Ideal MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \qquad \rho \frac{d\vec{v}}{dt} = -\nabla p + \frac{1}{\mu_0}\left(\nabla \times \vec{B}\right) \times \vec{B},$$

$$\frac{d\boldsymbol{e}}{dt} + (\gamma_{\mathrm{g}} - 1)\boldsymbol{e}\nabla \cdot \vec{v} = 0, \qquad \frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B}\right).$$

$$\frac{d}{dt}\left(p\rho^{-\gamma_{\mathrm{g}}}\right) = \frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p + \gamma_{\mathrm{g}} p \nabla \cdot \vec{v} = \frac{dp}{dt} + \gamma_{\mathrm{g}} p \nabla \cdot \vec{v} = 0$$

$$\frac{d\boldsymbol{e}}{dt} + (\gamma_{\mathrm{g}} - 1)\boldsymbol{e}\nabla \cdot \vec{v} = 0$$

- **Maxwell's equations** describe evolution of electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$ in response to current density $\mathbf{j}(\mathbf{r},t)$ and space charge $\tau(\mathbf{r},t)$:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \textit{(Faraday)} \tag{1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \quad c \equiv (\epsilon_0 \mu_0)^{-1/2}, \qquad \textit{('Ampère')} \tag{2}$$

$$\nabla \cdot \mathbf{E} = \frac{\tau}{\epsilon_0}, \qquad \textit{(Poisson)} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0. \qquad \textit{(no monopoles)} \tag{4}$$

- **Gas dynamics equations** describe evolution of density $\rho(\mathbf{r},t)$ and pressure $p(\mathbf{r},t)$:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{v} \equiv \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \textit{(mass conservation)} \tag{5}$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} + \gamma p \nabla \cdot \mathbf{v} \equiv \frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0, \qquad \textit{(entropy conservation)} \tag{6}$$

  where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

  is the *Lagrangian time-derivative* (moving with the fluid).

- **Coupling between system described by $\{\mathrm{E}, \mathrm{B}\}$ and system described by $\{\rho, p\}$** comes about through equations involving the velocity $\mathbf{v}(\mathbf{r}, t)$ of the fluid:
  'Newton's' *equation of motion for a fluid element* describes the acceleration of a fluid element by pressure gradient, gravity, and electromagnetic contributions,

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{F} \equiv -\nabla p + \rho \mathbf{g} + \mathbf{j} \times \mathbf{B} + \tau \mathbf{E}\,; \qquad \textit{(momentum conservation)} \qquad (7)$$

  'Ohm's' law (for a perfectly conducting moving fluid) expresses that *the electric field $\mathbf{E}'$ in a co-moving frame vanishes*,

$$\mathbf{E}' \equiv \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0\,. \qquad \textit{('Ohm')} \qquad (8)$$

- Equations (1)–(8) are complete, but inconsistent for *non-relativistic velocities*:

$$v \ll c\,. \qquad (9)$$

  $\Rightarrow$ We need to consider **pre-Maxwell equations**.

1. *Maxwell's displacement current negligible* [$\mathcal{O}(v^2/c^2)$] for non-relativistic velocities:

$$\frac{1}{c^2}\left|\frac{\partial \mathbf{E}}{\partial t}\right| \sim \frac{v^2}{c^2}\frac{B}{l_0} \quad \ll \quad \mu_0|\mathbf{j}| \approx |\nabla \times \mathbf{B}| \sim \frac{B}{l_0} \qquad \text{[using Eq. (8)]},$$

indicating length scales by l_0 and time scales by t_0, so that $v \sim l_0/t_0$.

$\Rightarrow$ Recover original Ampère's law:

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B}\,. \tag{10}$$

2. *Electrostatic acceleration is also negligible* [$\mathcal{O}(v^2/c^2)$]:

$$\tau|\mathbf{E}| \sim \frac{v^2}{c^2}\frac{B^2}{\mu_0 l_0} \quad \ll \quad |\mathbf{j} \times \mathbf{B}| \sim \frac{B^2}{\mu_0 l_0} \qquad \text{[using Eqs. (3), (8), (10)]}\,.$$

$\Rightarrow$ Space charge effects may be ignored and Poisson's law (3) can be dropped.

3. *Electric field then becomes a secondary quantity,* determined from Eq. (8):

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}\,. \tag{11}$$

$\Rightarrow$ For non-relativistic MHD, $|\mathbf{E}| \sim |\mathbf{v}||\mathbf{B}|$, i.e. $\mathcal{O}(v/c)$ smaller than for EM waves.

- Exploiting these approximations, and eliminating $\mathbf{E}$ and $\mathbf{j}$ through Eqs. (10) and (11), the basic equations of ideal MHD are recovered *in their most compact form*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{12}$$

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla p - \rho \mathbf{g} - \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = 0\,, \tag{13}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \tag{14}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \tag{15}$$

$\Rightarrow$ *Set of eight nonlinear partial differential equations (PDEs) for the eight variables $\rho(\mathbf{r},t)$, $\mathbf{v}(\mathbf{r},t)$, $p(\mathbf{r},t)$, and $\mathbf{B}(\mathbf{r},t)$*.

- The magnetic field equation (15)(b) is to be considered as a initial condition: once satisfied, it remains satisfied for all later times by virtue of Eq. (15)(a).

(a) Young stellar object (YSO)

$(R_d \sim 1\,\mathrm{AU}\,,\, H_d \sim 0.01\,\mathrm{AU}\,,$
$M_* \sim 1 M_\odot\,,\, n = 10^{18}\,\mathrm{m}^{-3}\,)$:

$$\begin{aligned} |\mathbf{F}_B| &= 5.3 \times 10^{-12}\,, \\ |\mathbf{F}_g^{\mathrm{ex}}| &= 1.0 \times 10^{-9}\,, \\ |\mathbf{F}_g^{\mathrm{in}}| &= 2.9 \times 10^{-19}\,. \end{aligned} \qquad (22)$$

(b) Active galactic nucleus (AGN)

$(R_d \sim 50\,\mathrm{kpc}\,,\, H_d \sim 120\,\mathrm{pc}\,,$
$M_* \sim 10^8 M_\odot\,,\, n = 10^{12}\,\mathrm{m}^{-3}\,)$:

$$\begin{aligned} |\mathbf{F}_B| &= 2.2 \times 10^{-21}\,, \\ |\mathbf{F}_g^{\mathrm{ex}}| &= 1.0 \times 10^{-27}\,, \\ |\mathbf{F}_g^{\mathrm{in}}| &= 6.4 \times 10^{-22}\,. \end{aligned} \qquad (23)$$

$$\vec{j} \times \vec{B} = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B}$$

$$= \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B} - \nabla \left(\frac{B^2}{2\mu_0} \right)$$

$$\vec{j} \times \vec{B} = \frac{1}{\mu_0} \left(B \vec{b} \cdot (\nabla_\parallel + \nabla_\perp) \right) \left(B \vec{b} \right) - \nabla_\parallel \left(\frac{B^2}{2\mu_0} \right) - \nabla_\perp \left(\frac{B^2}{2\mu_0} \right) =$$

$$= \frac{1}{\mu_0} \frac{B^2}{R_\mathrm{C}} \vec{\mathrm{e}}_\mathrm{n} - \nabla_\perp \left(\frac{B^2}{2\mu_0} \right),$$

$$\rho \frac{d\vec{v}}{dt} = -\nabla p - \nabla_\perp \left(\frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} \frac{B^2}{R_\mathrm{C}} \vec{\mathrm{e}}_\mathrm{n} + \rho \vec{g}$$

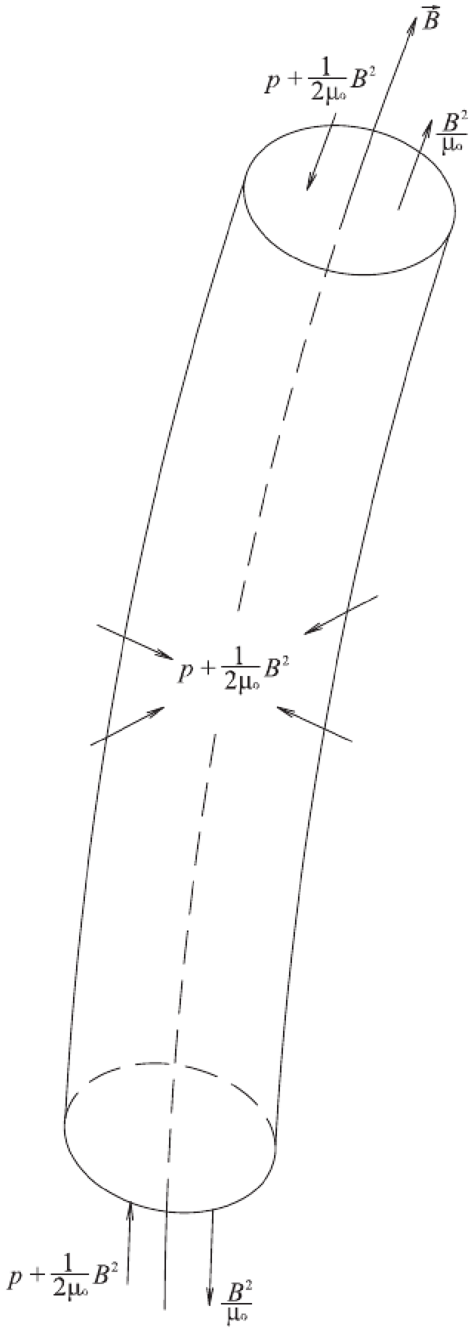
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\pi} = 0, \qquad \frac{\partial \vec{\pi}}{\partial t} + \nabla \cdot \hat{\mathrm{T}} = \vec{0},$$

$$\frac{\partial H}{\partial t} + \nabla \cdot \vec{U} = 0, \quad \frac{\partial \vec{B}}{\partial t} + \nabla \cdot \hat{\mathrm{Y}} = \vec{0},$$

$$\vec{\pi} = \rho\vec{v},\ \hat{\mathrm{T}} = \rho\vec{v}\vec{v} + \left(p + \tfrac{1}{2\mu_0}B^2\right)\hat{\mathrm{I}} - \tfrac{1}{\mu_0}\vec{B}\vec{B},\ H = \tfrac{1}{2}\rho v^2 + \tfrac{1}{\gamma_{\mathrm{g}}-1}p + \tfrac{1}{2\mu_0}B^2$$

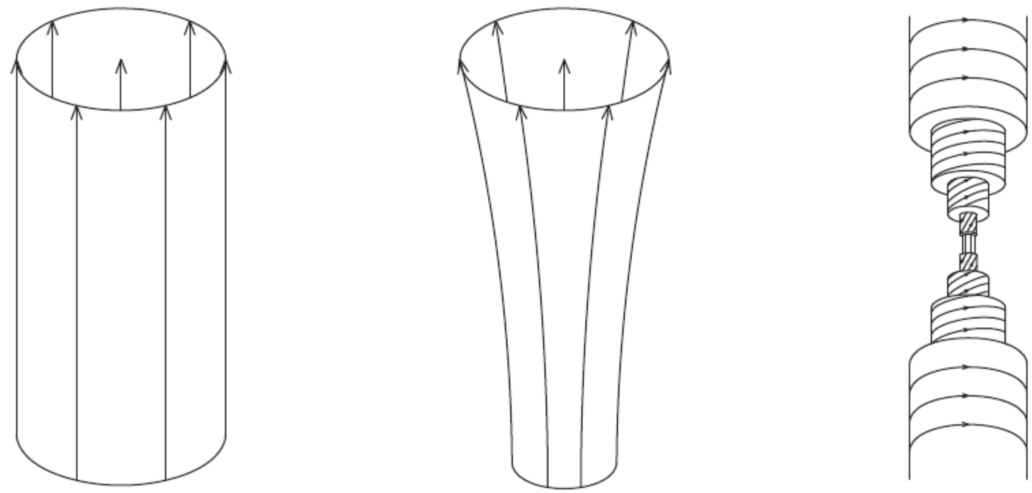
$$\vec{U} = \left(\tfrac{1}{2}\rho v^2 + \tfrac{\gamma_{\mathrm{g}}}{\gamma_{\mathrm{g}}-1}p\right)\vec{v} + \tfrac{1}{\mu_0}B^2\vec{v} - \vec{v}\cdot\tfrac{1}{\mu_0}\vec{B}\vec{B}$$

$$\hat{\mathrm{Y}} = \vec{v}\vec{B} - \vec{B}\vec{v}$$



$$\hat{\mathrm{T}} = \begin{pmatrix} p+\frac{B^2}{2\mu_0} & 0 & 0 \\ 0 & p+\frac{B^2}{2\mu_0} & 0 \\ 0 & 0 & p-\frac{B^2}{2\mu_0} \end{pmatrix} \begin{matrix} \perp \\ \perp \\ \parallel \end{matrix}$$

Magnetic flux tubes



$$\vec{B} \cdot \vec{e}_n = 0$$

Magnetic fields confining plasmas are essentially *tubular structures*: The magnetic field equation

$$\nabla \cdot \mathbf{B} = 0 \tag{28}$$

is not compatible with spherical symmetry. Instead, magnetic flux tubes become the essential constituents.

$$\frac{d\mathcal{F}_{\vec{F}}}{dt} = \frac{d}{dt}\int_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \int_{\mathcal{S}} \left(\frac{\partial \vec{F}}{\partial t} - \nabla \times (\vec{v} \times \vec{F}) + \vec{v} \nabla \cdot \vec{F} \right) \cdot d\vec{S}$$

$$\mathcal{F}_{\vec{B}} = \int_{\mathcal{S}} \vec{B} \cdot d\vec{S} = \text{const}$$

$$B_1 S_1 = B_2 S_2,\ S = 4\pi R^2$$

$$R_1 = 10^6 \text{ km} \qquad R_2 = 10 \text{ km} \qquad B_1 = 100 \text{ G} = 0.01 \text{ T}$$

$$B_2 = B_1 (R_1/R_2)^2 = 10^{12} \text{ G} = 10^8 \text{ T}$$

$$B_2 = B_1 (\rho_2/\rho_1)^{2/3}$$

Line stretching

$$B_1 dS_1 = B_2 dS_2$$

$$\rho_1 dS_1 l_1 = \rho_2 dS_2 l_2$$

$$B_2 = B_1(\rho_2/\rho_1)(l_2/l_1)$$

$$\vec{E} = -\vec{u} \times \vec{B} + \\ + \frac{\vec{j}}{\sigma} + \frac{\vec{j} \times \vec{B}}{|q_e| n_e} - \frac{\nabla p_e}{|q_e| n_e} + \frac{m_e}{q_e^2 n_e} \left(\frac{\partial \vec{j}}{\partial t} + \nabla \cdot \left(\vec{u}\vec{j} + \vec{j}\vec{u} - \frac{\vec{j}\vec{j}}{|q_e| n_e} \right) \right)$$

$$\vec{E} = -\vec{u} \times \vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = -\frac{1}{\mu_0 \sigma} \left(\nabla \times \left(\nabla \times \vec{B} \right) \right) + \nabla \times \left(\vec{v} \times \vec{B} \right)$$

$$\vec{E} = \vec{j}/\sigma - \vec{u} \times \vec{B}$$

$$\frac{1}{\mu_0 \sigma} \Delta \vec{B} + \nabla \times \left(\vec{v} \times \vec{B} \right)$$

$$R_m \equiv \mu_0 \boldsymbol{L} \sigma_0 u_0$$

$$\frac{\partial \bar{\vec{B}}}{\partial \bar{t}} = \frac{1}{R_m} \frac{\bar{\Delta} \bar{\vec{B}}}{\bar{\sigma}} + \bar{\nabla} \times (\bar{\vec{v}} \times \bar{\vec{B}})$$

$$\vec{\bar{E}} = -\vec{\bar{u}} \times \vec{\bar{B}} + A_1 \frac{\vec{\bar{j}}}{\bar{\sigma}} + A_2 \frac{\vec{\bar{j}} \times \vec{\bar{B}}}{\bar{n}_{\mathrm{e}}} - A_3 \frac{\bar{\nabla} \bar{p}}{\bar{n}_{\mathrm{e}}} + \frac{A_4}{\bar{n}_{\mathrm{e}}} \left(\frac{\partial \vec{\bar{j}}}{\partial \bar{t}} + \bar{\nabla} \cdot (\vec{\bar{u}}\vec{\bar{j}} - \vec{\bar{j}}\vec{\bar{u}}) \right) - \frac{A_5}{\bar{n}_{\mathrm{e}}} \bar{\nabla} \cdot \left(\frac{\vec{\bar{j}}\vec{\bar{j}}}{\bar{n}_{\mathrm{e}}} \right),$$

$$A_1 = 1/R_{\mathrm{m}}, \; R_{\mathrm{m}} \equiv \mu_0 \boldsymbol{L} \sigma_0 u_0,$$

$$A_2 = (2/\beta_0)(r_{\mathrm{cp0}}/\boldsymbol{L})$$

$$A_3 = \omega/\omega_{\mathrm{cp_0}} = r_{\mathrm{cp0}}/\boldsymbol{L},$$

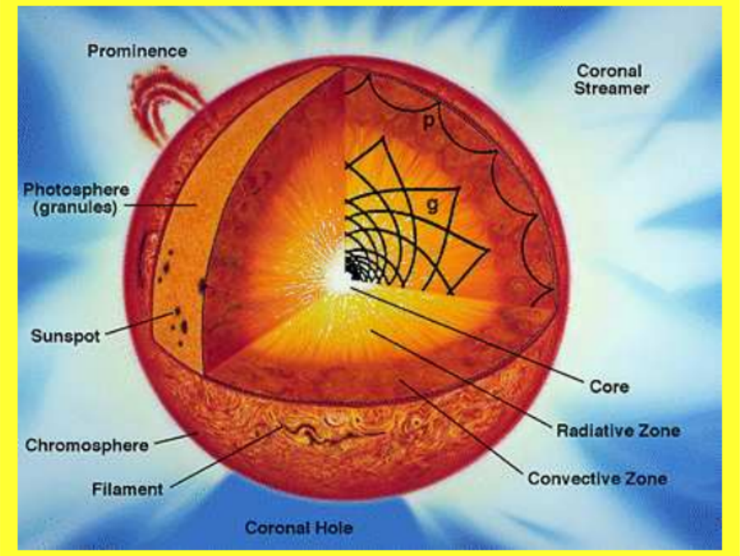
$$A_4 = m_{\mathrm{e}} j_0 (q_{\mathrm{e}}^2 n_0 E_0 t_0) = (c/\omega_{\mathrm{pe0}})^2/\boldsymbol{L}^2,$$

$$A_5 = (2/\beta_0)(r_{\mathrm{cp0}}/\boldsymbol{L})(c/\omega_{\mathrm{pe0}})^2/\boldsymbol{L}^2.$$

Recall structure of the Sun:

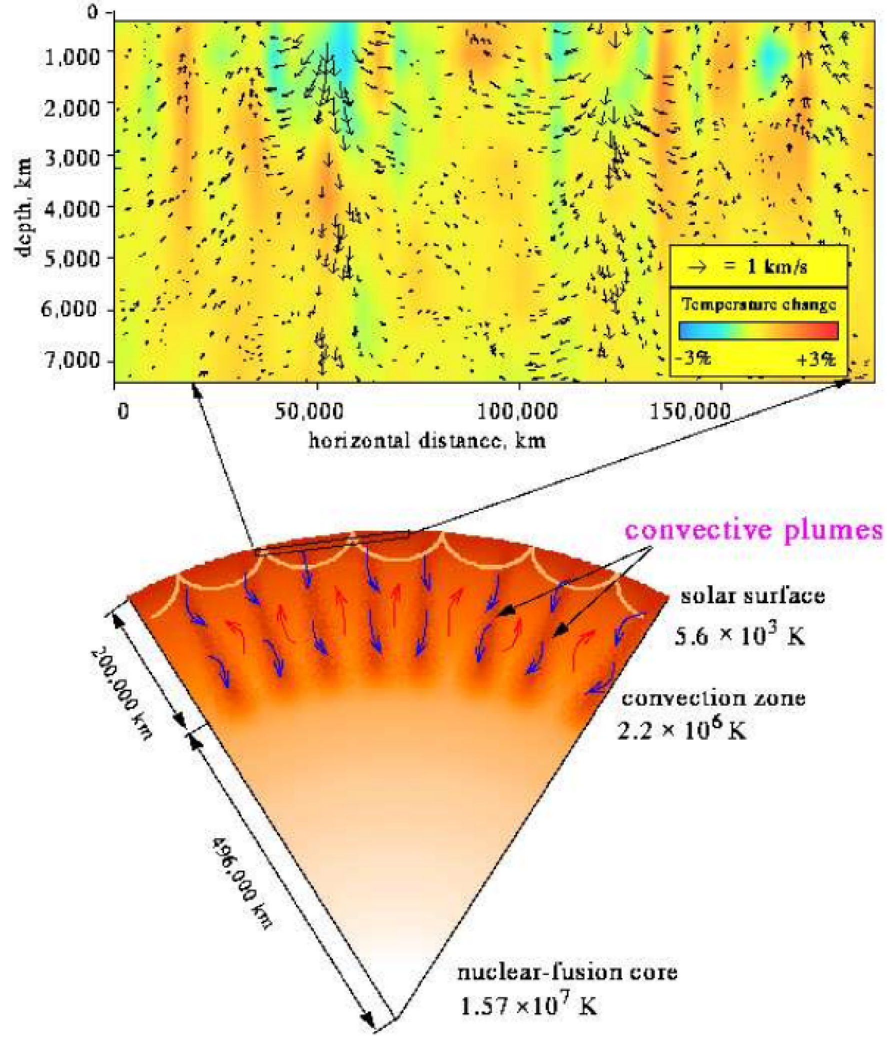
- core, $r \leq 0.25 R_\odot$:
  thermonuclear conversion of hydrogen into helium;
- radiative zone, $0.25 R_\odot \leq r \leq 0.71 R_\odot$:
  outward radiative transport of produced energy;
- **convection zone,** $0.71 R_\odot \leq r \leq R_\odot$:
  temperature gradient so steep that the plasma is convectively unstable

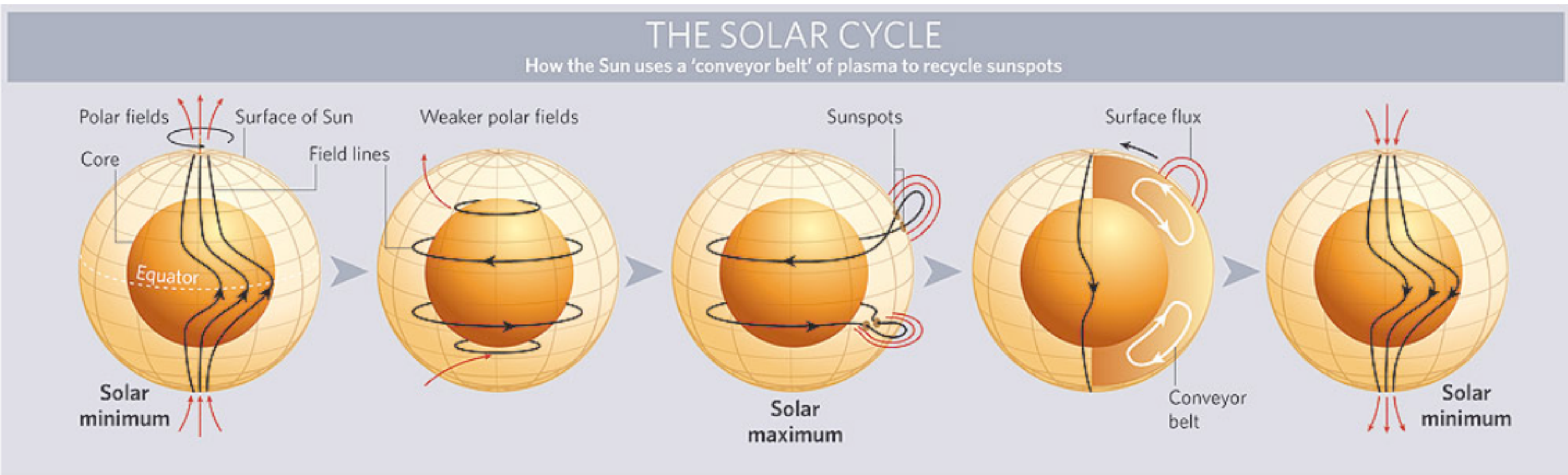
  $\Rightarrow$ **seat of the solar dynamo!**

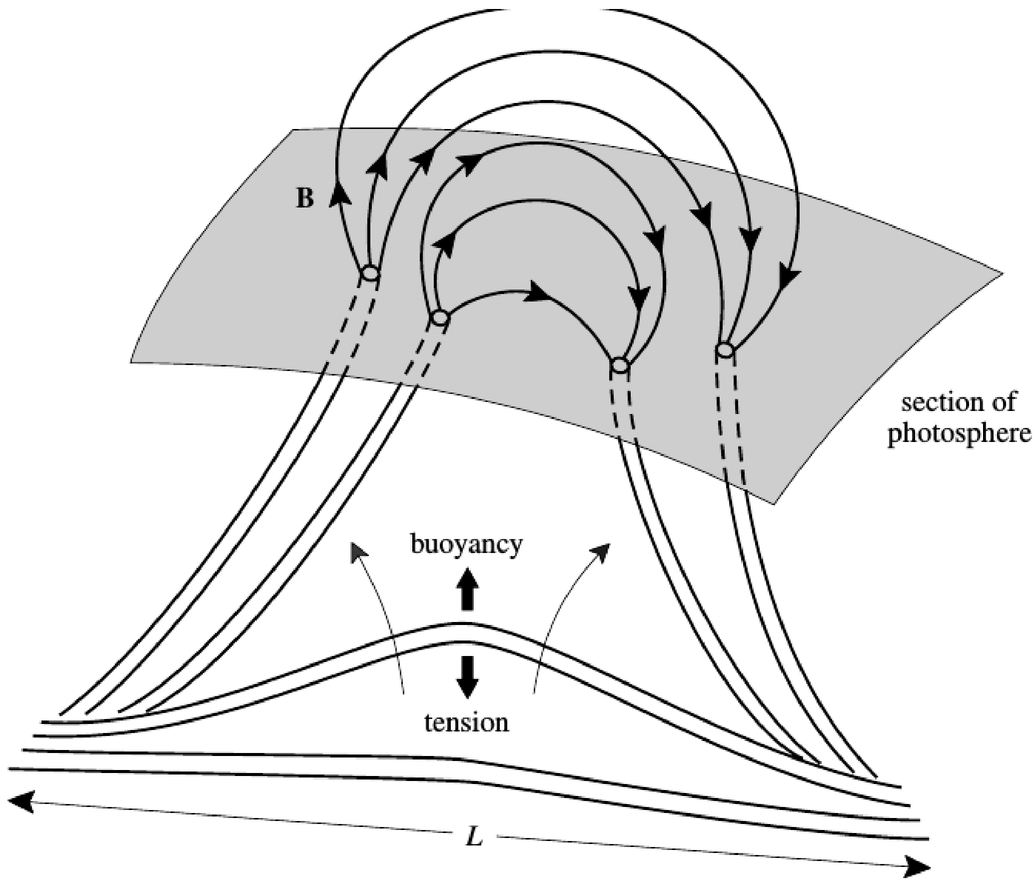


(from SOHO web site)

Convective Flows Below The Sun's Surface




THE SOLAR CYCLE
How the Sun uses a 'conveyor belt' of plasma to recycle sunspots
Polar fields
Surface of Sun
Core
Field lines
Equator
Solar minimum
Weaker polar fields
Sunspots
Solar maximum
Surface flux
Conveyor belt
Solar minimum



Buoyancy of flux tubes from the convective zone to the surface of the Sun.

- Dark spots in the (visible) photosphere that are cooler (darker) than surroundings.
- Can last days to months and rotate West–East across the disk in bands up to $\pm 35^\circ$ about the equator.
- Reveal existence of *several 1000 Gauss magnetic field!*

(from SOHO web site)