Tvrđenje 7 (Gausove sume): $g(\chi_n') = \left(\frac{-2}{n}\right)(2+2i)n$

desna str. (6) $= \left(\frac{-2}{n}\right) \cdot (8n^2)^{1-s} \pi^{s-2} \Gamma(2-s) L(E_n, 2-s)$

$$N = 32n^2$$

$$\frac{\sqrt{N}}{2} = 2^{\frac{3}{2}} n$$

Pomnožimo (6) sa $\left(\frac{\sqrt{N}}{2}\right)^s$:

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(E_n, s) =$$

$$= \left(\frac{-2}{n}\right) \cdot 8^{1-s} \left(\frac{N}{32}\right)^{1-s} \left(\frac{\sqrt{N}}{2}\right)^s \frac{1}{\pi^{2-s}} \Gamma(2-s) L(E_n, 2-s)$$

$$= \left(\frac{-2}{n}\right) \cdot \left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s) L(E_n, 2-s)$$

$\blacksquare$

kraj dokaza Teoreme 1

Gausove sume na $\mathbb{Z}[i]$

$$\nu = \begin{cases} (2+2i)n, & n \text{ neparno} \\ 2n, & n \text{ parno} \end{cases}$$

$\chi_n' : \left(\mathbb{Z}[i]/\langle \nu \rangle\right)^{\times} \longrightarrow \mathbb{C}^*$ Dirichlet-ov karakter konduktora $\langle \nu \rangle$

(D) Definišemo aditivni karakter na $\mathbb{Z}[i]/\langle \nu \rangle$

$$\psi(\gamma) = e^{2\pi i \, \mathrm{Re}\left(\frac{\gamma}{\nu}\right)}$$

(D) Gauss-ova suma za Dir. karakter χ_n' je definisana sa

$$g(\chi_n') := \sum_{\gamma \in \mathbb{Z}[i]/\langle \nu \rangle} \chi_n'(\gamma) \, e^{2\pi i \, \mathrm{Re}\left(\frac{\gamma}{\nu}\right)}$$

- Želimo da izračunamo vrednosti ovih Gauss-ovih suma

<u>Zadatak 1</u>

$$g(\chi_1') = 2+2i$$
$$g(\chi_2') = 4i$$

- Za m neparan bezkvadratan $\in \mathbb{Z}_{>0}$ označimo sa

$$\left(\frac{\cdot}{m}\right) : \left(\mathbb{Z}[i]/\langle m \rangle\right)^{\times} \to \mathbb{C}^{\times}$$

multiplikativni karakter definisan sa $\gamma \longmapsto \left(\frac{N\gamma}{m}\right)$

Pokazali smo:

$$\chi_n' = \begin{cases} \chi_1' \cdot \left(\frac{\cdot}{n}\right), & n \text{ neparno} \\ \chi_2' \cdot \left(\frac{\cdot}{n_0}\right), & n = 2n_0 \text{ parno} \end{cases}$$

- Gaussova suma karaktera $\left(\frac{\cdot}{m}\right)$ je definisana sa

$$g\left(\left(\frac{\cdot}{m}\right)\right) := \sum_{\gamma \in \mathbb{Z}[i]/\langle m\rangle} \left(\frac{N\gamma}{m}\right) e^{2\pi i \operatorname{Re}\left(\frac{\gamma}{m}\right)}$$

$\ulcorner$ $\gamma \mapsto 2\gamma$ je bijekcija na $(\mathbb{Z}[i]/\langle m\rangle)^\times$

$N(2\gamma) = 4N\gamma$, pa je $\left(\frac{N(2\gamma)}{m}\right) = \left(\frac{N\gamma}{m}\right)$

$Tr(\gamma) = \gamma + \bar{\gamma}$ $\quad (Tr : \mathbb{Q}(i) \to \mathbb{Q})$

$\operatorname{Re}\left(\frac{2\gamma}{m}\right) = \frac{Tr(\gamma)}{m}$ $\lrcorner$

$$g\left(\left(\frac{\cdot}{m}\right)\right) = \sum_{\gamma \in \mathbb{Z}[i]/\langle m\rangle} \left(\frac{N\gamma}{m}\right) e^{2\pi i \frac{Tr\gamma}{m}}$$

<u>Lema 2</u> Za n neparno važi

$$\boxed{g(\chi_n') = \left(\frac{-2}{n}\right) g(\chi_1')\, g\left(\left(\frac{\cdot}{n}\right)\right)}$$

γ_1 prolazi klasama ostataka $\mathbb{Z}[i]/\langle n\rangle$

γ_2 prolazi klasama ostataka $\mathbb{Z}[i]/\langle 2+2i\rangle$

Kineski t. o ostacima:

$\gamma = (2+2i)\gamma_1 + n\gamma_2$ prolazi klasama ostataka $\mathbb{Z}[i]/\langle(2+2i)n\rangle = \mathbb{Z}[i]/\langle\nu\rangle$

$$\chi_n'(\gamma) = \chi_1'(\gamma)\left(\frac{N\gamma}{n}\right) = \chi_1'(n\gamma_2)\left(\frac{N((2+2i)\gamma_1)}{n}\right)$$

$$\chi_1'(n) = \begin{Bmatrix} 1, & n\equiv 1\ (4) \\ -1, & n\equiv 3\ (4)\end{Bmatrix} = \left(\frac{-1}{n}\right)$$

$$n\cdot i^s \equiv 1 \pmod{2+2i}$$

$$\left(\frac{N((2+2i)\gamma_1)}{n}\right) = \left(\frac{8N\gamma_1}{n}\right) = \left(\frac{2N\gamma_1}{n}\right) = \left(\frac{2}{n}\right)\left(\frac{N\gamma_1}{n}\right)$$

$$Re\left(\frac{\gamma}{\nu}\right) = Re\left(\frac{(2+2i)\gamma_1 + n\gamma_2}{(2+2i)n}\right) = Re\left(\frac{\gamma_1}{n} + \frac{\gamma_2}{2+2i}\right)$$

Pa je

$$g(\chi_n') = \left(\frac{-1}{n}\right)\left(\frac{2}{n}\right)\sum_{\substack{\gamma_1\in\mathbb{Z}[i]/\langle n\rangle \\ \gamma_2\in\mathbb{Z}[i]/\langle 2+2i\rangle}} \chi_1'(\gamma_2)\left(\frac{N\gamma_1}{n}\right)e^{2\pi i\, Re\left(\frac{\gamma_1}{n}\right) + 2\pi i\, Re\frac{\gamma_2}{2+2i}}$$

$$= \left(\frac{-2}{n}\right)\underbrace{\sum_{\gamma_1\in\mathbb{Z}[i]/\langle n\rangle}\left(\frac{N\gamma_1}{n}\right)e^{2\pi i\, Re\left(\frac{\gamma_1}{n}\right)}}_{=g\left(\left(\frac{\cdot}{n}\right)\right)} \cdot \underbrace{\sum_{\gamma_2\in\mathbb{Z}[i]/\langle 2+2i\rangle}\chi_1'(\gamma_2)e^{2\pi i\, Re\frac{\gamma_2}{2+2i}}}_{=g(\chi_1')}$$

Zadatak 3 Za $n = 2n_0$ parno (n_0 neparno) važi

$$\boxed{g(\chi_n') = \left(\frac{-1}{n_0}\right) g(\chi_2')\, g\left(\left(\frac{\cdot}{n_0}\right)\right)}$$

Lema 4 Za m neparan beskvadratan, za svaku faktorizaciju $m = m_1 m_2$ važi

$$g\left(\left(\frac{\cdot}{m}\right)\right) = g\left(\left(\frac{\cdot}{m_1}\right)\right) g\left(\left(\frac{\cdot}{m_2}\right)\right)$$

▽ Kineska t. o ostacima ▲

Zadatak 5 Neka je χ_2 multipl. karakter reda 2 na konačnom polju $\mathbb{F}_q$ (q stepen neparnog prostog). Dokažite da je

$$g(\chi_2)^2 = (-1)^{\frac{q-1}{2}} q$$

gde je $g(\chi_2)$ Gauss-ova suma na konačnom polju.

Lema 6 Ako je p neparan prost broj, onda je

$$\boxed{g\left(\left(\frac{\cdot}{p}\right)\right) = p}$$

▽ **Slučaj $p \equiv 1\ (4)$** $\quad p = \beta \cdot \overline{\beta}, \quad \beta = a + bi$

$r_1, r_2 \in \{0, 1, 2, \ldots, p-1\}$ skup klasa ostataka $\mathbb{Z}[i]/\langle\beta\rangle$ i $\mathbb{Z}[i]/\langle\overline{\beta}\rangle$

β i $\bar\beta$ su uzajamno prosti $\to$ Kineska + o ostacima:

$\gamma = \gamma_1 \beta + \gamma_2 \bar\beta$ prolazi klasama ostataka $\mathbb{Z}[i]/\langle p\rangle$

$$N\gamma = \gamma\cdot\bar\gamma = (\gamma_1\beta + \gamma_2\bar\beta)(\gamma_1\bar\beta + \gamma_2\beta) \equiv 2\gamma_1\gamma_2 \operatorname{Re}(\beta^2) \pmod p$$

$$\operatorname{Re}(\beta^2) = a^2 - b^2 = 2a^2 - p \equiv 2a^2 \pmod p$$

$$\operatorname{Re}\gamma = \gamma_1 a + \gamma_2 a$$

pa imamo

$$g\left(\left(\frac{\cdot}{p}\right)\right) = \sum_{\gamma\in\mathbb{Z}[i]/\langle p\rangle} \left(\frac{N\gamma}{p}\right) e^{2\pi i \operatorname{Re}\left(\frac{\gamma}{p}\right)}$$

$$= \sum_{\gamma_1,\gamma_2\in\mathbb{Z}/p\mathbb{Z}} \left(\frac{2\gamma_1\gamma_2\cdot 2a^2}{p}\right) e^{2\pi i\left(\frac{a\gamma_1}{p} + \frac{a\gamma_2}{p}\right)}$$

$$= \Bigg(\underbrace{\sum_{\gamma_1\in\mathbb{Z}/p\mathbb{Z}} \left(\frac{\gamma_1}{p}\right) e^{2\pi i\frac{\gamma_1}{p}}}_{g_{\mathbb{F}_p}(\chi_2)}\Bigg)^2 = g(\chi_2)^2 \overset{\text{Zad 5}}{=} (-1)^{\frac{p-1}{2}} p = p$$

<u>Slučaj $p\equiv 3\ (4)$</u> $\quad \mathbb{Z}[i]/\langle p\rangle \cong \mathbb{F}_{p^2}$.

Gauss-ovu sumu $g\left(\left(\frac{\cdot}{p}\right)\right)$ na $\mathbb{Z}[i]$ možemo da prepoznamo kao Gaussovu sumu na $\mathbb{F}_{p^2}$ ~~ko~~ za par karaktere nastao podizanjem multiplikativnog $\left(\frac{\cdot}{p}\right) = \chi_2$ i aditivnog karaktere $e^{2\pi i\frac{\cdot}{p}}$ sa $\mathbb{F}_p$ na $\mathbb{F}_{p^2}$ (koristeći $N:\mathbb{F}_{p^2}\to\mathbb{F}_p$, $Tr:\mathbb{F}_{p^2}\to\mathbb{F}_p$)

Primenite Hasse-Davenportovu relaciju i Zadatak 5

Tvrdnja 7 Za svako beskvadratno n imamo

$$g(\chi_n') = \begin{cases} \left(\frac{-2}{n}\right)\nu, & n \text{ neparno} \\ \left(\frac{-1}{n_0}\right)i\nu, & n = 2n_0 \text{ parno} \end{cases}$$

$\triangledown$ Spojite prethodne leme i zadatke $\triangle$

Lema 8 $S_m = S_{(m_1,m_2)} = \sum_{0 \le a,b < 4n} \chi_n'(a+bi) e^{\frac{2\pi i}{4n} m\cdot(a,b)}$

Onda je

$$S_m = \begin{cases} 0, & \text{ako } m_1+m_2 i \notin \langle 1+i\rangle \\ 2\chi_n'(\gamma) g(\chi_n'), & \text{ako } m_1+m_2 i = (1+i)\gamma \in \langle 1+i \rangle \end{cases}$$

$\triangledown$ slučaj $(1+i) \nmid m_1+m_2 i \iff m_1, m_2$ suprotne parnosti tj. m_1+m_2 neparno

Sumacija u S_m ima $(4n)^2 = 16n^2$ članova, a konduktor $\nu = (2+2i)n$ ima normu $N(\nu) = N(\langle\nu\rangle) = 8n^2$

$\rightarrow$ lako se pokaže da kad $0 \le a, b < 4n$, $a+bi$ prolazi klasama ostataka $\mathbb{Z}[i]/\langle\nu\rangle$ tačno 2 puta.

- Ako su $a_1+b_1 i$, $a_2+b_2 i$ u različitim klasama modulo $4n$, ali u istoj klasi modulo $\nu = (2+2i)n$, onda je

$$\chi_n'(a_1+b_1 i) = \chi_n'(a_2+b_2 i)$$

ali iz

$$(a_1+b_1 i)-(a_2+b_2 i) \equiv (2+2i)n \pmod{4n}$$

slijedi da je

$$e^{2\pi i \frac{m\cdot((a_1,b_1)-(a_2,b_2))}{4n}} = e^{2\pi i \frac{m\cdot(2n,2n)}{4n}} = e^{\pi i (m_1+m_2)} = -1$$

pa se članovi u sumi S_m koji odgovaraju (a_1,b_1) i (a_2,b_2) pokrate i $S_m=0$.

<u>Slučaj $m_1+m_2 i=(1+i)\gamma$</u>

$$m\cdot(a,b) = (m_1,m_2)\cdot(a,b) = m_1 a + m_2 b =$$

$$= \operatorname{Re}\left((m_1-m_2 i)(a+bi)\right) = \operatorname{Re}\left((1-i)\overline{\gamma}(a+bi)\right)$$

Aditivni karakter iz sume S_m je onda

$$e^{2\pi i \frac{m\cdot(a,b)}{4n}} = e^{2\pi i \operatorname{Re}\left(\frac{(1-i)\overline{\gamma}(a+bi)}{(1-i)(1+i)2n}\right)} = e^{2\pi i \operatorname{Re}\left(\frac{\overline{\gamma}(a+bi)}{(1+i)\cdot 2n}\right)}$$

$$= e^{2\pi i \operatorname{Re}\left(\frac{\overline{\gamma}(a+bi)}{\nu}\right)} \qquad \nu=(2+2i)n$$

- χ_n' je <u>primitivni</u> karakter modulo $(2+2i)n$
- Imamo:

$$S_m = \sum_{0\le a,b<4n} \dots = 2 \sum_{a+bi\in \mathbb{Z}[i]/\langle\nu\rangle} \chi_n'(a+bi)\cdot e^{2\pi i \operatorname{Re}\left(\frac{\overline{\gamma}(a+bi)}{\nu}\right)} =$$

(8. domaći zadatak)

$$= 2\,\overline{\chi_n'}(\overline{\gamma})\, g(\chi_n') = 2\,\chi_n'(\gamma)\, g(\chi_n')$$