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Lema (❀): Ako su x_i, $i=1,\dots,n$ čvorovi KF koja je tačna za sve polinome stepena $2n-1$ onda je

$$\int_a^b p(x)\,\underbrace{\omega_n(x)\cdot P_{n-1}(x)}\,dx = 0$$

gde $P_{n-1}(x)$ proizvoljni poly. stepena $n-1$, a $\omega_n^{(x)}=\prod_{i=1}^{n}(x-x_i)$.

(D): KF tačna za poly st. $2n-1$

$st(\omega_n(x)\cdot P_{n-1}(x)) = 2n-1 \Rightarrow$ KF tačna i za njega

$$\int_a^b p(x)\,\omega_n(x)\cdot P_{n-1}(x)\,dx = \frac{b-a}{2}\sum_{i=1}^{n} A_i\cdot \underbrace{\omega_n(x_i)}_{=0}\cdot P_{n-1}(x_i)$$

$$= 0$$

lema primena

$$\omega_n(x) = \prod_{i=1}^{n}(x-x_i) = x^n + b_1\cdot x^{n-1} + \dots + b_{n-1}x + b_n$$

$$\int_a^b p(x)\,\omega_n(x)\,P_{n-1}(x)\,dx = 0 \iff \boxed{\underbrace{\int_a^b p(x)\,\omega_n(x)\,x^k\,dx = 0 \quad k=0,\dots,n-1}_{\text{poly}}}$$

Linearan sistem n jna sa n nepoznatih b_i, $i=1,\dots,n$

Rešavanjem sistema $\Rightarrow$ $\omega_n(x)$
njegove nule x_i, $i=0,\dots,n-1$

Kako odrediti nule :(
↳ numerika :)

- da li su sve nule jednostruke? :(
- da li su sve nule realne? :(
- da li sve nule $\in(a,b)$? :(

ORTOGONALNI POLINOMI

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$L^2(a,b)$

$(f,g) = \int_a^b p(x)\cdot f(x)\, g(x)\, dx$

$(f,f) = \int_a^b p(x)\, f^2(x)\, dx$ $\quad \left[\begin{array}{l} \|f\|^2 = (f,f) \\ p(x) \geq 0 \end{array} \right]$

$(f,g) = 0$ ortogonalne

$\{f_k(x)\},\ k = 0,1,2,\dots$ ortogonalan sistem $(f_i,f_j) = 0\ \forall i \neq j$

$f_k(x) = Q_k(x)$ sistem ortog. poly.

normiran poly. $\equiv$ moničan $(1\cdot x^n + \dots)$

(T) $\exists$ normirani polinomi $Q_k(x)$, $k = 0,1,\dots$ takvi da je $(Q_i,Q_j) = 0$, $i \neq j$. Ovi polinomi su jedinstveno određeni rekurentnom f-lom:

$Q_0(x) = 1$

$$Q_{k+1}(x) = \Big(x - \underbrace{\frac{(xQ_k,Q_k)}{(Q_k,Q_k)}}_{a_k}\Big)\cdot Q_k(x) - \underbrace{\frac{(Q_k,Q_k)}{(Q_{k-1},Q_{k-1})}}_{a_{k-1}}\cdot Q_{k-1}(x)$$

pri čemu je drugi sabirak $= 0$ za $k = 0$.

(D) pp. da važi za sve Q_j, $j \leq k$

$$Q_{k+1}(x) = (x - a_k)\, Q_k(x) + a_{k-1} Q_{k-1}(x) + \dots + a_0 \cdot Q_0(x) \quad \Big| \cdot Q_j \quad _{j=0,\dots,k}$$

$a_j = ?$ $\quad (Q_{k+1},Q_j) = 0 \quad \forall j = 0,\dots,k$

već važi (IH) $(Q_i,Q_j) = 0$ $i \neq j$, $i,j \leq k$

$$(Q_{k+1},Q_j) = ((x-a_k)Q_k, Q_j) + a_{k-1}(Q_{k-1},Q_j) + \dots + a_0 (Q_0,Q_j) \quad _{j=0,\dots,k}$$

$j = 0,\dots,k-1$: $\quad (Q_{k+1},Q_j) = ((x-a_k)Q_k,Q_j) + a_j(Q_j,Q_j)$

$$\overset{\parallel}{0} \quad = (xQ_k,Q_j) - a_k\underbrace{(Q_k,Q_j)}_{=0,\ j<k} + a_j(Q_j,Q_j)$$

$$= (xQ_k,Q_j) + a_j(Q_j,Q_j) = 0$$

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$$\Rightarrow \quad a_j = -\frac{(xQ_k, Q_j)}{(Q_j, Q_j)}, \quad j = 0, \dots, k-1$$

za $j = k$: $(Q_{k+1}, Q_k) = ((x - a_k)Q_k, Q_k) + 0 + \dots + 0$

$$\underset{0}{\parallel} \qquad = (xQ_k, Q_k) - a_k (Q_k, Q_k) = 0$$

$$\Rightarrow a_k = \frac{(xQ_k, Q_k)}{(Q_k, Q_k)}$$

$$(Q_j, Q_j) = \int_a^b \underbrace{p(x)}_{>0} \cdot \underbrace{Q_j^2(x)}_{\geq 0} dx = 0 \iff Q_j \equiv 0 \; ↯$$

IH: $Q_{j+1} = \left(x - \frac{(xQ_j, Q_j)}{(Q_j, Q_j)}\right) Q_j - \frac{(Q_j, Q_j)}{(Q_{j-1}, Q_{j-1})} \cdot Q_{j-1}, \quad j+1 \leq k, \; j \leq k-1$

$$x \cdot Q_j = Q_{j+1} + \frac{(xQ_j, Q_j)}{(Q_j, Q_j)} \cdot Q_j + \frac{(Q_j, Q_j)}{(Q_{j-1}, Q_{j-1})} \cdot Q_{j-1} \quad / \cdot Q_k$$

$$(xQ_j, Q_k) = (Q_{j+1}, Q_k) + 0 + 0$$

$$\overset{\parallel}{\int_a^b} p(x) \cdot x \cdot Q_j \cdot Q_k \, dx = \int_a^b p(x) \underset{\sim}{Q_j} \cdot \underset{\sim}{x \cdot Q_k} \, dx = (Q_j, xQ_k)$$

$$= (xQ_k, Q_j)$$

$$\Rightarrow a_j = -\frac{(xQ_k, Q_j)}{(Q_j, Q_j)} = \frac{(Q_{j+1}, Q_k)}{(Q_j, Q_j)} = \begin{cases} 0, & j < k-1 \\ -\frac{(Q_k, Q_k)}{(Q_j, Q_j)}, & j = k-1 \end{cases}$$

jedinstven iz normiranosti. $\boxtimes$

1. $x^2 + 2x + 3 = 0$

$2x^2 + 4x + 6 = 0$

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$\{Q_i\}$ baza

$$P_m(x) = \sum_{i=1}^{m} a_i Q_i(x) \qquad / \cdot Q_n \qquad (Q_i, Q_n) = 0_{\; i<n}$$

$$(P_m, Q_n) = \int_a^b p(x) \cdot P_m(x) \cdot \underbrace{Q_n(x)}_{\text{iz (T)}} dx = 0, \quad m<n$$

Q_n jedinstven! (moničan)

Lema ❀
$$\int_a^b p(x) P_m(x) \underbrace{w_n(x)}_{\text{moničan}} dx = 0$$

$$\Rightarrow w_n(x) = Q_n(x)$$

(T) Koreni x_i, $i = 1, \dots, n$, (normiranog) ort. poly. $Q_n(x)$ su realne, jednostruke i pripadaju (a, b)

(D) $x_1, \dots, x_j$ realni koreni od Q_n koji $\in (a,b)$ i koji su neparne višestrukosti.

$$a < x_1 < \dots < x_j < b$$

$$P(x) = \prod_{i=1}^{j} (x - x_i)$$

$P(x) \cdot Q_n(x)$

$x_1, \dots, x_j$

$\underbrace{\text{jednostruke} \quad x_1, \dots, x_j \text{ neparne viš.}}$

$x_1, \dots, x_j$ parne višestrukosti

$$(x-x_1)^{p_1} \cdot (x-x_2)^{p_2} \cdot \dots \cdot (x-x_j)^{p_j}, \quad p_k - \text{parni}$$

preostale nule od Q_n su parne višestrukosti

$$= \left[(x-x_1)^{\frac{p_1}{2}} \cdot (x-x_2)^{\frac{p_2}{2}} \cdot \dots \cdot (x-x_j)^{\frac{p_j}{2}} \cdot \dots \cdot (x-x_k)^{\frac{p_k}{2}} \right]^2 \geq 0$$

$$(Q_n, P) = \int_a^b p(x) \cdot Q_n(x) \cdot P(x)\, dx \neq 0$$

$$(Q_n, P) \neq 0 \Rightarrow st(P) = st(Q_n) = n$$

$st(Q) = n$
nep. višestr. $\Rightarrow 1$

$$\Rightarrow P = \prod_{i=1}^{n} (x - x_i) \qquad \Rightarrow x_i \text{ su realni, jednostruki i } \in (a,b) \quad \boxtimes$$

$\Rightarrow$ nule od Q —||—

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Greška GKF:

$$|R_{2n-1}(f)| \leq \frac{1}{(2n)!} \cdot \max_{\xi \in [a,b]} |f^{(2n)}(\xi)| \cdot \int_a^b p(x)\, \omega_n^2(x)\, dx$$

3 tačke $\Rightarrow$ NJtn Kotos. P_3 (uz poboljšanu tačnost)

GKF P_5

10 tačaka $\Rightarrow$ NjKKF P_9

GKF P_{19}

Da li GKF može da bude tačna i za P_{2n} ?

$$P_{2n}(x) = \prod_{i=1}^{n} (x-x_i)^2 = \omega_n^2(x)$$

$$I(P_{2n}) = \int_a^b \underbrace{p(x)}_{>0} \underbrace{(x-x_1)^2}_{>0} \cdots (x-x_n)^2\, dx > 0$$

$\not\parallel$

$$S_{2n-1}(P_{2n}) = \frac{b-a}{2} \sum_{i=1}^{n} A_i (x_i-x_1)^2 \cdots \underbrace{(x_i-x_i)^2}_{0} \cdots (x_i-x_n)^2$$

$$= 0$$