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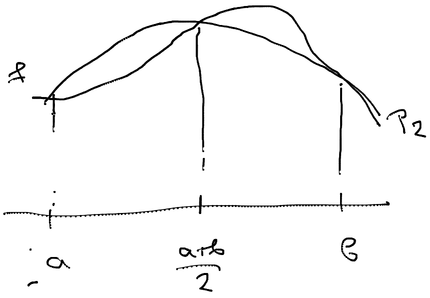
Ogrouna

KF. Simpsona ($n=2$, $t_0=-1$, $t_1=0$, $t_2=1$)



$$\sum_{i=0}^{2} A_i f(x_i)$$

$$A_0 = \int_{-1}^{1} \frac{\cancel{t-t_1}}{t_0-t_1} \cdot \frac{t-t_2}{t_0-t_2}\, dt$$

$$= \int_{-1}^{1} \frac{t-0}{1} \cdot \frac{t-1}{-2}\, dt = \dots = \frac{1}{3}$$

$$A_1 = \int_{-1}^{1} \frac{\cancel{t-t_0}}{t_1-t_0} \cdot \frac{t-t_2}{t_1-t_2}\, dt =$$

$$= \int_{-1}^{1} \frac{t+1}{1} \cdot \frac{t-1}{-1}\, dt = \dots = \frac{4}{3}$$

$$A_2 = \dots = A_0$$

$$\boxed{S_2(f) = \frac{b-a}{2} \cdot \frac{1}{3}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)}$$

$$|R_2(f)| \le \frac{1}{4!} \cdot \left(\frac{b-a}{2}\right)^5 \cdot \max_{x\in[a,b]} |f^{IV}(x)| \cdot \int_{-1}^{1} t^2 |t^2-1|\, dt$$

$$= \frac{1}{90}\left(\frac{b-a}{2}\right)^5 \cdot \max_{x\in[a,b]} |f^{IV}(x)|$$

Ilmaah: pravougaonik:

$$\int_a^b \underbrace{f(x)}_{\to P_0}\, dx \approx C \cdot f\left(\frac{a+b}{2}\right)$$

$$f \equiv x^0 = 1 : \int_a^b f(x)\, dx = C \cdot f\left(\frac{a+b}{2}\right)$$

$$\int_a^b 1 \cdot dx = (b-a) = C$$

trapez: $f \equiv x^0 = 1$: , $f = x^1 = x$ $\qquad \int_a^b f(x)dx \approx c_1 \cdot f(a) + c_2 f(b)$

$$f \equiv 1 : \int_a^b f(x)\, dx = c_1 \underbrace{f(a)}_{1} + c_2 \cdot \underbrace{f(b)}_{1}$$

$$(b-a) = c_1 + c_2 \qquad (*)$$

simetrija $c_1 = c_2 = \frac{1}{2}(b-a)$

$$f = x : \int_a^b x \cdot dx = c_1 \cdot \overset{=f(a)}{a} + c_2 \cdot \overset{=f(b)}{b}$$

$$\frac{1}{2}(b^2-a^2) = a \cdot c_1 + b \cdot c_2 \qquad (**)$$

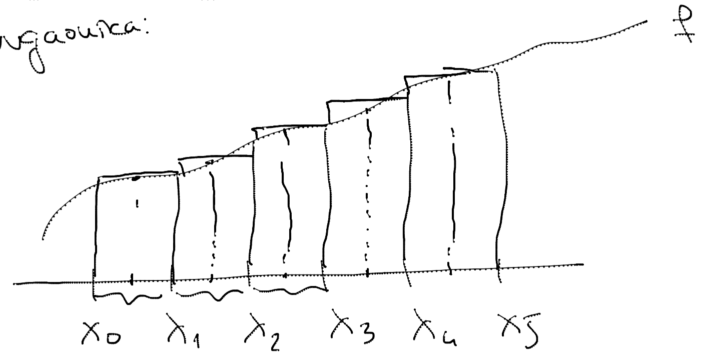
Simpson:
$$\int_a^b f(x)dx \simeq c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$
$$= \text{ za } f(x)=1,\ f(x)=x,\ f(x)=x^2$$

$$f(x)=1: \quad (b-a) = c_1 + c_2 + c_3$$
$$f(x)=x: \quad \tfrac{1}{2}(b^2-a^2) = c_1 \cdot a + c_2 \cdot \frac{a+b}{2} + c_3 \cdot b$$
$$f(x)=x^2: \quad \frac{1}{3}(b^3-a^3) = a^2 \cdot c_1 + \left(\frac{a+b}{2}\right)^2 \cdot c_2 + b^2 c_3$$

$c_1 = c_3$ (simetrija) $\qquad c_2 = \frac{4}{3}(b-a)$

$c_1 = \frac{b-a}{6}$

Pravougaonika:



$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

Opšta KF Njutn-Kotesovog tipa

n - intervala, dužine h

Pravougaonik:
$$\int_a^b f(x)dx \approx \overset{(b-a)/n}{h} \cdot \sum_{i=1}^{n} f_{i-1/2} = S_0^h(f)$$
$$|R_0^h| \le \sum_{i=1}^{n} \frac{h^3}{24} \cdot \max_{\xi_i \in [x_{i-1},x_i]} |f''(\xi_i)| = \frac{n \cdot h^3}{24} \cdot \frac{1}{n} \sum_{i=1}^{n} \max_{[x_{i-1},x_i]} |f''(\xi_i)|$$
$$n \cdot h = (b-a)$$
$$\le \frac{n \cdot h^3}{24} \cdot \max_{\xi \in [a,b]} |f''(\xi)| = \frac{(b-a) \cdot h^2}{24} \cdot \max_{\xi \in [a,b]} |f''(\xi)|$$

Trapez:
$$\int_a^b f(x)dx = \sum_{i=1}^{n} \frac{h}{2}(f_{i-1} + f_i) = \frac{h}{2}\left(f_0 + f_1 + f_1 + f_2 + f_2 + f_3 + \dots + f_{n-1} + f_n\right)$$
$$= \frac{h}{2}\left(f_0 + 2 \cdot \sum_{i=1}^{n-1} f_i + f_n\right) = S_1^h(f)$$

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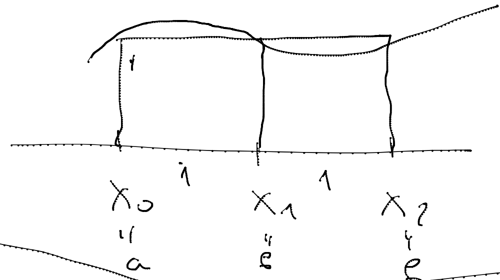
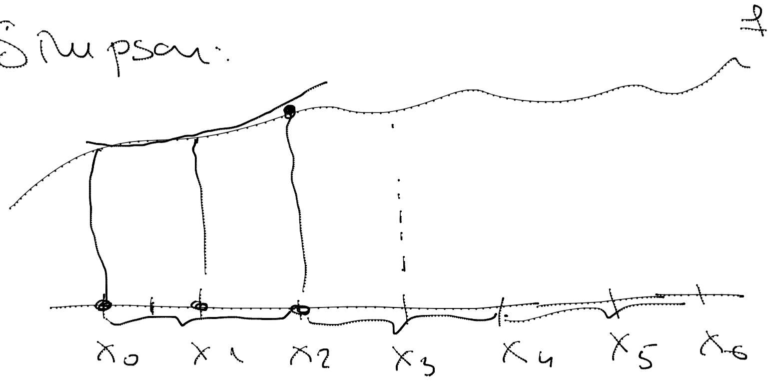
$$|R_1^h| \le \sum_{i=1}^{m} \frac{h^3}{12} \cdot \max_{[x_{i-1}, x_i]} |f''(\xi_i)| = \ldots \le \frac{(b-a)\cdot h^2}{12} \cdot \max_{\xi\in[a,b]} |f''(\xi)|$$



$(b-a) = 2$

$(b-a) = 1$

Simpson:



paran br. intervala

$h = \frac{b-a}{2m}$ $\quad n = 2m$

$n+1$ tačka

$$\int_a^b f(x)\,dx = \sum_{i=1}^{m} \underbrace{\int_{x_{2(i-1)}}^{x_{2i}} f(x)\,dx} \approx \frac{h}{3} \sum_{i=1}^{m} \left(f_{2(i-1)} + 4 f_{2i-1} + f_{2i}\right)$$

$$|R_2^h(f)| \le \sum_{i=1}^{m} \frac{h^5}{90} \cdot \max_{\xi\in[x_{2i-2}, x_{2i}]} |f^{IV}(\xi_i)| = \frac{h^5}{90} \cdot m \cdot \frac{1}{m} \sum_{i=1}^{m} \max_{[x_{2i-2}, x_i]} |f^{IV}(\xi_i)|$$

$$\le \frac{(b-a)\cdot h^4}{180} \cdot \max_{x\in[a,b]} |f^{IV}(x)| \qquad \le \frac{h^5 \cdot m}{90} \cdot \max_{[a,b]} |f^{IV}(x)|$$

$$\frac{h^4 \cdot \frac{b-a}{2m} \cdot m}{90} \cdot \max_{[a,b]} |f^{IV}(x)|$$

Rungeova ocena greške:

$$I = I_h + M_1 \cdot h^k$$

$$I = I_H + M_2 \cdot H^k \quad (-)$$

$$M_1 = M_2 = M$$

$$I_h - I_H = M(H^k - h^k) \Rightarrow M = \frac{I_h - I_H}{H^k - h^k}$$

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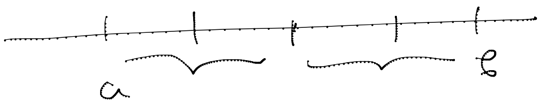
$$R_h \approx M \cdot h^k = \frac{I_h - I_H}{H^k - h^k} \cdot h^k = \frac{I_h - I_H}{\cancel{h^k}\left(\frac{H^k}{h^k} - 1\right)} \cdot \cancel{h^k}$$

$$= \frac{I_h - I_H}{\left(\frac{H}{h}\right)^k - 1}$$

$H = 2h$: $\boxed{R_h \approx \frac{I_h - I_H}{2^k - 1}}$

$k = 2$ (trapez, pravoug.)

$k = 4$ (Simps.)



I_H

$I_h \to R_h$

$I_{\frac{h}{2}} \to R_{\frac{h}{2}}$

Greška računa KF:

$R < tol$

$R = R_M + \underset{?}{\underline{R_R}} < tol$

$S_n(f) = \sum_i A_i \cdot \underline{f_i}$, f_i dato sa greškom ε

$R_R \le \sum_i |A_i| \cdot \varepsilon = \varepsilon \cdot \underbrace{\sum_{i=1}^{n} |A_i|}$

$f \equiv 1$: $\int_a^b 1 \cdot dx = (b-a) = \sum_i A_i \cdot 1 \Rightarrow \sum_i A_i = b - a$

$\Rightarrow \boxed{R_R \le \varepsilon \cdot (b-a)}$

$$\int_0^\infty \ldots = \underbrace{\int_0^M \ldots}_{KF} + \underbrace{\int_M^\infty \ldots}_{\le \ldots}$$

Gausove KF

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$$S_n(f) = \frac{b-a}{2} \sum_{i=1}^{n} A_i \cdot f(x_i)$$

$A_i = ?$ (n) $x_i = ?$ (n) $\rightarrow$ $2n$ nepoznatih

$2n-1$ – stepen polinoma

$$I(P_m) = \int_a^b p(x) \cdot \underline{P_m(x)}\, dx = \frac{b-a}{2} \sum_{i=1}^{n} A_i P_m(x_i) \equiv S_n(P_m)$$

$$m \leq 2n-1$$

$$\{x^k\} \vdots \quad \int_a^b p(x) \cdot x^k\, dx = \frac{b-a}{2} \sum_{i=1}^{n} A_i x_i^k \quad , k = 0, \dots, 2n-1$$

:(sistem nije linearan

Lema 1: Da bi KF bila tačna za proizvoljni polinom stepena m $\left(P_m = \sum_{k=0}^{m} a_k \cdot x^k\right)$ potrebno je i dovoljno da ona bude tačna za sve funkcije x^k, $k = 0, \dots, m$

$\boxed{\Rightarrow}$ $S_n(P_m) = I(P_m) \overset{?}{\Longrightarrow} S_n(x^k) = I(x^k)$, $k = 0, \dots, m$

$\boxed{\Leftarrow}$ $S_n(x^k) = I(x^k) \overset{?}{\Longrightarrow} S_n(P_m) = I(P_m)$

$$S_n(P_m) = \frac{b-a}{2} \cdot \sum_{i=1}^{n} A_i \cdot \underbrace{P_m(x_i)}$$

$$= \frac{b-a}{2} \sum_{i=1}^{n} A_i \cdot \sum_{k=0}^{m} a_k x_i^k$$

$$= \sum_{k=0}^{m} a_k \cdot \underbrace{\left(\frac{b-a}{2} \sum_{i=1}^{n} A_i \cdot x_i^k \right)}_{S_n(x^k)}$$

$$= \sum_{k=0}^{m} a_k \cdot \underbrace{S_n(x^k)}$$

$$= \sum_{k=0}^{m} a_k \cdot I(x^k)$$

$$= \sum_{k=0}^{m} a_k \cdot \int_a^b p(x) \cdot x^k\, dx = \int_a^b p(x) \cdot \underbrace{\sum_{k=0}^{m} a_k \cdot x^k}_{P_m} dx$$

$$= \int_a^b p(x) \cdot P_m(x)\, dx = I(P_m) \quad \boxed{\times}$$