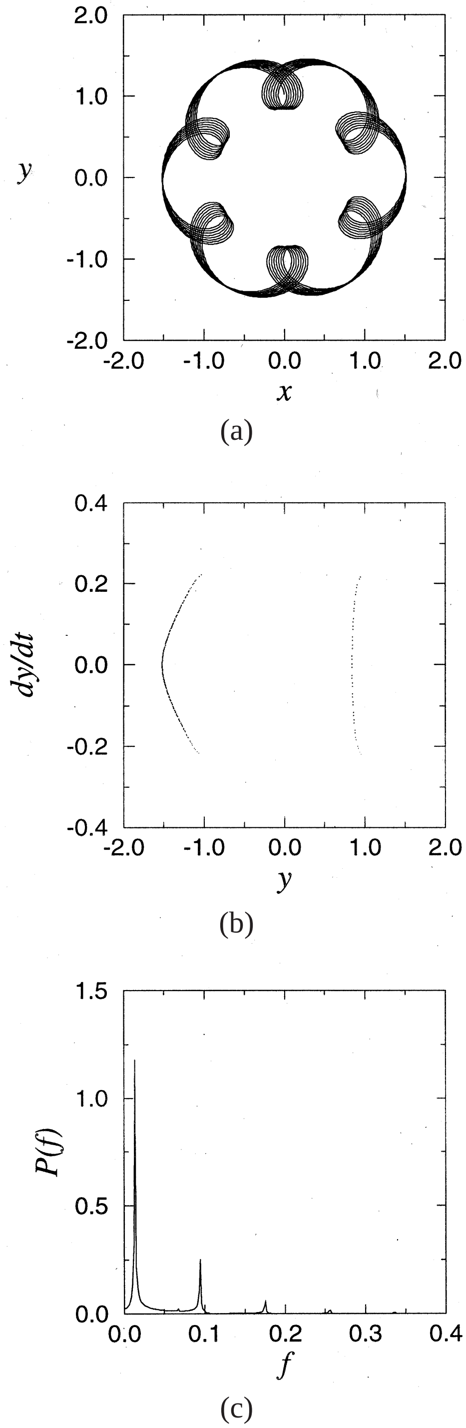

Fig. 5. Charged particle moving in the field of a magnetic dipole. Typical motion in the equatorial plane. The initial conditions are $(x, \dot{x}, y, \dot{y}) = (0, 0, 1, 0.225)$. (a) Trajectory in the (x, y)-plane. (b) Poincaré section in the $(y, \dot{y})$-plane for $x = 0$ and $\dot{x} > 0$. (c) Power spectrum obtained from the time series of $x(t)$.

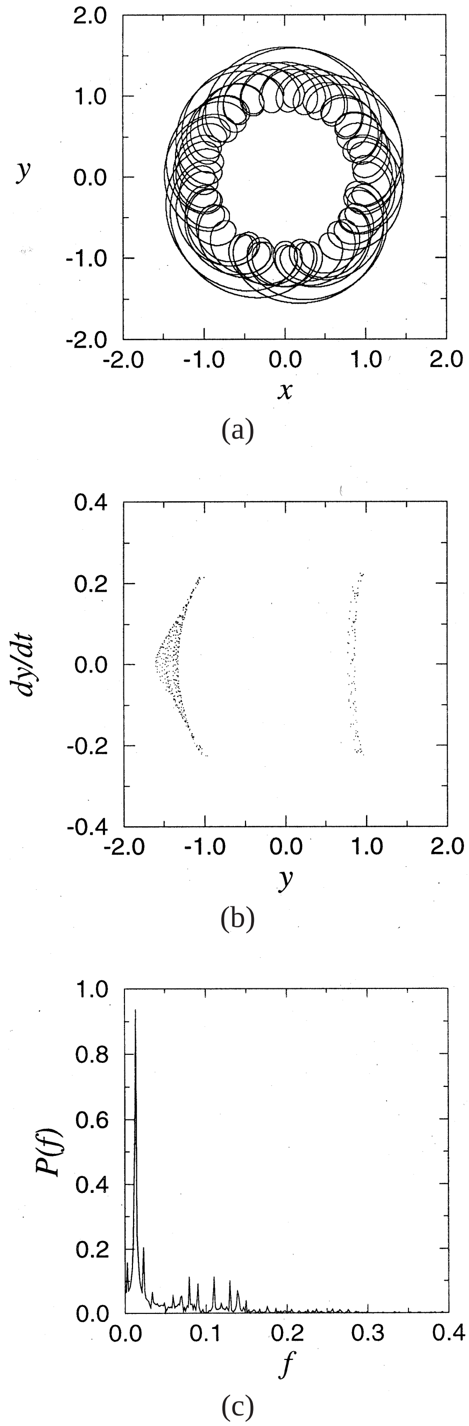

Fig. 6. Typical motion in the equatorial plane with an oscillatory perturbation in the dipole moment. Same initial conditions as Fig. 5; $\varepsilon = 0.2$ and $f = 0.01$. (a) Trajectory of the particle. (b) Poincaré section in the $(y, \dot{y})$-plane for $x = 0$ and $\dot{x} > 0$. (c) Power spectrum obtained from the time series of $x(t)$.

(this frequency is about one tenth the smallest natural frequency of the unperturbed system, so the oscillations are relatively slow; the initial conditions are the same as in Fig. 5). The Poincaré section [Fig. 6(b)] is scattered, and the power spectrum displays strong background noise — both characteristics of chaos. The largest Lyapunov exponent is now positive: $\lambda = (0.0021, 0, -0.0021)$. Notice that even in the presence of a time-dependent field the system remains conservative, and hence the sum of the Lyapunov exponents must be zero (as our numerical results confirm). Incidentally, motion in the meridian plane remains qualitatively similar to that in Fig. 4, even in the presence of a sinusoidal perturbation on the dipole moment. However, the boundary of the allowed zone now fluctuates in time.

4. Conclusion

We have studied the motion of a charged particle in the field of a magnetic dipole. In the familiar naive picture, the motion of a bounded particle becomes attached to a magnetic line, and spirals around the line while sliding along it towards one of the poles, and then bouces back along the same line as it moves towards the opposite pole. The actual picture is quite different. For low energies the motion in the meridian plane is quasiperiodic, with the particles moving from pole to pole in a spiraling fashion while at the same time moving laterally. As the energy increases we observe a transition from quasiperiodic to chaotic behavior, in which the motion changes from spiraling with lateral displacement to irregular motion. With further increases in the energy the motion changes from chaotic (where the largest Lyapunov exponent is positive) to hyperchaotic behavior (where the two largest Lyapunov exponents are positive). If the motion is confined to the equatorial plane, the trajectories are quasiperiodic. However, a time-dependent perturbation of the dipole moment can induce chaotic motion in the equatorial plane. We observe that even in the presence of an oscillating dipole field, in the low energy regime, the motion of the particle remains quasiperiodic, and it becomes chaotic only for high energies. Qualitatively similar conclusions presumably hold for the motion of a charge in the field of a higher magnetic multipole, as long as the field is symmetric about the z axis.

In the relativistic version of Störmer's problem, if radiation losses are neglected, the only modification is the introduction of a factor $\gamma = (1 - v^2/c^2)^{-1/2}$ in the equations of motion. Since v is a constant this is equivalent to a change in the value of M, and in the units we have used this alters nothing. Therefore, the motion is essentially the same as the nonrelativistic case.

Acknowledgments

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