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(T) Ako je neka norma matrice B (saglasna sa nekom normom vektora) manja od 1, tada iterativni proces $x_{k+1} = Bx_k + c$ konvergira.

(D) $\|Bx\| \le \|B\| \cdot \|x\|$ (saglasnost normi)

λ_i - s. vred B

v_i - s. vektor λ_i

$$B v_i = \lambda_i v_i$$

$$|\lambda_i| \cdot \cancel{\|v_i\|} = \|\lambda_i v_i\| = \|B v_i\| \le \|B\| \cdot \cancel{\|v_i\|}, \quad \|v_i\| \ne 0$$

$$|\lambda_i| \le \underbrace{\|B\| < 1}$$

$$\|B\| < 1 \Rightarrow |\lambda_i| < 1$$

$\Rightarrow$ konv. $\boxtimes$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}\right]$$

metoda proste iteracije (Jakobijeva)

Gauss-Zajdelova metoda

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(-a_{12}x_2^{(k)} - \dots - a_{1n}x_n^{(k)}\right) + \frac{b_1}{a_{11}}$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(-a_{21}x_1^{\overset{(k+1)}{(k)}} - a_{23}x_3^{(k)} - \dots - a_{2n}x_n^{(k)}\right) + \frac{b_2}{a_{22}}$$

$$\vdots$$

$$x_n^{(k+1)} = \underbrace{\frac{1}{a_{nn}}\left(-a_{n1}x_1^{\overset{(k+1)}{(k)}} - \dots - a_{n,n-1}x_{n-1}^{\overset{(k+1)}{(k)}}\right)} + \frac{b_n}{a_{nn}}$$

$$\boxed{x^{(k+1)} = B \cdot x^{(k)} + c}$$

$$B = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & \dots & -\frac{a_{1n}}{a_{11}} \\ \vdots & & & \\ -\frac{a_{n1}}{a_{nn}} & \dots & \dots & 0 \end{bmatrix}, \quad c = \begin{bmatrix} \frac{b_1}{a_{11}} \\ \vdots \\ \frac{b_n}{a_{nn}} \end{bmatrix}$$

$B = [b_{ij}]$, $A = [a_{ij}]$

$$\|B\|_\infty = \max_i \sum_j^n |b_{ij}| = \max_i \sum_{\substack{j=1 \\ j \ne i}}^n \left| \frac{a_{ij}}{a_{ii}} \right| < 1$$

$$\Rightarrow \boxed{\sum^n |a_{ij}| < 1 \cdot |a_{ii}|}, \quad \forall i = 1, \dots, n$$

dijagonalna dominantnost (po vrstama)

$$\Rightarrow \boxed{\sum_{j=1,\, j \neq i}^{n} |a_{ij}| < 1 \cdot |a_{ii}|} \quad , \forall i = 1, \ldots, n$$

dijagonalna dominantnost (po vrstama)

$$\frac{a_{21}}{a_{22}} + \frac{a_{31}}{a_{33}} + \ldots + \frac{a_{n1}}{a_{nn}} \rightarrow \text{kolone} \qquad (\|B\|_1)$$

$$\frac{a_{12}}{a_{11}} + \frac{a_{13}}{a_{11}} + \ldots + \frac{a_{1n}}{a_{11}} \rightarrow \text{vrste} \qquad (\|B\|_\infty)$$

$q = ?$

$$X = \underbrace{Bx + C}$$

$$F(x) = Bx + C$$

$$F(x) - F(y) = Bx + \cancel{C} - By - \cancel{C}$$
$$= B(x-y)$$

$$\|F(x) - F(y)\| = \|B(x-y)\|$$
$$\leq \underbrace{\|B\|}_{<1} \cdot \|x-y\| \Rightarrow \|B\| = \underline{q}$$

$$\boxed{\|B\|_\infty = \underline{q}}$$

- apriorna / aposteriorna ocena greške

Numerička stabilnost

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$Ax = b$

male promene A, b → male promene x (stabilan)
→ velike –||– (nestabilan)

$\det(A) = 0$ ↗ nema rešenja
↘ ∞ mnogo rešenja

A, $\det(A) \neq 0$ → jedinstveno rešenje

$A_1, A_2, A_3, \ldots$ $\det(A_i) \approx 0$

Da li $\det(A)$ može da bude pokazatelj?

NE ;)

$\text{broj}^4 \cdot \det(A)$; $\det(A) = 0{,}01$

$A_1 = 10 \cdot A_{3\times3}$ $\det(A_1) = 10^3 \cdot \det(A) = 10$

$\text{cond}(A) = \|A\| \cdot \|A^{-1}\|$, cond(A) malo → bolje uslovljena (stabilnija)

cond(A) veliko → loše –||– (nestabilno)

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x^* - tačno rješ.
x' - približno rješ.

$$Ax^* = b$$
$$Ax' = b'$$
$b \neq b'$

$$A(x'-x^*) = b'-b$$
$$x'-x^* = A^{-1}(b'-b)$$
$$\|x'-x^*\| = \|A^{-1}(b'-b)\|$$
$$\leq \|A^{-1}\| \cdot \|b'-b\| \quad / : \|b\|$$

$$\frac{\|x'-x^*\|}{\|b\|} \leq \|A^{-1}\| \cdot \frac{\|b'-b\|}{\|b\|}, \quad Ax^* = b$$

$$\frac{\|x'-x^*\|}{\|A\| \cdot \|x^*\|} \leq \|A^{-1}\| \cdot \frac{\|b'-b\|}{\|b\|}$$

$$\underbrace{\frac{\|x'-x^*\|}{\|x^*\|}}_{\text{Rel. greška rješenja}} \leq \underbrace{\|A\| \cdot \|A^{-1}\|}_{\text{cond}(A)} \cdot \underbrace{\frac{\|b'-b\|}{\|b\|}}_{\text{Rel. greška vektora desne strane}}$$

Slično ako dozvolimo i A da se mijenja.

$$\rightarrow A'$$

$$\text{cond}(A), \ \text{cond}(A')$$

$$\frac{\|x'-x^*\|}{\|x^*\|} \leq \text{cond}(A) \left(\ldots\ldots \ \text{cond}(A') \ldots \right)$$

Nelinearne jednačine

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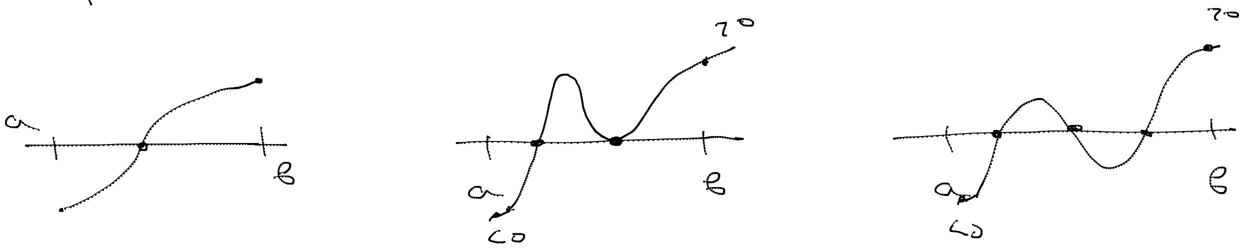
$f(x)=0$ (1 jna, 1 nepoznata!)

* Lokalizacija rešenja

$[a,b] \ni$ tačno 1 rešenje

(T) Ako $f \in C[a,b]$, $f(a)\cdot f(b)<0 \Rightarrow f(x)=0$ ima bar 1 nule $x^* \in (a,b)$



(T) Ako $f \in C(a,b)$ i monotona na $[a,b]$ i $f(a)\cdot f(b)<0$

$\Rightarrow f(x)=0$ ima jedinstveno nule $x^* \in (a,b)$

Metoda proste iteracije

$x^{(n+1)} = g(x^{(n)})$, x_0 — nije jednoznačna!

$x_1, x_2, x_3, \ldots \to x^*$ } na $[a,b]$

dvoslojna, stacionarna, reda 1 $\Rightarrow$ Banahova (T)

(T) Ako je funkcija $g(x)$ diferencijabilna, $|g'(x)| \le q < 1$, $x \in B$ i $g: B \to B$ tada iterativni proces konvergira $x_n \to x^*$, $n \to \infty$ i važi:

$$|x_n - x^*| \le \frac{q^n}{1-q} |x_1 - x_0| \quad \text{apriorna}$$

$$|x_n - x^*| \le \frac{q}{1-q} |x_n - x_{n-1}| \quad \text{aposteriorna}$$

(D) Lagranžova t. o srednjoj vrednosti:

$$|g(x)-g(y)| = |g'(\xi)|\cdot|x-y| \quad \xi\in(a,b)$$

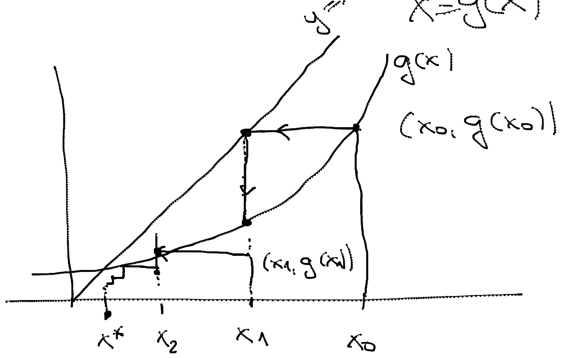
$$\le \underbrace{\max_{\xi\in(a,b)}|g'(\xi)|}_{<1} \cdot |x-y| \Rightarrow q = \max_{\xi\in[a,b]}|g'(\xi)|$$

Ostalo iz Banahove t. ☒

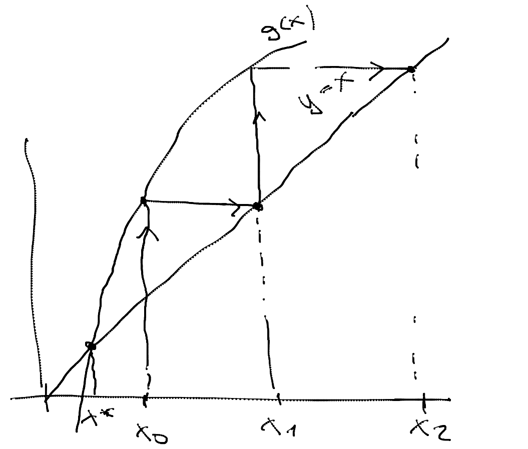
$x = g(x)$ nije $|g'(\xi)|<1$ → smanjiti $[a,b]$

$x = g_2(x)$ . . . .

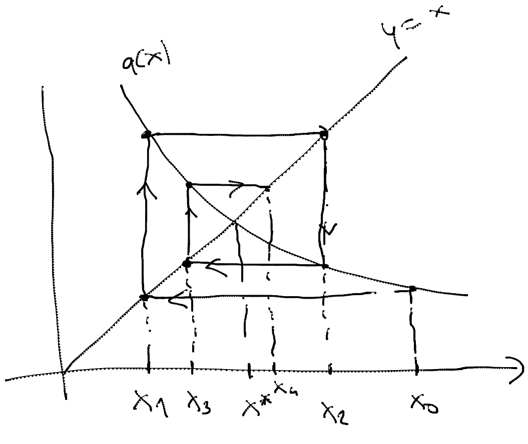
$x = g(x)$

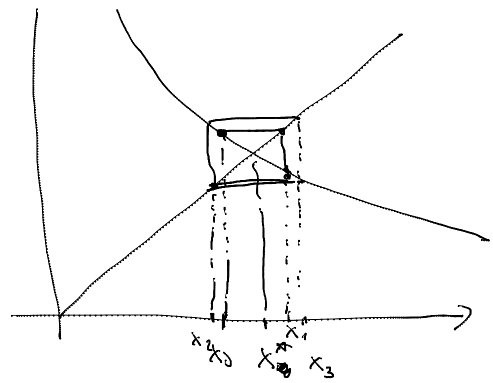


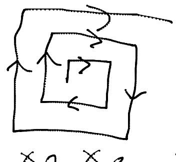
$0 < q < 1$



$q > 1$







samokorigujuća metoda (proizvoljno x_0)

$x_0, x_1, x_2, x_3 \to x^*$

pogrešiti (a,b)

$= x_0', x_1', x_2' \to x^*$