# Kinetic theory and fluid models

$$\mathcal{K}_\alpha(\vec{r},\vec{v},t) = \sum_{i=1}^{N_\alpha} \delta(\vec{r}-\vec{r}_{\alpha,i})\delta(\vec{v}-\vec{v}_{\alpha,i})$$

$$\vec{r}_{\alpha,i} = \vec{r}_{\alpha,i}(t)$$

$$\vec{v}_{\alpha,i} = \vec{v}_{\alpha,i}(t)$$

$$\frac{\partial \mathcal{K}_\alpha}{\partial t} = -\sum_{\imath=1}^{N_\alpha}\Big(\vec{v}_{\alpha,i}\cdot\nabla\delta(\vec{r}-\vec{r}_{\alpha,i})\delta(\vec{v}-\vec{v}_{\alpha,i}) + \dot{\vec{v}}_{\alpha,i}\cdot\nabla_v\delta(\vec{r}-\vec{r}_{\alpha,i})\delta(\vec{v}-\vec{v}_{\alpha,i})\Big)$$

$$m_\alpha \dot{\vec{v}}_{\alpha,i} = q_\alpha \left( \vec{E}^{\mathrm{m}}(\vec{r}_{\alpha,i}, t) + \vec{v}_{\alpha,i} \times \vec{B}^{\mathrm{m}}(\vec{r}_{\alpha,i}, t) \right)$$

$$\nabla \cdot \vec{E}^{\mathrm{m}}(\vec{r}, t) = \frac{\rho^{\mathrm{el,m}}(\vec{r}, t)}{\varepsilon_0}, \quad \nabla \times \vec{E}^{\mathrm{m}}(\vec{r}, t) = -\frac{\partial \vec{B}^{\mathrm{m}}(\vec{r}, t)}{\partial t},$$

$$\nabla \cdot \vec{B}^{\mathrm{m}}(\vec{r}, t) = 0, \quad \nabla \times \vec{B}^{\mathrm{m}}(\vec{r}, t) = \mu_0 \vec{j}^{\,\mathrm{m}}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial \vec{E}^{\mathrm{m}}(\vec{r}, t)}{\partial t}$$

$$\rho^{\mathrm{el,m}}(\vec{r}, t) = \sum_\alpha q_\alpha \int_{V_{\vec{v}}} \mathcal{K}_\alpha(\vec{r}, \vec{v}, t) d^3\vec{v}, \quad \vec{j}^{\,\mathrm{m}}(\vec{r}, t) = \sum_\alpha q_\alpha \int_{V_{\vec{v}}} \vec{v} \mathcal{K}_\alpha(\vec{r}, \vec{v}, t) d^3\vec{v}$$

$$\frac{\partial \mathcal{K}_\alpha}{\partial t} + \vec{v} \cdot \nabla \mathcal{K}_\alpha + \frac{q_\alpha}{m_\alpha} \left( \vec{E}^{\mathrm{m}} + \vec{v} \times \vec{B}^{\mathrm{m}} \right) \cdot \nabla_v \mathcal{K}_\alpha = 0$$

$$\mathcal{K}_\alpha = f_\alpha + \delta\mathcal{K}_\alpha$$

$$\begin{aligned}\vec{E}^{\mathrm{m}}(\vec{r},t) &= \vec{E}(\vec{r},t) + \delta\vec{E}^{\mathrm{m}}(\vec{r},t)\\ \vec{B}^{\mathrm{m}}(\vec{r},t) &= \vec{B}(\vec{r},t) + \delta\vec{B}^{\mathrm{m}}(\vec{r},t)\end{aligned}$$

$$\langle\delta\mathcal{K}_\alpha\rangle_{\mathrm{ANS}} = \langle\delta\vec{E}^{\mathrm{m}}\rangle_{\mathrm{ANS}} = \langle\delta\vec{B}^{\mathrm{m}}\rangle_{\mathrm{ANS}} = 0$$

$$\frac{\partial f_\alpha(\vec{r},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla f_\alpha(\vec{r},\vec{v},t) + \frac{q_\alpha}{m_\alpha}\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\nabla_v f_\alpha(\vec{r},\vec{v},t) = \mathcal{I}_\alpha,$$

$$\mathcal{I}_\alpha = -\frac{q_\alpha}{m_\alpha}\left\langle\left(\delta\vec{E}^{\mathrm{m}} + \vec{v}\times\delta\vec{B}^{\mathrm{m}}\right)\cdot\nabla_v\delta\mathcal{K}_\alpha\right\rangle_{\mathrm{ANS}},$$

$$\frac{df_\alpha(\vec{r},\vec{v},t)}{dt} = \mathcal{I}_\alpha$$

$$\int_{V_{\vec{v}}} d^3\vec{v}$$

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \vec{u}_\alpha) = 0$$

$$\frac{df_\alpha(\vec{r}, \vec{v}, t)}{dt} = \not{\mathcal{I}}_\alpha$$

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{u}_\alpha) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

$$\frac{df_\alpha(\vec{r}, \vec{v}, t)}{dt} = \mathcal{I}_\alpha$$

transport theory

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{u}_\alpha) = \mathcal{C}_{1,\alpha}, \quad \mathcal{C}_{1,\alpha} = m_\alpha \int_{V_{\vec{v}}} \mathcal{I}_\alpha d^3\vec{v}$$

$$\sum_\alpha \mathcal{C}_{1,\alpha} = 0$$

$$\frac{\partial \rho_\alpha^{\mathrm{el}}}{\partial t} + \nabla \cdot \vec{j}_\alpha = 0, \quad \rho_\alpha^{\mathrm{el}} = n_\alpha q_\alpha, \quad \vec{j}_\alpha = \rho_\alpha^{\mathrm{el}} \vec{u}_\alpha$$

$$\int_{V_{\vec{v}}} m_\alpha \vec{v} d^3\vec{v}$$

$$\begin{aligned}\rho_\alpha^{\mathrm{el}} &= q_\alpha n_\alpha \\ \vec{j}_\alpha &= \rho_\alpha^{\mathrm{el}} \vec{u}_\alpha\end{aligned}$$

$$\rho_\alpha \frac{d\vec{u}_\alpha}{dt} = -\nabla \cdot \hat{\mathrm{P}}_\alpha + \rho_\alpha^{\mathrm{el}} \vec{E} + \vec{j}_\alpha \times \vec{B} + \vec{\mathcal{C}}_{2,\alpha} - \vec{u}_\alpha \mathcal{C}_{1,\alpha}, \quad \vec{\mathcal{C}}_{2,\alpha} = m_\alpha \int_{V_{\vec{v}}} \vec{v} \mathcal{I}_\alpha d^3\vec{v}$$

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \nabla \cdot (\rho \vec{u}\vec{u}) = \rho \frac{d\vec{u}}{dt} = -\nabla \cdot \hat{\mathrm{P}} + \vec{j} \times \vec{B}$$

$$\int_{V_{\vec{v}}} \frac{1}{2} m_\alpha v^2 d^3\vec{v}$$

$$\mathcal{C}_{3,\alpha} = \frac{1}{2} m_\alpha \int_{V_{\vec{v}}} v^2 \mathcal{I}_\alpha d^3\vec{v}$$

$$\frac{3}{2}\frac{dp_\alpha}{dt} + \frac{3}{2} p_\alpha \nabla \cdot \vec{u}_\alpha + (\hat{\mathrm{P}}_\alpha \cdot \nabla) \cdot \vec{u}_\alpha + \nabla \cdot \vec{q}_\alpha = \mathcal{C}_{3,\alpha} - \vec{u}_\alpha \cdot \vec{C}_{2,\alpha}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e + \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right) + \\ + \nabla \cdot \left(\left(\frac{1}{2}\rho u^2 + \rho e\right)\vec{u} + \hat{\mathrm{P}} \cdot \vec{u} + \vec{q} + \vec{\mathcal{S}}\right) = 0$$

adiabatic vs. isothermal approximation

# Geleralized Ohms law

$$\vec{E} = -\vec{u} \times \vec{B} + \\ + \frac{\vec{j}}{\sigma} + \frac{\vec{j} \times \vec{B}}{|q_e| n_e} - \frac{\nabla p_e}{|q_e| n_e} + \frac{m_e}{q_e^2 n_e} \left( \frac{\partial \vec{j}}{\partial t} + \nabla \cdot \left( \vec{u}\vec{j} + \vec{j}\vec{u} - \frac{\vec{j}\vec{j}}{|q_e| n_e} \right) \right)$$

$$\vec{E} = -\vec{u} \times \vec{B}$$

$$\vec{E} = \vec{j}/\sigma - \vec{u} \times \vec{B}$$

$$\vec{E} = -\vec{u}_e \times \vec{B}$$

$$n_{\mathrm{e}} = n_{\mathrm{p}} = n_0$$

$$n_{\mathrm{e}}(\infty) = n_{\mathrm{p}}(\infty) = n_0$$

$$n_{\mathrm{e}}(\vec{r}),\ n_{\mathrm{p}}(\vec{r})$$

$$\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r})$$

$$n_{\mathrm{p}}(\vec{r}) = n_0 e^{-|q_{\mathrm{e}}|\Phi(\vec{r})/(kT)}$$

$$n_{\mathrm{e}}(\vec{r}) = n_0 e^{|q_{\mathrm{e}}|\Phi(\vec{r})/(kT)}$$

$$\rho^{\mathrm{el}}(\vec{r}) = -|q_{\mathrm{e}}|\big(n_{\mathrm{e}}(\vec{r}) - n_{\mathrm{p}}(\vec{r})\big) + |q_{\mathrm{e}}|\delta(\vec{r})$$

$$\nabla\cdot\vec{E}(\vec{r}) = \rho^{\mathrm{el}}(\vec{r})/\varepsilon_0$$

$$\Delta\Phi(\vec{r}) - \frac{|q_{\mathrm{e}}|n_0}{\varepsilon_0}\left(e^{\frac{|q_{\mathrm{e}}|\Phi(\vec{r})}{kT}} - e^{-\frac{|q_{\mathrm{e}}|\Phi(\vec{r})}{kT}}\right) = -\frac{|q_{\mathrm{e}}|}{\varepsilon_0}\delta(\vec{r})$$

$$e^{\pm|q_e|\Phi/(kT)} \approx 1 \pm |q_e|\Phi/(kT)$$

$$\Delta\Phi(\vec{r}) - \frac{1}{r_{D_H}^2}\Phi(\vec{r}) = -\frac{|q_e|}{\varepsilon_0}\delta(\vec{r}),\ \ r_{D_H} = \sqrt{\frac{\varepsilon_0 kT}{2n_0 q_e^2}}$$

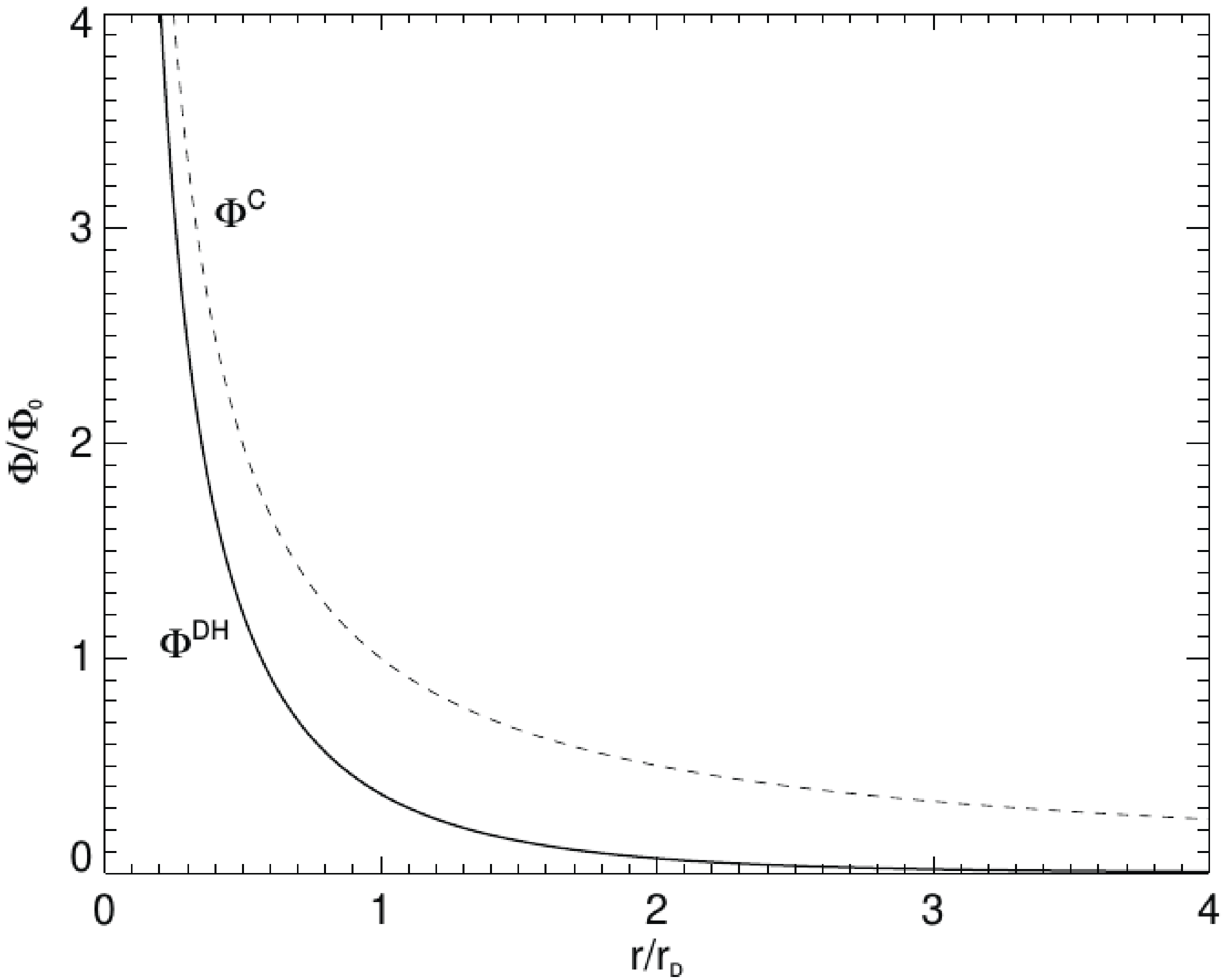
spherical symetry

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right) - \frac{1}{r_{D_H}^2}\Phi(r) = 0$$

$$\Phi(r) = \frac{|q_e|}{4\pi\varepsilon_0}\frac{F(r)}{r}$$

$$\frac{d^2F(r)}{dr^2} - \frac{1}{r_{D_H}^2}F(r) = 0$$

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{|q_e|}{r}e^{-r/r_{D_H}}$$



4
3
2
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$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \vec{u}_\alpha) = 0, \; m_\alpha \left( \frac{\partial \vec{u}_\alpha}{\partial t} + \vec{u}_\alpha \cdot \nabla \vec{u}_\alpha \right) = q_\alpha \vec{E}$$

Cold 2 component plasma

Thermal motion is slow

No scalar pressure

No B field

Without dissipation

No global motion

Infinite volume

$$\nabla \cdot \vec{E} = \frac{\rho^{\mathrm{el}}}{\varepsilon_0} = \frac{|q_\mathrm{e}|}{\varepsilon_0} (n_\mathrm{p} - n_\mathrm{e})$$

$$\vec{E}_0 = \vec{0}$$

$$n_{\mathrm{e}1}(\vec{r}, t) \ll n_0$$

$$\vec{u}_\mathrm{e} = \vec{u}_{\mathrm{e}1}$$

$$m_\mathrm{e} \ll m_\mathrm{p}$$

$$\vec{u}_\mathrm{p} = \vec{u}_{\mathrm{p}1} = \vec{0}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + n_0 \nabla \cdot \vec{u}_{\mathrm{e}1} = 0, \quad m_\mathrm{e} \frac{\partial \vec{u}_{\mathrm{e}1}}{\partial t} = -|q_\mathrm{e}| \vec{E}_1, \quad \nabla \cdot \vec{E}_1 = -\frac{|q_\mathrm{e}|}{\varepsilon_0} n_{\mathrm{e}1}$$

$$\frac{\partial^2 n_{\mathrm{e}1}}{\partial t^2} = -n_0 \nabla \cdot \frac{\partial \vec{u}_{\mathrm{e}1}}{\partial t} = \frac{n_0 |q_\mathrm{e}|}{m_\mathrm{e}} \nabla \cdot \vec{E}_1 = -\omega_{\mathrm{pe}}^2 n_{\mathrm{e}1}, \quad \omega_{\mathrm{pe}} = \sqrt{\frac{n_0 q_\mathrm{e}^2}{\varepsilon_0 m_\mathrm{e}}}$$

Plasma oscillations

Warm plasma – same as before, but with scalar pressure

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha \vec{u}_\alpha\right) = 0, \quad n_\alpha m_\alpha \left(\frac{\partial \vec{u}_\alpha}{\partial t} + \vec{u}_\alpha \cdot \nabla \vec{u}_\alpha\right) + \nabla p_\alpha = -n_\alpha q_\alpha \nabla \Phi,$$

$$\frac{\partial p_\alpha}{\partial t} + \vec{u}_\alpha \cdot \nabla p_\alpha + \gamma_{\mathrm{g},\alpha} p_\alpha \nabla \cdot \vec{u}_\alpha = 0, \quad \nabla^2 \Phi = -\frac{|q_\mathrm{e}|}{\varepsilon_0} \left(n_\mathrm{p} - n_\mathrm{e}\right),$$

plane waves

$$\frac{\partial n_{\alpha 1}}{\partial t} + n_{\alpha 0} \nabla \cdot \vec{u}_{\alpha 1} = 0 \;\Rightarrow\; \vec{k} \cdot \check{\vec{u}}_{\alpha 1} = \frac{\omega \check{n}_{\alpha 1}}{n_{\alpha 0}},$$

$$\frac{\partial p_{\alpha 1}}{\partial t} + \gamma_{\mathrm{g},\alpha} p_{\alpha 0} \nabla \cdot \vec{u}_{\alpha 1} = 0 \;\xrightarrow{\vec{k}\cdot\check{\vec{u}}_{\alpha 1}}\; \check{p}_{\alpha 1} = \frac{\gamma_{\mathrm{g},\alpha} p_{\alpha 0} \check{n}_{\alpha 1}}{n_{\alpha 0}},$$

$$m_\alpha n_{\alpha 0} \frac{\partial \vec{u}_{\alpha 1}}{\partial t} = -q_\alpha n_{\alpha 0} \nabla \Phi_1 - \nabla p_{\alpha 1} \;\xrightarrow[\vec{k}\cdot\check{\vec{u}}_{\alpha 1}]{\check{p}_{\alpha 1}}\; \check{n}_{\alpha 1} = \frac{q_\alpha n_{\alpha 0}}{m_\alpha} \frac{k^2 \check{\Phi}_1}{\omega^2 - \gamma_{\mathrm{g},\alpha} k^2 v_{\mathrm{th},\alpha}^2},$$

$$\nabla^2 \Phi_1 = -\frac{1}{\varepsilon_0} \sum_\alpha n_{\alpha 1} q_\alpha \;\Rightarrow\; k^2 \check{\Phi}_1 = \frac{1}{\varepsilon_0} \sum_\alpha \check{n}_{\alpha 1} q_\alpha$$

$$\left(1-\sum_{\alpha}\frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2-\gamma_{\mathrm{g},\alpha}k^2v_{\mathrm{th},\alpha}^2}\right)\check{\Phi}_1=0,\quad \omega_{\mathrm{p}\alpha}^2=\frac{n_{\alpha 0}q_\alpha^2}{\varepsilon_0 m_\alpha}$$

$$(1+\chi_\mathrm{e}+\chi_\mathrm{p})\,\check{\Phi}_1=0,\quad \chi_\alpha=-\frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2-\gamma_{\mathrm{g},\alpha}k^2v_{\mathrm{th},\alpha}^2},\quad \alpha=\mathrm{e},\mathrm{p}$$

$$\check{\Phi}_1\neq 0$$

$$1+\chi_\mathrm{e}+\chi_\mathrm{p}=0$$

$$\omega/k \gg v_{\mathrm{th},\alpha} \qquad\qquad (k^2/\omega^2)v_{\mathrm{th},\alpha}^2 \ll 1$$

$$\omega^2 = \omega_{\mathrm{pe}}^2 + \gamma_{\mathrm{g,e}} k^2 v_{\mathrm{th,e}}^2 = \omega_{\mathrm{pe}}^2 (1 + \gamma_{\mathrm{g,e}} k^2 r_{\mathrm{De}}^2)$$

$$\omega/k \ll v_{\mathrm{th},\alpha}$$

Dynamical screening

$$v_{\mathrm{th,p}} \ll \omega/k \ll v_{\mathrm{th,e}}$$

Ion-acoustic waves

$$\omega^2 = \frac{k^2 v_{\mathrm{s,p}}^2}{1 + \gamma_{\mathrm{g,e}} k^2 r_{\mathrm{De}}^2} + \gamma_{\mathrm{g,p}} k^2 v_{\mathrm{th,p}}^2$$

$$v_{\mathrm{s,p}}^2 = \gamma_{\mathrm{g,e}} \omega_{\mathrm{pp}}^2 r_{\mathrm{De}}^2$$

$$m_\alpha n_\alpha \left( \frac{\partial \vec{u}_\alpha}{\partial t} + \vec{u}_\alpha \cdot \nabla \vec{u}_\alpha \right) = -\nabla p_\alpha + n_\alpha q_\alpha \vec{E}, \quad \alpha = \mathrm{e}, \mathrm{p}$$

$$\vec{E} = -\nabla \Phi - \partial \vec{A} / \partial t$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E}_1 = -\nabla \Phi_1 - \partial \vec{A}_1 / \partial t, \ \vec{E}_0 = \vec{0}, \ \vec{B}_0 = \vec{0}$$

$$\nabla \cdot \vec{A} = 0$$

$$m_\alpha n_{\alpha 0} \frac{\partial \vec{u}_{\alpha 1}}{\partial t} = -\nabla p_{\alpha 1} - q_\alpha n_{\alpha 0} \nabla \Phi_1 - q_\alpha n_{\alpha 0} \frac{\partial \vec{A}_1}{\partial t}$$

$$\frac{\partial}{\partial t} \Big( \nabla \times \left( m_\alpha n_{\alpha 0} \vec{u}_{\alpha 1} \right) \Big) = -q_\alpha n_{\alpha 0} \frac{\partial \vec{B}_1}{\partial t}$$

$$\nabla \times \vec{j}_1 = -\varepsilon_0 \omega_\mathrm{p}^2 \vec{B}_1, \quad \vec{j}_1 = \sum_\alpha n_{\alpha 0} q_\alpha \vec{u}_{1\alpha}, \quad \omega_\mathrm{p}^2 = \sum_\alpha \omega_{\mathrm{p}\alpha}^2 \approx \omega_\mathrm{pe}^2$$

$$\vec{j}_1 = \frac{1}{\mu_0} \nabla \times \vec{B}_1 - \varepsilon_0 \frac{\partial \vec{E}_1}{\partial t}$$

$$\nabla \times \left( \nabla \times \vec{B}_1 - \frac{1}{c^2} \frac{\partial \vec{E}_1}{\partial t} \right) = -\frac{\omega_\mathrm{p}^2}{c^2} \vec{B}_1, \quad \frac{1}{\mu_0} = \varepsilon_0 c^2$$

$$\Delta \vec{B}_1 = \frac{1}{c^2} \frac{\partial^2 \vec{B}_1}{\partial t^2} + \frac{\omega_\mathrm{p}^2}{c^2} \vec{B}_1$$

$$\omega^2 = \omega_\mathrm{p}^2 + k^2 c^2$$

$$\omega^2 < \omega_\mathrm{p}^2$$

No propagation