Orbital method

Qualitative description

Collisionless, magnetized plasmas

Follow the path of individual charged particles in known fields

Interplanetary, interstellar, intergalactic plasmas

Known fields – particles does not influence fields (but they do in reality...)

Sources o fields are outside of the volume of interest

$$\nabla \cdot \vec{E} \approx 0,\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$
$$\nabla \cdot \vec{B} = 0,\ \nabla \times \vec{B} \approx \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}.$$

$$m\ddot{\vec{r}} = q\vec{E}(\vec{r},t) + q\dot{\vec{r}} \times \vec{B}(\vec{r},t) + \vec{F}_{\mathrm{ne}}(\vec{r},\dot{\vec{r}},t)$$

nonrelativistic

relativistic

$$\frac{d}{dt}\left(\gamma m\vec{v}\right) = \vec{F}(\vec{r},\vec{v},t), \;\; \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

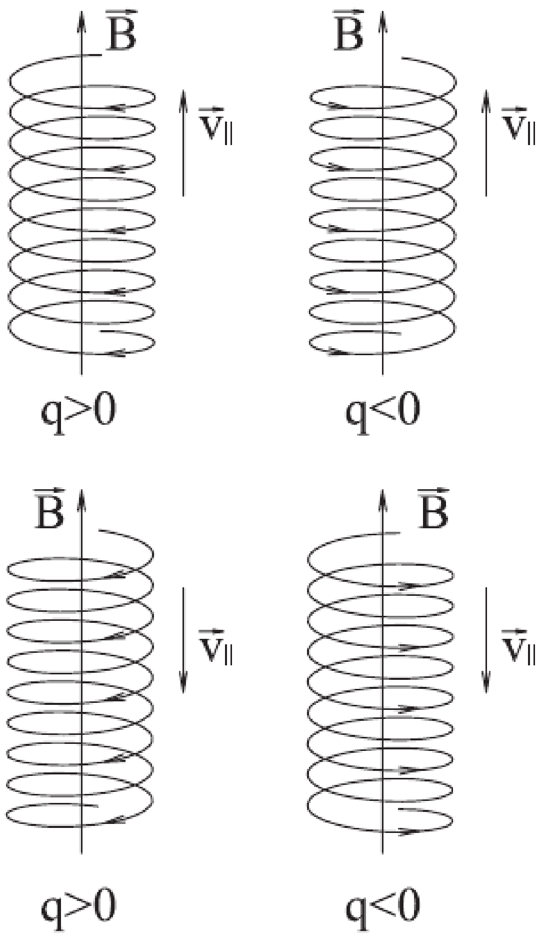
$$\frac{d}{dt}\left(\gamma mc^2\right) = \vec{F}\cdot\vec{v}$$

$$\boxed{\gamma m\frac{d\vec{v}}{dt} + \frac{\vec{v}}{c^2}\vec{F}\cdot\vec{v} = \vec{F}}$$

Const. B

$$m_{\rm p}\ddot{\vec{r}} = |q_{\rm e}|(\dot{\vec{r}} \times \vec{B}) \qquad \vec{b} \equiv \vec{B}/B = \vec{e}_z$$

$$\ddot{x} = \frac{|q_{\rm e}|B}{m_{\rm p}}\dot{y},\ \ddot{y} = -\frac{|q_{\rm e}|B}{m_{\rm p}}\dot{x},\ \ddot{z} = 0$$



$$\left.\begin{array}{l} x = r_{\rm cp}\big(1-\cos(\omega_{\rm cp}t)\big) \\ y = r_{\rm cp}\sin(\omega_{\rm cp}t) \end{array}\right\} \qquad (x - r_{\rm cp})^2 + y^2 = r_{\rm cp}^2,$$

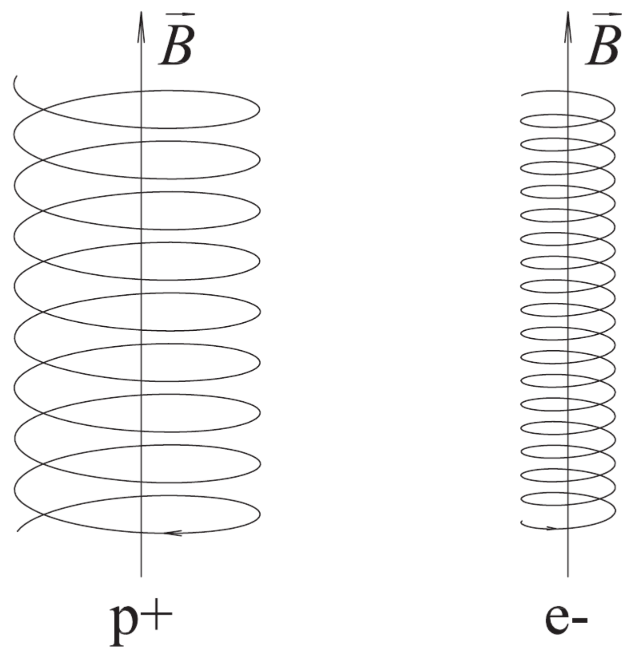
$$z = v_\parallel^0 t,$$

$$\omega_{\rm cp} = |q_{\rm e}|B/m_{\rm p}$$

$$\vec{v}(0) = (0\ v_\perp^0\ v_\parallel^0)^{\rm T}$$

$$r_{\rm cp} = v_\perp^0/\omega_{\rm cp}$$

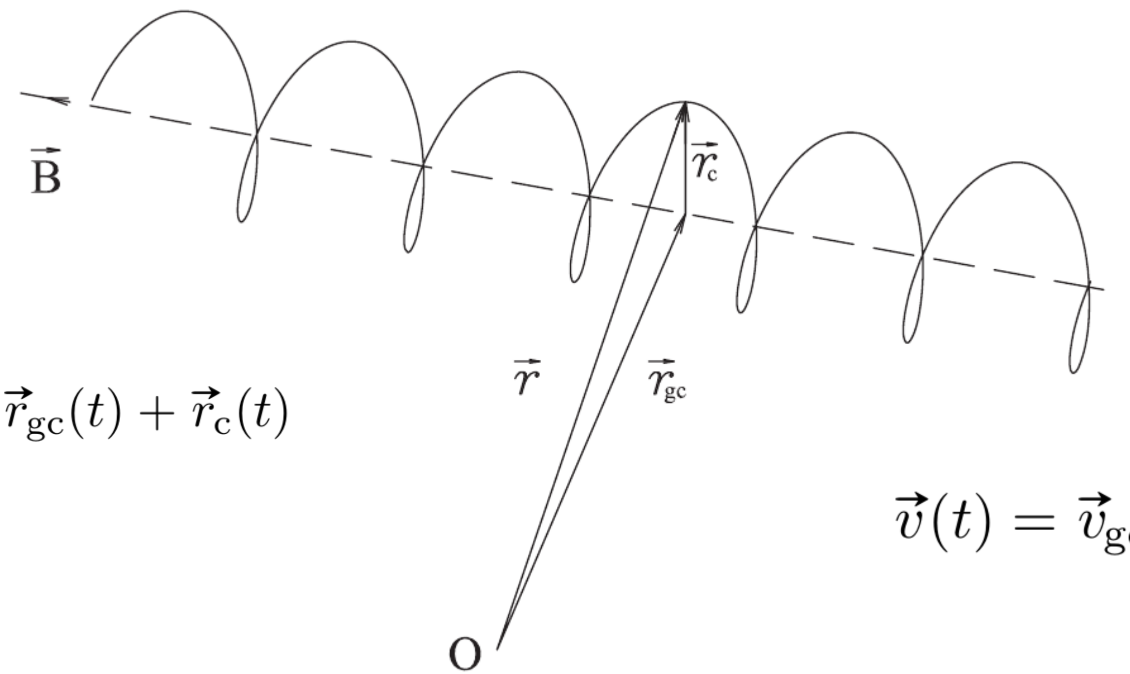
$$m\ddot{\vec{r}} = q\left(\dot{\vec{r}} \times \vec{B}(\vec{r})\right) \ \Big/ \cdot \dot{\vec{r}} \ \Rightarrow \ \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = 0$$



$$\omega_{\rm c}^{\rm rel} = |q|B/(\gamma m)$$

Guiding center

Cyclotron rotation + Guiding center motion



$$\vec{r}(t) = \vec{r}_{\rm gc}(t) + \vec{r}_{\rm c}(t)$$

$$\vec{v}(t) = \vec{v}_{\rm gc}(t) + \vec{v}_{\rm c}(t)$$

Additional longitudinal const. force – just a change in parallel motion – stretching of the helix

But, if we have const. $\vec{F} = \vec{F}_\perp = F_\perp \vec{\mathrm{e}}_x$, then:

$$\ddot{x} - \omega_\mathrm{c}\dot{y} - \frac{F_\perp}{m} = 0, \;\; \ddot{y} + \omega_\mathrm{c}\dot{x} = 0$$

$$\vec{v}(0) = (0 \; v_\perp^0 \; v_\parallel^0)^\mathrm{T}$$

$$x = \left(\frac{v_\perp^0}{\omega_\mathrm{c}} + \frac{F_\perp}{m\omega_\mathrm{c}^2}\right)\Big(1 - \cos(\omega_\mathrm{c} t)\Big), \;\; y = \left(\frac{v_\perp^0}{\omega_\mathrm{c}} + \frac{F_\perp}{m\omega_\mathrm{c}^2}\right)\sin(\omega_\mathrm{c} t) - \frac{F_\perp}{m\omega_\mathrm{c}} t,$$

$$\left(x - \left(\frac{v_\perp^0}{\omega_\mathrm{c}} + \frac{F_\perp}{m\omega_\mathrm{c}^2}\right)\right)^2 + \left(y + \frac{F_\perp}{m\omega_\mathrm{c}} t\right)^2 = \left(\frac{v_\perp^0}{\omega_\mathrm{c}} + \frac{F_\perp}{m\omega_\mathrm{c}^2}\right)^2$$

Switch to the guiding center (inertial) reference frame – only gyro-motion

$$\vec{v}_{\rm gc} = \vec{v}_{\|} - F_{\perp}/(m\omega_{\rm c})\vec{\rm e}_y$$

$$\tilde{v}_{\|} \; = \; 0$$

$$\vec{\tilde{r}} \; = \; \vec{r} \; - \; \vec{v}_{\rm gc} t$$

$$\tilde{v}_{\perp} \; \equiv \; \sqrt{\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2} \; =$$

$$v_{\rm c} = v_{\perp}^{0} + F_{\perp}/(m\omega_{\rm c}) = {\rm const}$$

Particle velocity can be written as:

guiding center drift

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp} = \vec{v}_{\parallel} + \vec{v}_{\mathrm{c}} + \vec{V}_{\mathrm{D}}$$

$$\vec{V}_{\mathrm{D}} = -F_{\perp}/(m\omega_{\mathrm{c}})\vec{\mathrm{e}}_y$$

gyro-motion

Zero order guiding center drifts

More generally for zero order drifts in 3D:

$$\vec{V}_{\mathrm{D}} = \frac{1}{q}\frac{\vec{F}_{\perp} \times \vec{B}}{B^2}$$

$$\vec{V}_{\mathrm{D}}^{\vec{E}\times\vec{B}} = \frac{\vec{E}_{\perp} \times \vec{B}}{B^2}$$

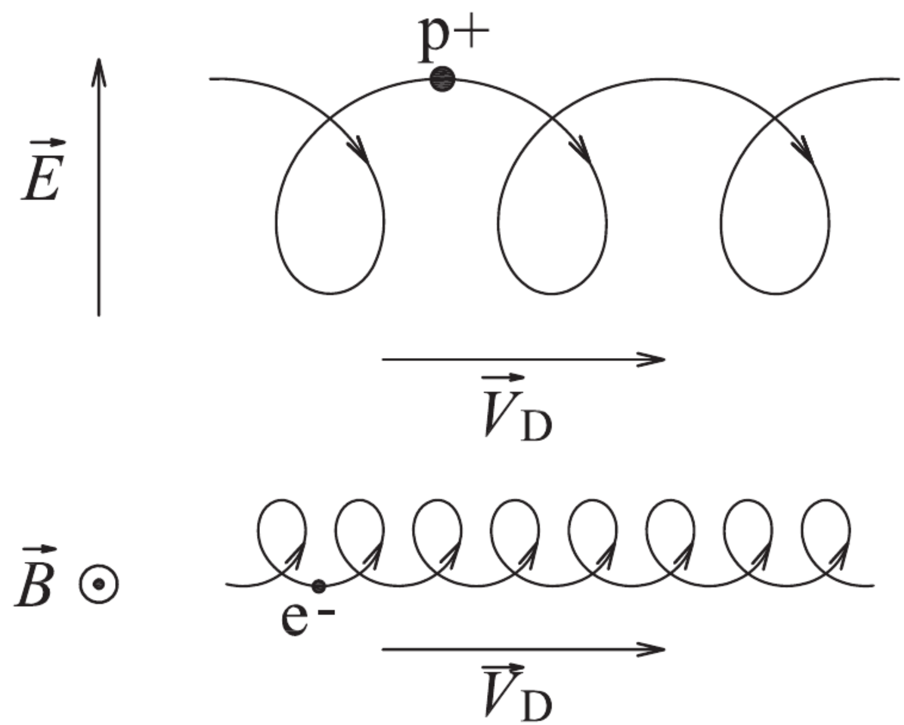
Electric drift

additional constant electric field

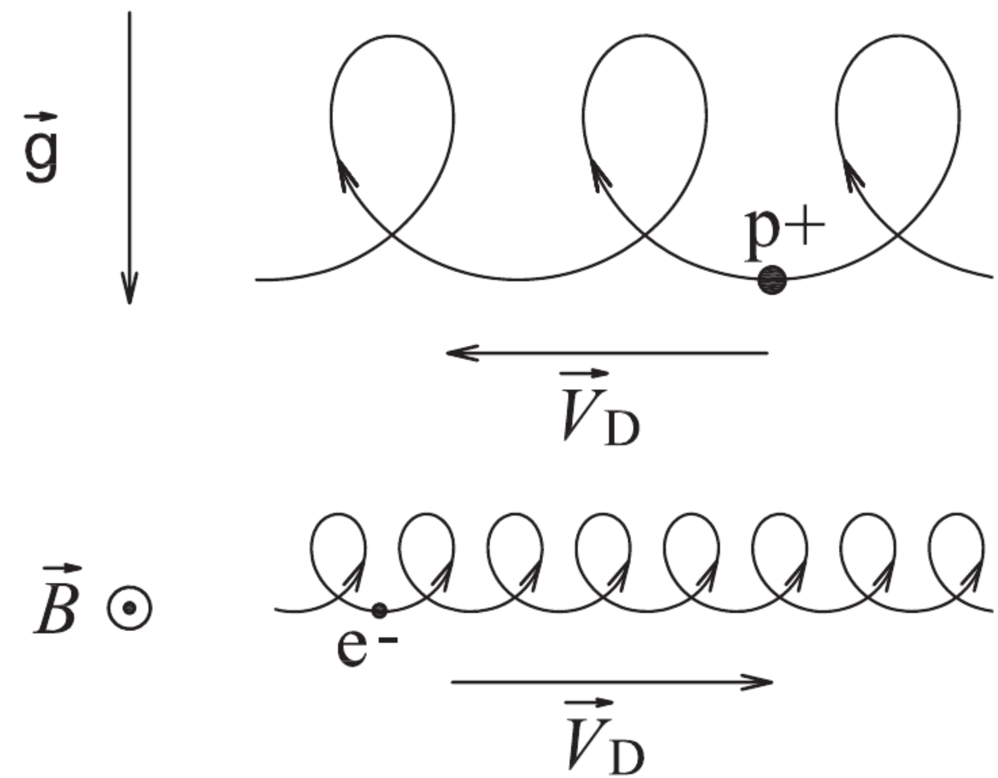
$$\vec{V}_{\mathrm{D}}^{m\vec{g}} = \frac{m}{q}\frac{\vec{g} \times \vec{B}}{B^2}$$

gravitational drift

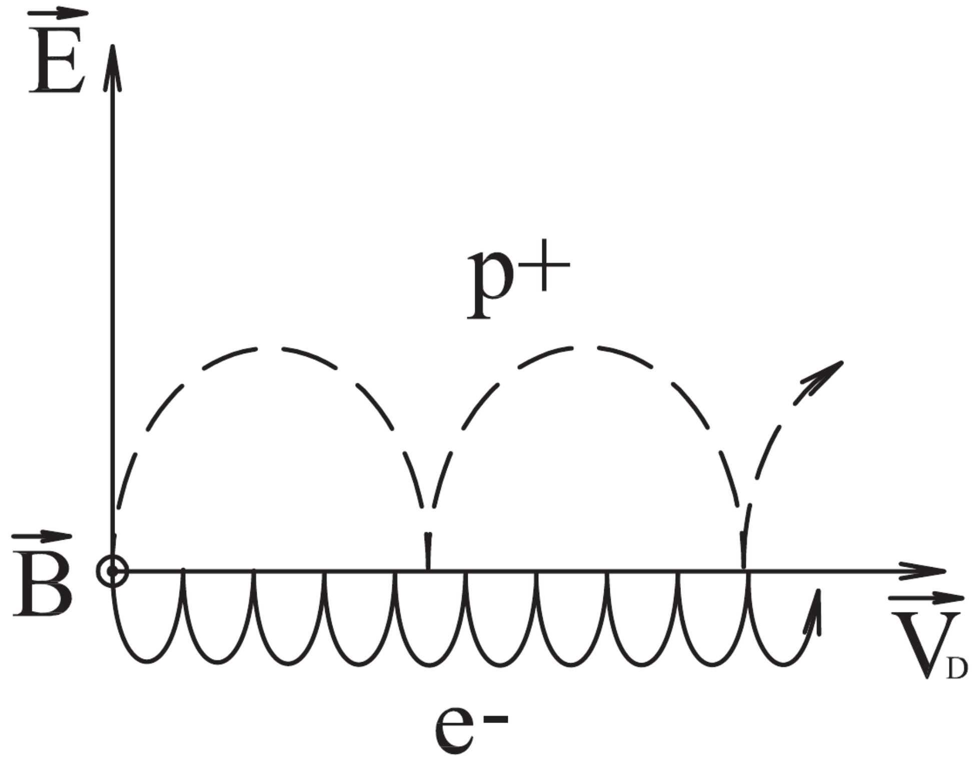
additional constant gravitational field



p+
$\vec{E}$
$\vec{V}_D$
$\vec{B}$ ⊙
e-
$\vec{V}_D$



$\vec{g}$
p+
$\vec{V}_D$
$\vec{B}$
e-
$\vec{V}_D$



Different initial conditions

Switch to the guiding center (inertial) reference frame – only gyro-motion

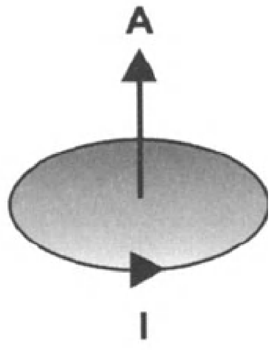
Assume only orthogonal electric field – electric drift

$$\|\tilde{\vec{E}}_\perp\| = \|\vec{E}_\perp + \vec{V}_\mathrm{D}^{\vec{E}\times\vec{B}} \times \vec{B}\| = 0$$

Cyclotron rotation viewed as a current loop

Magnetic moment of a current loop

$$\vec{M} = -M\vec{b} \qquad M = \mathrm{w}_{\perp}^{\mathrm{c}}/B \qquad \mathrm{w}_{\perp}^{\mathrm{c}} = mv_{\perp}^{2}/2$$



Hydrogen plasma occupy a finite part of the space

e.g. atmosphere at some hight *z*

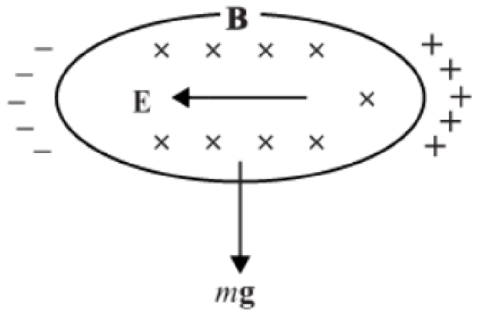
A gravitational force acting perpendicular to the magnetic field does not directly cause the plasma to "fall" – the induced drift velocity is perpendicular to the applied force

But… the plasma does indeed fall...

Unlike the E × B drift, the drift velocity produced by a gravitational force causes positive and negative charges to move in opposite directions, thereby causing a current

For a plasma of finite size this current produces a polarization charge at the boundaries of the plasma which, in turn, produces an electric field perpendicular to the gravitational force

This electric field causes an E × B drift in the direction of the gravitational force



A downward gravitational field $m\mathbf{g}$ causes electrons and ions to drift in opposite directions, thereby producing polarization charges at the boundaries of the plasma. The resulting electric field $\mathbf{E}$ then causes a downward $\mathbf{E} \times \mathbf{B}$ drift.

Tensors in Euclid R^3 space

$$\vec{T}_1 = \begin{pmatrix} T_{11} \\ T_{12} \\ T_{13} \end{pmatrix}, \ \vec{T}_2 = \begin{pmatrix} T_{21} \\ T_{22} \\ T_{23} \end{pmatrix}, \ \vec{T}_3 = \begin{pmatrix} T_{31} \\ T_{32} \\ T_{33} \end{pmatrix}$$

$$\hat{\mathrm{T}} = \begin{pmatrix} \vec{\mathrm{e}}_1 \cdot \vec{T}_1 & \vec{\mathrm{e}}_1 \cdot \vec{T}_2 & \vec{\mathrm{e}}_1 \cdot \vec{T}_3 \\ \vec{\mathrm{e}}_2 \cdot \vec{T}_1 & \vec{\mathrm{e}}_2 \cdot \vec{T}_2 & \vec{\mathrm{e}}_2 \cdot \vec{T}_3 \\ \vec{\mathrm{e}}_3 \cdot \vec{T}_1 & \vec{\mathrm{e}}_3 \cdot \vec{T}_2 & \vec{\mathrm{e}}_3 \cdot \vec{T}_3 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}$$

dyadics

$$\vec{A}\vec{B} = \begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}$$

local tensor

$$\nabla \vec{A} = \begin{pmatrix} \partial A_x/\partial x & \partial A_y/\partial x & \partial A_z/\partial x \\ \partial A_x/\partial y & \partial A_y/\partial y & \partial A_z/\partial y \\ \partial A_x/\partial z & \partial A_y/\partial z & \partial A_z/\partial z \end{pmatrix}$$

(a) Divergence terms:

$$\partial B_x/\partial x, \quad \partial B_y/\partial y, \quad \partial B_z/\partial z$$

(b) Gradient terms:

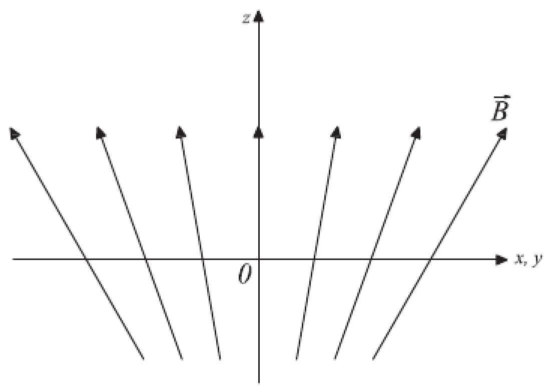
$$\partial B_z/\partial x, \quad \partial B_z/\partial y$$

(c) Curvature terms:

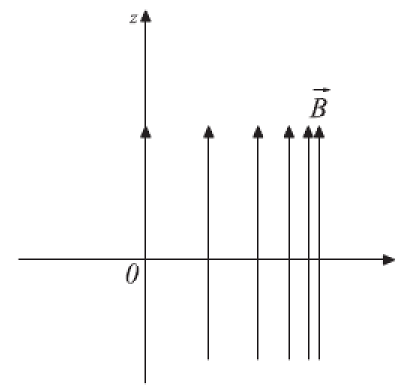
$$\partial B_x/\partial z, \quad \partial B_y/\partial z$$

(d) Shear terms:

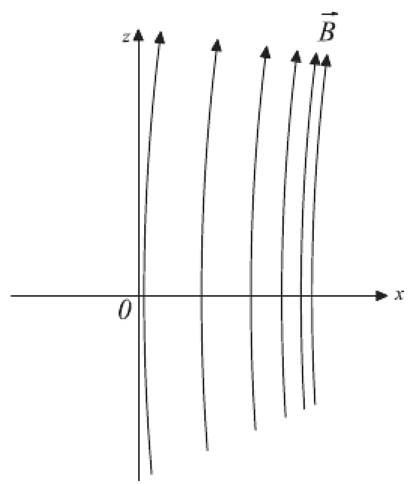
$$\partial B_x/\partial y, \quad \partial B_y/\partial x$$



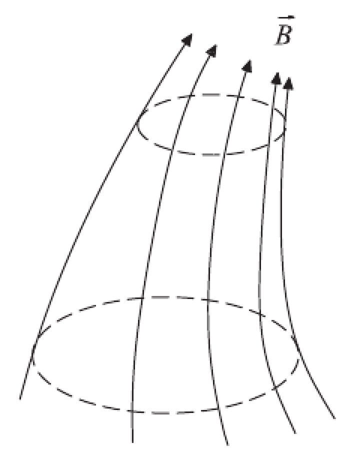
(a)



(b)

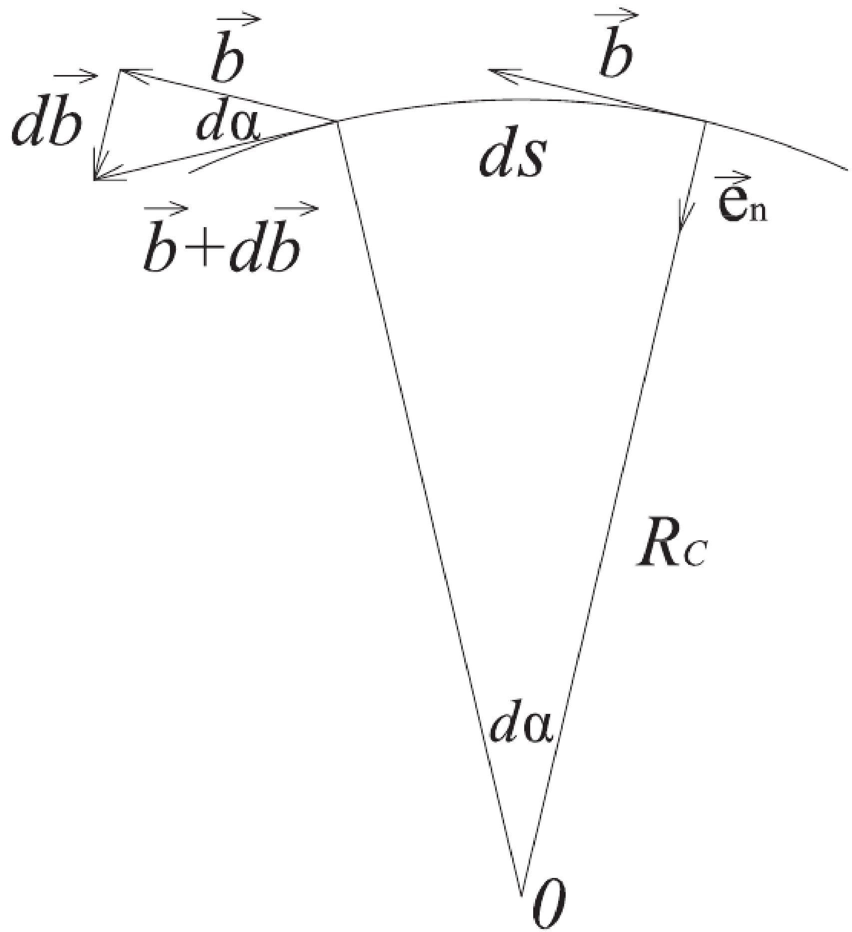


(c)



(d)

$$\frac{d\vec{b}}{ds} = (\vec{b} \cdot \nabla)\vec{b} = \frac{\vec{e}_n}{R_C}$$



Guiding center approximation

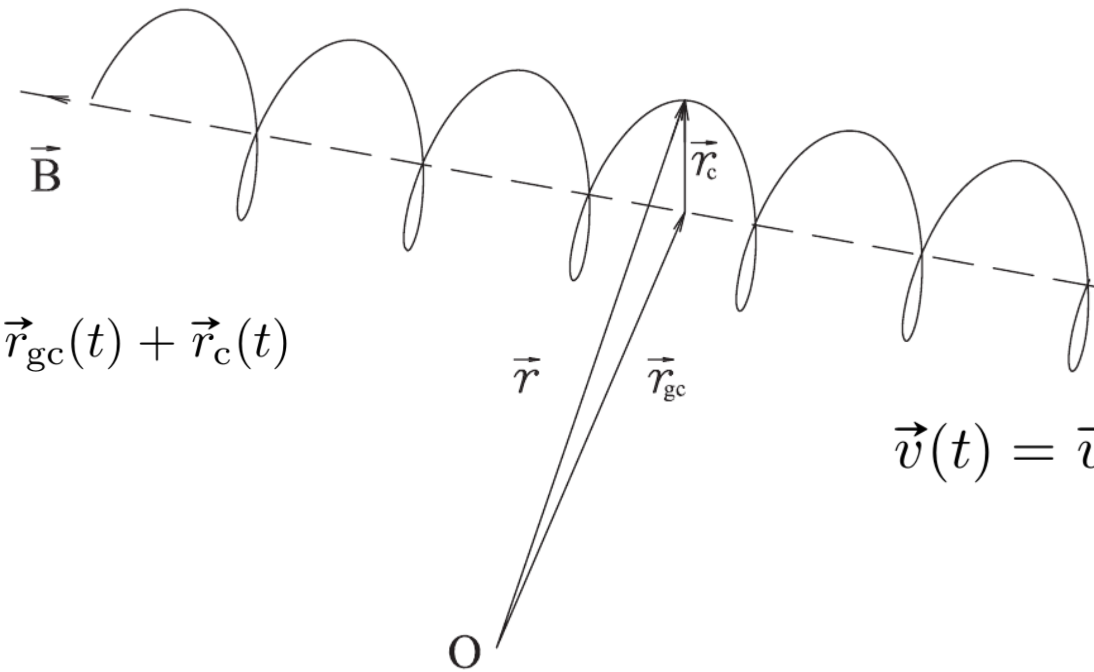
In many cases: small gyro-radius and high gyro-frequency in comparison to the characteristic length/time scales

$$r_{\mathrm{c}} \ll \boldsymbol{L}, \ \omega_{\mathrm{c}}^{-1}/\tau \ll 1$$

$$v_{\mathrm{gc}} \ll v_{\mathrm{c}} \qquad V_{\mathrm{D}} \ll v_{\mathrm{th}}$$

$$\tau_{\mathrm{c}} \ll r_{\mathrm{c}}/V_{\mathrm{D}} \ll \tau_{\mathrm{coll}}$$

Particle trajectory can be approximated with:



$$\vec{r}(t) = \vec{r}_{\mathrm{gc}}(t) + \vec{r}_{\mathrm{c}}(t)$$

$$\vec{v}(t) = \vec{v}_{\mathrm{gc}}(t) + \vec{v}_{\mathrm{c}}(t)$$

if:

$$\left| r_{\mathrm{c}} \frac{\|\nabla_{\perp} B\|}{B} \right| \ll 1, \quad \left| \frac{v_{\parallel}}{\omega_{\mathrm{c}}} \frac{\|\nabla_{\parallel} B\|}{B} \right| \ll 1, \quad \left| \frac{1}{\omega_{\mathrm{c}}} \frac{\partial \ln B}{\partial t} \right| \ll 1$$

$$m\dot{\vec{v}} = q\vec{v} \times \vec{B}(\vec{r})$$

ONLY
weak inhomogeneity

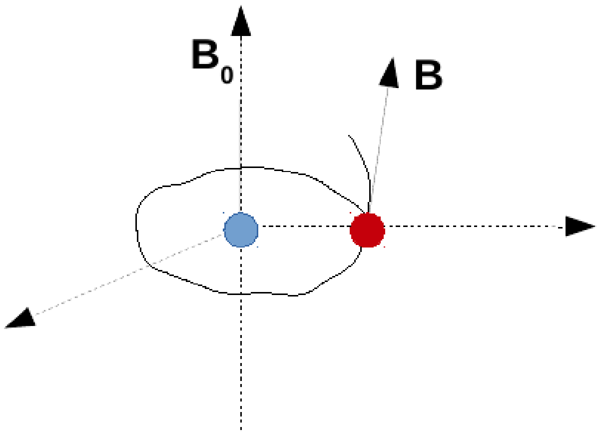
$$\langle \cdots \rangle_{\mathrm{c}} \equiv \frac{1}{\tau_{\mathrm{c}}} \int_0^{\tau_{\mathrm{c}}} (\cdots)\, dt$$

average over gyro-period

$$\vec{B} = \vec{B}_0 + \vec{B}_1$$

$$\vec{B}_1 = \vec{r}_{\mathrm{c}} \cdot \nabla_0 \vec{B}$$

weak inhomogeneity can be treated as a small perturbation – Taylor series – only first order is important



...

$$m\left(\frac{d\vec{V}_{\mathrm{D}}}{dt}\right)_{\perp} = q\vec{V}_{\mathrm{D}} \times \vec{B} + \langle \vec{F}_{\perp} \rangle,$$

$$\langle \vec{F}_{\perp} \rangle \equiv -M\nabla_{\perp}B + \vec{F}_{\mathrm{cf}},\ \ \vec{F}_{\mathrm{cf}} \equiv -mv_{\parallel}^2(\vec{b}\cdot\nabla)\vec{b} = -\frac{2\mathrm{w}_{\parallel}}{R_{\mathrm{C}}}\vec{\mathrm{e}}_{\mathrm{n}}$$

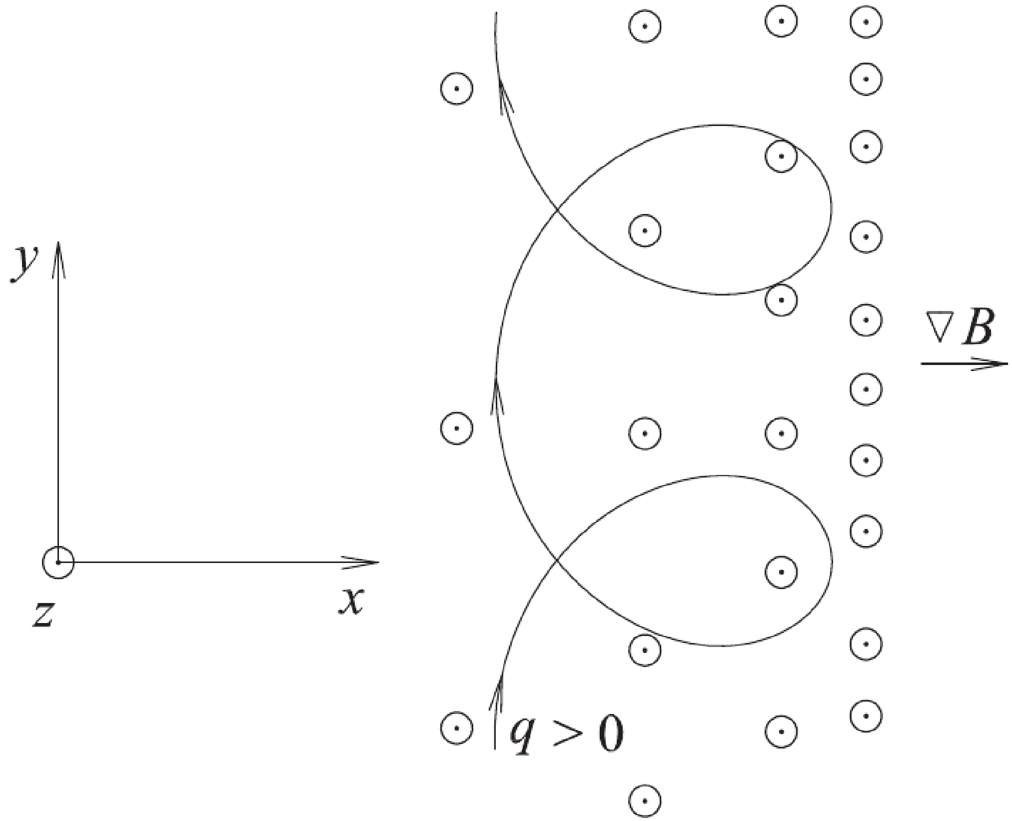
$$\vec{V}_{\mathrm{D}} = \frac{\vec{\mathbb{F}}_{\perp} \times \vec{B}}{qB^2}, \ \vec{\mathbb{F}}_{\perp} \equiv \langle \vec{F}_{\perp} \rangle - m \left(\frac{d\vec{V}_{\mathrm{D}}}{dt} \right)_{\perp}$$

$$\vec{V}_{\mathrm{D}}^{\mathrm{mag}} = \frac{1}{q} \frac{\langle \vec{F}_{\perp} \rangle \times \vec{B}}{B^2} = \frac{M}{qB^2} \vec{B} \times \nabla B + \frac{2\mathrm{w}_{\parallel}}{qB^2} \vec{B} \times (\vec{b} \cdot \nabla) \vec{b}$$

gradient + centrifugal drift = magnetic drift

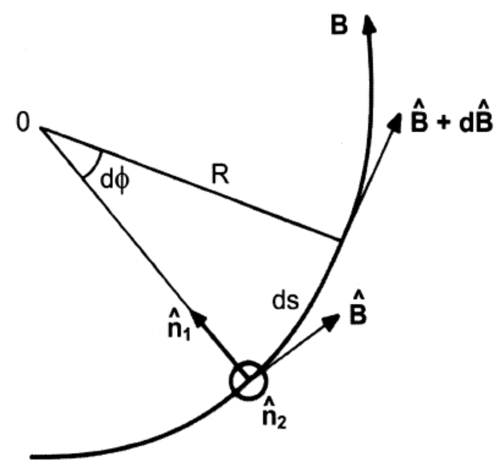
the first order drifts – depend on particles energy

Gradient drift



$$\omega_{\mathrm{cp}} = |q_{\mathrm{e}}|B/m_{\mathrm{p}}$$

$$r_{\mathrm{cp}} = v_{\perp}^{0}/\omega_{\mathrm{cp}}$$



$$\vec{F}_{\mathrm{cf}} \equiv -mv_{\parallel}^{2}(\vec{b}\cdot\nabla)\vec{b} = -\frac{2\mathrm{w}_{\parallel}}{R_{\mathrm{C}}}\vec{\mathrm{e}}_{\mathrm{n}}$$

If we add electric field – zero and first term in Taylor expansion lead only to the ExB drift

$$\nabla \times \vec{E}(\vec{r}) = \vec{0}$$

$$\vec{V}_{\mathrm{D}}^{(\mathrm{I})} = \frac{1}{q}\frac{\langle\vec{F}_{\perp}\rangle \times \vec{B}}{B^2}, \ \langle\vec{F}_{\perp}\rangle \equiv q\vec{E} - M\nabla_{\perp}B - \frac{2\mathrm{w}_{\parallel}}{R_{\mathrm{C}}}\vec{\mathrm{e}}_{\mathrm{n}}$$

But if few add the second term:

$$\vec{V}_{\mathrm{D}}^{\vec{E}(\vec{r})} = \left(1 + \frac{r_{\mathrm{c}}^2}{4}\Delta\right)\frac{\vec{E} \times \vec{B}}{B^2}$$

the second order drift

the second order drift due to the small time dependence

$$\vec{V}_{\mathrm{D}} = \frac{\vec{\mathbb{F}}_{\perp} \times \vec{B}}{qB^2}, \ \vec{\mathbb{F}}_{\perp} \equiv \langle \vec{F}_{\perp} \rangle - m\left(\frac{d\vec{V}_{\mathrm{D}}}{dt}\right)_{\perp}$$

$$\vec{V}_{\mathrm{D}} = \vec{V}_{\mathrm{D}}^{(\mathrm{I})} + \vec{V}_{\mathrm{D}}^{(\mathrm{II})}$$

$$\vec{V}_{\mathrm{D}}^{(\mathrm{II})} = -\frac{m}{qB^2}\frac{d\vec{V}_{\mathrm{D}}^{(\mathrm{I})}}{dt} \times \vec{B}$$

polarization drift

$$\vec{V}_{\mathrm{D}}^{\mathrm{p,\,E}} = -\frac{m}{qB^2}\frac{d\vec{V}_{\mathrm{D}}^{\vec{E}\times\vec{B}}}{dt} \times \vec{B} = \frac{m}{qB^2}\dot{\vec{E}}_{\perp}$$

Ordered motion – Drift currents

$$\vec{j}_{\mathrm{D}} = \sum_{\alpha} n_\alpha q_\alpha \langle \vec{V}_{\mathrm{D}} \rangle_\alpha$$

$$\vec{j}_{\mathrm{D}}^{\,\mathbf{grad}B} = \frac{\mathrm{W}_\perp}{B^3}\,\vec{B}\times\nabla B,$$

$$j_{\mathrm{D}}^{\vec{E}\times\vec{B}} \;=\; 0$$

$$\vec{j}_{\mathrm{D}}^{\,\mathrm{cf}} = \frac{2\mathrm{W}_\parallel}{B^2}\,\vec{B}\times(\vec{b}\cdot\nabla)\vec{b},$$

$$\vec{j}_{\mathrm{D}}^{\,m\vec{g}} = \frac{\rho}{B^2}\,\vec{g}\times\vec{B}$$

$$\vec{j}_{\mathrm{D}}^{\,\mathrm{p,\,E}} = \frac{\rho}{B^2}\,\dot{\vec{E}}_\perp,$$

$$\mathrm{W}_\perp \;=\; \textstyle\sum_\alpha n_\alpha \langle \mathrm{w}_\perp \rangle_\alpha \;\;\mathrm{i}\;\; \mathrm{W}_\parallel \;=\; \sum_\alpha n_\alpha \langle \mathrm{w}_\parallel \rangle_\alpha$$

Diamagnetic current

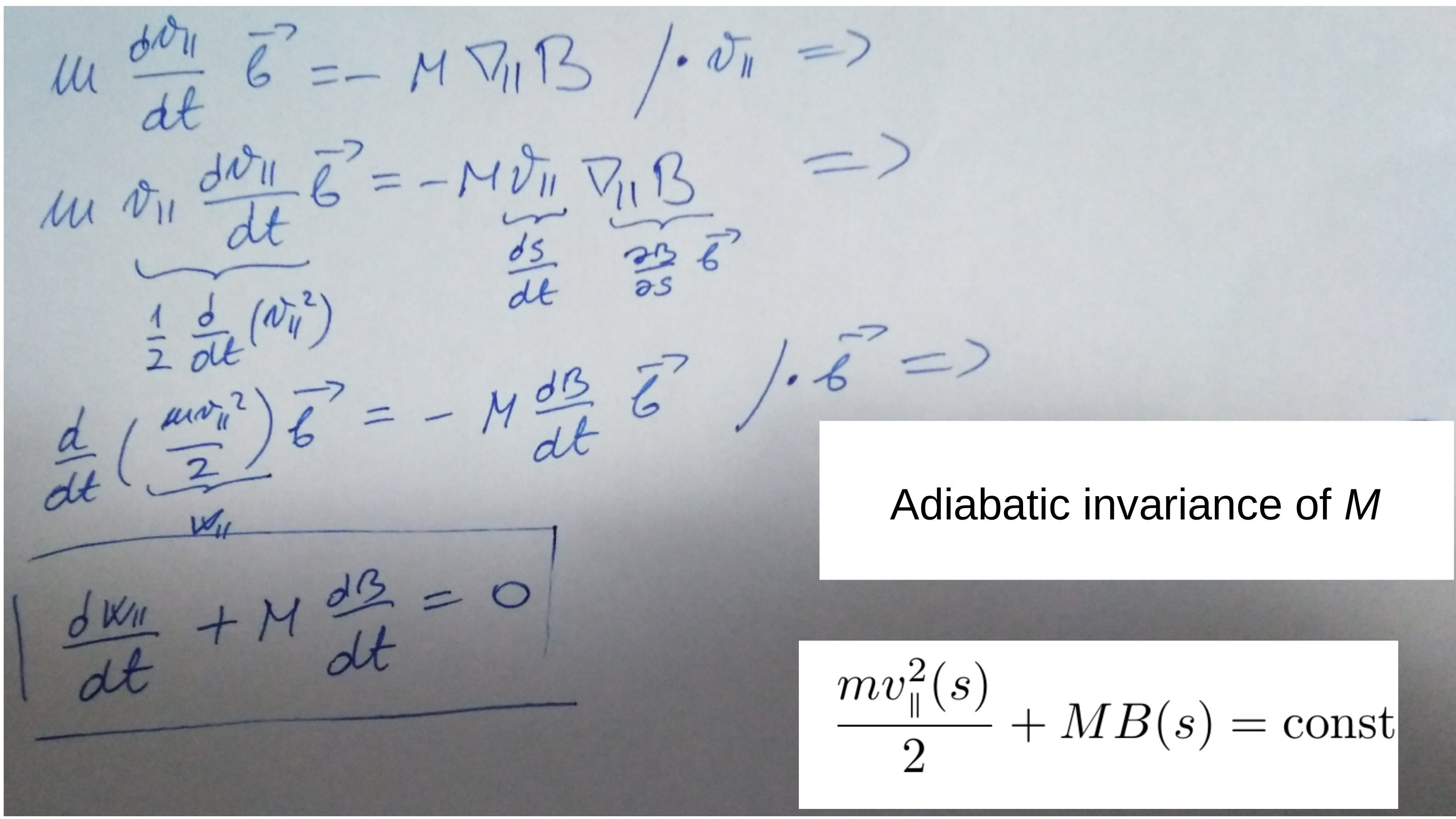
$$\vec{\mathcal{M}} = \sum_{\alpha} n_\alpha \langle \vec{M} \rangle_\alpha = -\frac{\mathrm{W}_\perp}{B}\vec{b} \equiv -\frac{p_\perp}{B}\vec{b}$$

$$\vec{j}_{\vec{\mathcal{M}}} = \nabla \times \vec{\mathcal{M}} = -\nabla \times \left(\frac{p_\perp}{B}\vec{b}\right)$$

Parallel motion

$$m\frac{dv_\parallel}{dt}\vec{b} = -M\nabla_\parallel B + m\vec{v}_\parallel\left(\vec{V}_\mathrm{D}\cdot\frac{\vec{\mathrm{e}}_\mathrm{n}}{R_\mathrm{C}}\right)$$

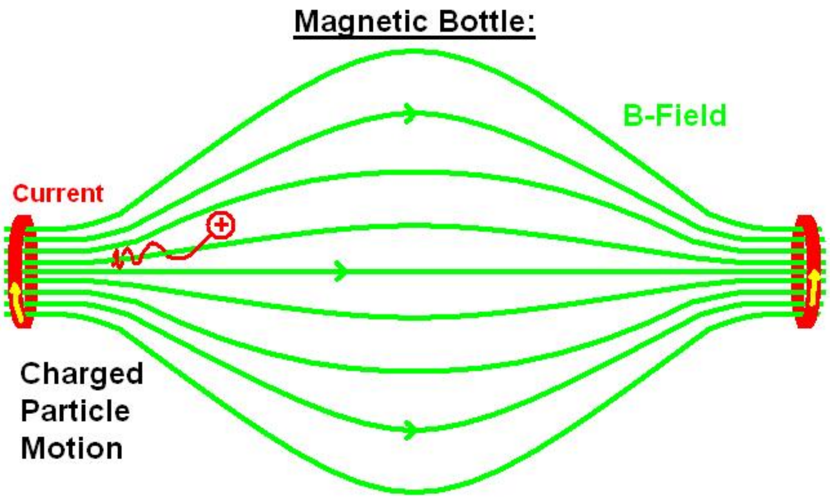
$$m\frac{dv_\parallel}{dt}\vec{b} = -M\nabla_\parallel B \longrightarrow \frac{d\mathrm{w}_\parallel}{dt} + M\frac{dB}{dt} = 0$$

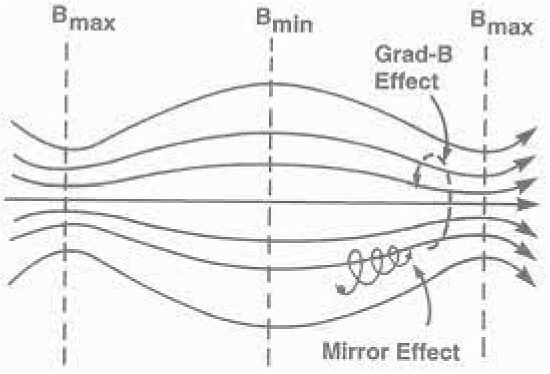


Adiabatic invariance of *M*

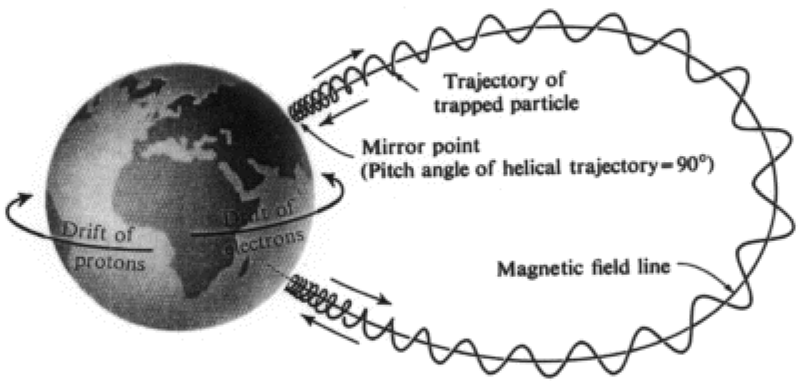
$$\frac{mv_\parallel^2(s)}{2} + MB(s) = \mathrm{const}$$

Magnetic mirrors





$$m\frac{dv_{\|}}{dt}\vec{b} = -M\nabla_{\|}B$$



$$r_c = v_c/|\omega_c| \propto B^{-1/2} \qquad v_\perp^2 \approx 2w_\perp^c/m \propto B$$

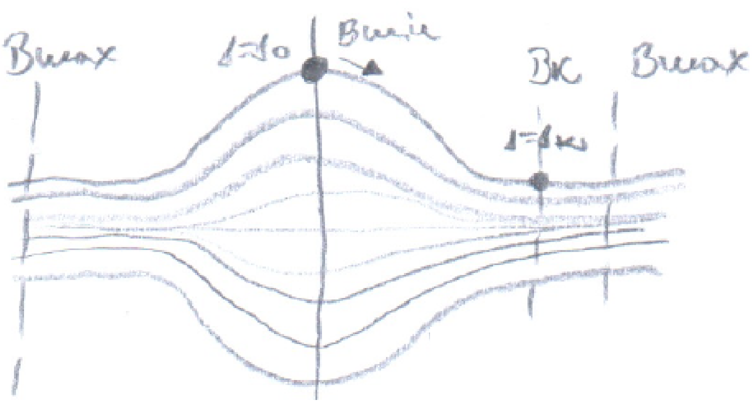
$$\frac{mv_\parallel^2(s)}{2} + MB(s) = MB_k$$

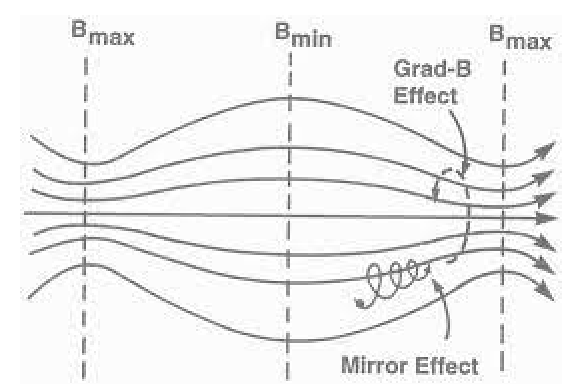
$$\mathrm{tg}\alpha = \frac{v_\perp}{v_\parallel} \approx \frac{1}{\sqrt{\frac{B_k}{B(s)} - 1}} \qquad \sin\alpha \approx \sqrt{B(s)/B_k}$$

$$B_k \leq B_{\max}$$

$$\downarrow$$

$$\sin\alpha_0 \geq \sqrt{\frac{B_{\min}}{B_{\max}}} \equiv \sin\alpha_{\mathrm{loss}}$$





The loss cone