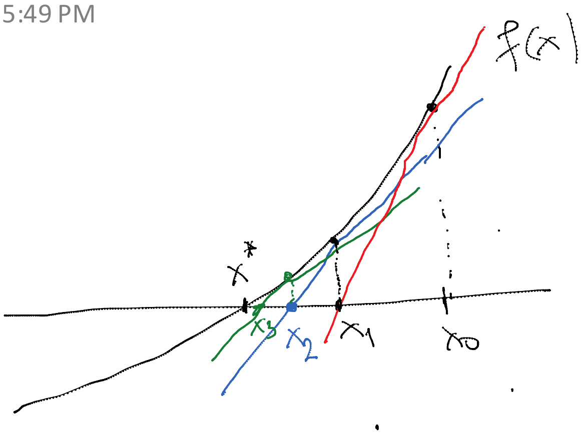
Newton

Tuesday, May 18, 2021
5:49 PM



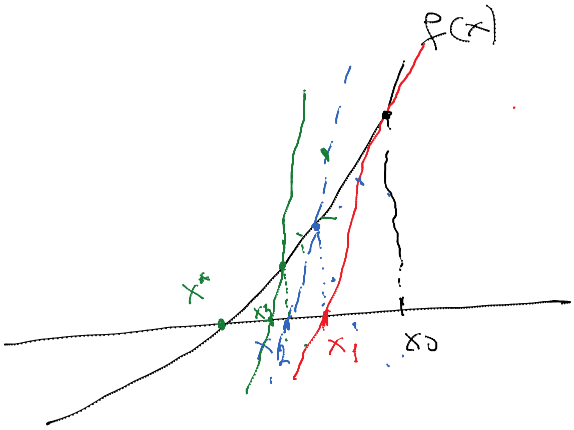
$f(x^*) = 0$

$(x_n, f(x_n))$:

$$\begin{cases} y = f'(x_n)(x - x_n) + f(x_n) \\ y = 0 \end{cases}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Modifikacija Njutnove metode:



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}$$

(T) Ako $f : [a,b] \to \mathbb{R}$ ima sl. osobine:

a) $f \in C[a,b]$

b) $f(a) \cdot f(b) < 0$

c) $\forall x \in [a,b],\ \exists f''(x)$

d) na $[a,b]$, $f'(x)$, $f''(x)$ ne menjaju znak, $f'(x) \neq 0$

e) $x_0 \in [a,b]$ t.d. $f(x_0) \cdot f''(x_0) > 0$

Onda niz $\{x_n\}$ sa prvim članom x_0, određen formulom $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ konvergira ka rešenju $x^* \in [a,b]$ jednačine $f(x^*) = 0$.

Tuesday, May 18, 2021
5:49 PM

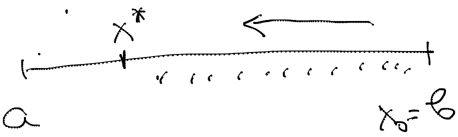
(D) (a), (b) $\Rightarrow$ $f(x)=0$ ima rešenje $x^* \in [a,b]$
(d) $f' \gtrless 0 \Rightarrow f$ monotona

$\Big\}$ $\exists!$ rešenje $x^* \in [a,b]$

pp. $f(a)<0$, $f(b)>0$, $f'(x)>0$, $f''(x)>0$, $\forall x \in [a,b]$

(e) biramo x_0 : $\underbrace{f(x_0)}\cdot\underbrace{f''(x_0)}_{>0}>0$

za $x_0=b$ ✓



$\{x_n\}$:

*) $x_n > x^*$ (Indukcijom)

$n=0$: $x_0 > x^*$ ✓

$x_k > x^*, \forall k \le n \overset{?}{\Rightarrow} x_{n+1} > x^*$

$$0 = f(x^*) = f(x_n + \underbrace{x^* - x_n})$$

$$= \underbrace{f(x_n) + f'(x_n)\cdot(x^*-x_n)} + \underbrace{\tfrac{1}{2}\underbrace{f''(\xi)}_{>0}\underbrace{(x^*-x_n)^2}_{\geq 0}}_{>0}, \quad \xi \in (x^*, x_n)$$

$$\Rightarrow f(x_n) + f'(x_n)(x^*-x_n) < 0$$

$$f(x_n) < -\underbrace{f'(x_n)}_{>0}(x^*-x_n) \quad /: f'(x_n)>0$$

$$\frac{f(x_n)}{f'(x_n)} < -x^* + x_n \quad (-1$$

$$\underbrace{x_n - \frac{f(x_n)}{f'(x_n)}} > x^*$$

$$x_{n+1} > x^*$$

$\Rightarrow \boxed{x_n > x^* \ \forall n}$

**) f je monotona, i $x_n > x^* \Rightarrow f(x_n) > 0$

$$x_{n+1} = x_n - \frac{f(x_n) \ {>0}}{f'(x_n) \ {>0}}$$

$\Rightarrow x_{n+1} < x_n$

$\Rightarrow \{x_n\} \searrow$

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Tuesday, May 18, 2021
5:49 PM

(*) i (**): $\{x_n\} \searrow x^*$ $\Rightarrow \exists \lim_{n\to\infty} x_n = \bar{x}$, $\bar{x} \in [a,b]$

$\bar{x} = x^*$?

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \Big\backslash \lim$$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n - \frac{f(\lim x_n)}{f'(\lim x_n)}$$

$$\bar{x} = \bar{x} - \frac{f(\bar{x})}{\underbrace{f'(\bar{x})}_{\neq 0}} \Rightarrow f(\bar{x}) = 0$$

$\Rightarrow \bar{x}$ rešenje jednačine $f(x)=0$

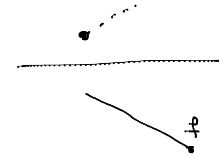
(zbog jedinstvenosti (sa početka)) $\Rightarrow \bar{x} = x^*$ ☒

Ocena greške:

T. o sr. vr.: $\underbrace{f(x^*)}_{=0} - f(x_n) = f'(\xi)(x^* - x_n)$, $\xi \in (x_n, x^*)$

$$|f(x_n)| = |f'(\xi)| \cdot |x^* - x_n|$$

$m_1 = \min_{x\in[a,b]} |f'(x)|$



$$|f(x_n)| \geq m_1 |x^* - x_n|$$

$$\boxed{|x^* - x_n| \leq \frac{|f(x_n)|}{m_1}}$$

Ocena greške ako je x_n određeno N. Njutnom:

$$f(x_n) = f(x_{n-1} + x_n - x_{n-1})$$

$$= \underbrace{f(x_{n-1}) + f'(x_{n-1}) \cdot (x_n - x_{n-1})}_{=0 \text{ (f-la N. met.)}} + \frac{1}{2} f''(\xi)(x_n - x_{n-1})^2$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$M_2 = \max_{x\in[a,b]} |f''(x)|$

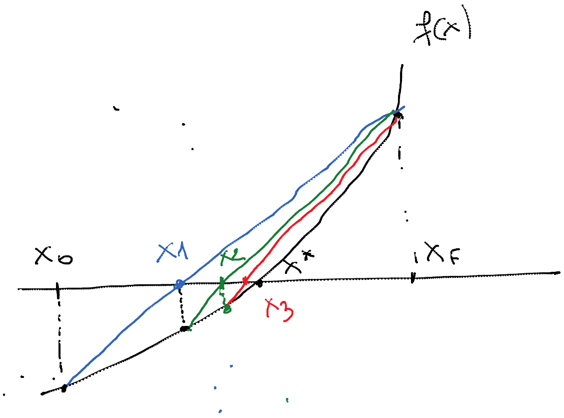
$$|f(x_n)| = \frac{1}{2} |f''(\xi)| \cdot |x_n - x_{n-1}|^2$$

$$\leq \frac{1}{2} M_2 |x_n - x_{n-1}|^2$$

$$\Rightarrow \boxed{|x^* - x_n| \leq \frac{M_2}{2m_1} \cdot |x_n - x_{n-1}|^2}$$

Regula Falsi (lažnog položaja)

Tuesday, May 18, 2021
5:49 PM



x_0, x_F

$(x_0, f(x_0))$, $(x_F, f(x_F))$

$$\begin{cases} \dfrac{y - f(x_n)}{f(x_F) - f(x_n)} = \dfrac{x - x_n}{x_F - x_n} \\ y = 0 \end{cases}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f(x_F) - f(x_n)}(x_F - x_n)}$$

Ocjena greške: $| - x^*$

$f(x^*) = 0$

$$x_{n+1} - x^* = x_n - x^* - \frac{f(x_n) - f(x^*)}{f(x_F) - f(x_n)}(x_F - x_n)$$

$\xi_1 \in [x^*, x_n]$

$\xi_2 \in [x_F, x_n]$

$$= x_n - x^* - \frac{f'(\xi_1)(x_n - x^*)}{f'(\xi_2)\cancel{(x_F - x_n)}} \cdot \cancel{(x_F - x_n)}$$

$$= (x_n - x^*)\left[1 - \frac{f'(\xi_1)}{f'(\xi_2)}\right]$$

$$= (x_n - x^* + x_{n+1} - x_{n+1}) \cdot [-\|-]$$

$$= (x_n - x_{n+1})[-\|-] + (x_{n+1} - x^*)[-\|-]$$

$$(x_{n+1} - x^*)\left(1 - 1 + \frac{f'(\xi_1)}{f'(\xi_2)}\right) = (x_n - x_{n+1}) \cdot \left(\frac{f'(\xi_2) - f'(\xi_1)}{f'(\xi_2)}\right)$$

$$x_{n+1} - x^* = (x_n - x_{n+1}) \cdot \frac{f'(\xi_2) - f'(\xi_1)}{\cancel{f'(\xi_2)}} \cdot \frac{\cancel{f'(\xi_2)}}{f'(\xi_1)}$$

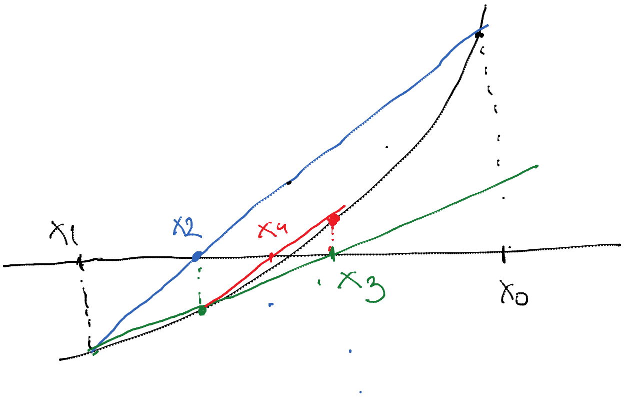
$m_1 = \min |f'(x)|$, $M_1 = \max |f'(x)|$

$0 \le m_1 \le |f'(x)| \le M_1 < \infty$

$$|x_{n+1} - x^*| \le \frac{M_1 - m_1}{m_1} \cdot |x_{n+1} - x_n|$$

Metoda sečice

Tuesday, May 18, 2021
5:49 PM



x_0, x_1

x_1, x_2

x_2, x_3

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_{n-1}) - f(x_n)} \cdot (x_{n-1} - x_n)$$

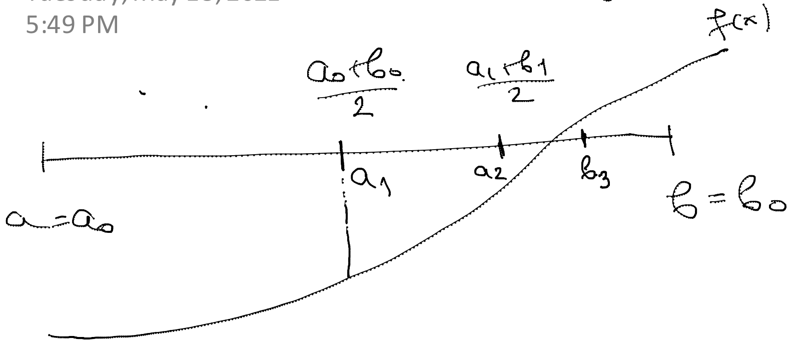
Sečica + Newton: $(x_n, f(x_n))$, $(\bar{x}_n, f(\bar{x}_n))$

$$\bar{x}_n = \bar{x}_{n-1} - \frac{f(\bar{x}_{n-1})}{f'(\bar{x}_{n-1})} \quad \text{(Newton)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f(\bar{x}_n) - f(x_n)} (\bar{x}_n - x_n)$$

Metoda polovljenja intervala

Tuesday, May 18, 2021
5:49 PM



$[a_0, b_0]$

$[a_1, b_0] = [a_1, b_1]$ iteracija

$[a_2, b_0] = [a_2, b_2]$

$[a_2, b_3] = [a_3, b_3]$

$$f\left(\frac{a_k+b_k}{2}\right) \approx 0 \Rightarrow x^* = \frac{a_k+b_k}{2}$$

$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \ldots \supset [a_n, b_n] \supset \ldots$$

$$f(a_n) \cdot f(b_n) < 0$$

$$b_n - a_n = \frac{1}{2^n} \cdot (b-a)$$

$$x^* \in [a_n, b_n]$$

(T) Neka $f \in C[a,b]$, $f(a) \cdot f(b) < 0$. Nizovi $\{a_n\}$ i $\{b_n\}$ levih i desnih krajeva intervala $[a_n, b_n]$ određenih na opisan način konvergiraju ka x^* koja je rešenje jednačine $f(x)=0$.

(D) $[a_0, b_0] \supset [a_1, b_1] \supset \ldots \supset [a_n, b_n] \supset \ldots$

$\Rightarrow$ $\underbrace{a_0 \leq a_1 \leq \ldots \leq a_n \leq \ldots}_{\text{monotono neopadajući}} \underbrace{< b_0}_{\text{ograničen odozgo}} \Rightarrow \exists \lim_{n\to\infty} a_n = A$

$\Rightarrow$ $\underbrace{b_0 \geq b_1 \geq \ldots \geq b_n \geq \ldots}_{\text{monotono nerastući}} \underbrace{> a_0}_{\text{ograničen odozdo}} \Rightarrow \exists \lim_{n\to\infty} b_n = B$

$$a_n \leq A \leq B \leq b_n$$

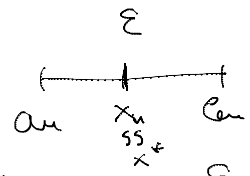
$$B - A = \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{1}{2^n}(b-a) = 0$$

$\Rightarrow A = B = x^*$ (jedinstvena, pripada svim $[a_n, b_n]$)

$f(x^*) = 0$?? : $\lim_{n\to\infty} \underbrace{\overset{\geq 0}{f(a_n)} \cdot \overset{\leq 0}{f(b_n)}}_{<0} = f(\lim_{n\to\infty} a_n) \cdot f(\lim_{n\to\infty} b_n) = f(A) \cdot f(B)$

$$= (f(x^*))^2 \leq 0 \quad \Rightarrow \quad f(x^*) = 0 \qquad \boxtimes$$

Tuesday, May 18, 2021
5:49 PM



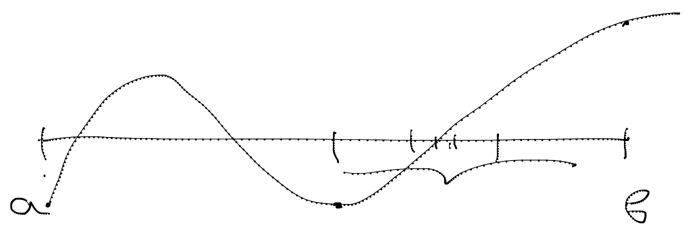
Ocena greške:

n-ta iteracija : $[a_n, b_n]$, $b_n - a_n < \varepsilon$

$$x_n = \frac{1}{2}(a_n + b_n)$$

$$|x^* - x_n| \le \frac{1}{2}\underbrace{(b_n - a_n)} = \frac{1}{2^{n+1}}(b - a)$$

$$n \geq \frac{\ln \frac{b-a}{\varepsilon}}{\ln 2}$$



$f(x) = x^2$

Metoda polovljenja intervala ne moze.
Sto se drugih metoda tice, ne ispunjava uslov f(a)f(b)<0 pa nece vaziti uslovi teorema.
Medjutim, teoreme za konvergenciju samo definisu uslove pod kojima sigurno postoji konvergencija. Ne vazi ekvivalencija. Drugim recima, konvergencija moze postojati cak i ako nisu ispunjeni uslovi, samo to ne mozemo dokazati teoremom. U praksi, mozemo probati primenu metode (Njutn, iterativna...) i videti da li ce konvergirati. U zavisnosti od f(x) i izbora x0 moze postojati konvergencija i u takvim slucajevima.