Integracija

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$$I = \int_a^b f(x)\,dx \approx \sum_{k=0}^{n} A_k f(x_k) = S_n(f) \qquad x_k \in [a,b]$$

$$I = \int_a^b f(x)\,dx \rightsquigarrow \int_a^b P_n(x)\,dx$$

$I(f) = S_n(f)$
KF tačna

$n, A_k, x_k, \; k = 0, \dots, n \to 2n+3$

$$f(x) = P_n(x) + R_n(x) \quad / \int$$

$$\underbrace{\int_a^b f(x)\,dx}_{} = \underbrace{\int_a^b P_n(x)\,dx}_{} + \underbrace{\int_a^b R_n(x)\,dx}_{R_n(f)}$$

greska integracije

$$R_n(f) = \int_a^b R_n(x)\,dx = \int_a^b \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot \omega_{n+1}(x)\,dx \quad , \xi \in [a,b]$$

$$|R_n(f)| \le \frac{M_{n+1}}{(n+1)!} \int_a^b |\omega_{n+1}(x)|\,dx$$

$$F(x) = \underbrace{\cos(x)}_{f(x)} \cdot \underbrace{\sqrt{x}}_{p(x)} = f(x) \cdot p(x)$$

$$I = \int_a^b p(x) \cdot \underbrace{f(x)}_{\rightsquigarrow L_n(x)}\,dx \quad , p(x) > 0$$

težinska f-ja

Njutn-Kotesove KF (imamo čvorove x_k)

$f(x) \approx L_n(x)$ Lagranž

$x \in [a,b] \rightsquigarrow t \in [-1,1]$

$$x = pt + q: \quad \left.\begin{aligned} a &= p \cdot (-1) + q \\ b &= p \cdot 1 + q \end{aligned}\right\} \quad \begin{aligned} \oplus \quad q &= \frac{a+b}{2} \\ \ominus \quad p &= \frac{b-a}{2} \end{aligned}$$

$$x = \frac{b-a}{2} t + \frac{a+b}{2}$$

$$L_n(x) = \sum_{i=0}^{n} \left(\prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j} \right) \cdot \underbrace{f(x_i)}_{\text{Const}} = \sum_{i=0}^{n} \left(\prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{t-t_j}{t_i-t_j} \right) \cdot \underbrace{f(x_i)}$$

$$I = \int_a^b p(x) \cdot f(x)\, dx \approx \int_a^b p(x) \cdot L_n(x)\, dx \qquad \uparrow \qquad \left\{ dx = \frac{b-a}{2} dt \right.$$

$$= \frac{b-a}{2} \int_{-1}^{1} \bar{p}(t) \cdot \underbrace{L_n\left(\frac{b+a}{2} + \frac{b-a}{2} t \right)} \cdot dt$$

$$= \frac{b-a}{2} \int_{-1}^{1} \bar{p}(t) \cdot \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{t-t_j}{t_i-t_j} \cdot \underbrace{f(x_i)}\, dt$$

$$A_i = \int_{-1}^{1} \bar{p}(t) \cdot \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{t-t_j}{t_i-t_j}\, dt$$

$$= \frac{b-a}{2} \sum_{i=0}^{n} A_i \cdot f(x_i)$$

$$R_n(f) = \int_a^b p(x) \cdot \underline{R_n(x)}\, dx = \int_a^b p(x) \cdot \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot \omega_{n+1}(x)\, dx$$

$$\left[|R_n(f)| \leq \frac{M_{n+1}}{(n+1)!} \cdot \left(\frac{b-a}{2} \right)^{n+2} \int_{-1}^{1} |\bar{p}(t) \cdot \bar{\omega}_{n+1}(t)|\, dt \right] \qquad dx \to dt$$

$$\omega_{n+1}(x) = \prod_{i=0}^{n} (x-x_i) = \left(\frac{b-a}{2} \right)^{n+1} \underbrace{\prod_{i=0}^{n} (t-t_i)}_{\bar{\omega}_{n+1}(t)}$$

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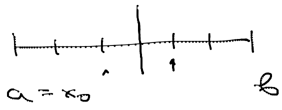
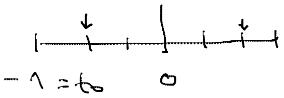
Lema 1 : Ako je $p(x)$ parna u odnosu na sredinu $[a,b]$, i čvorovi x_k simetrično raspoređeni u odnosu na sredinu intervala, onda $A_k = A_{n-k}$.



$x_0 \quad x_1 \quad \dots \quad x_{n-1} \quad x_n$

$A_0 \quad A_1 \quad A_{n-1} \quad A_n$



Ⓓ $\bar{p}(-t) = \bar{p}(t)$, $t_k = -t_{n-k} \overset{?}{\Rightarrow} A_k = A_{n-k}$

$t_k = -t_{n-k}$

$$A_k = \int_{-1}^{1} \bar{p}(t) \cdot \left(\prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{t-t_j}{t_k-t_j} \right) dt \overset{t=-s}{=} \int_{-1}^{1} \bar{p}(-s) \cdot \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{-s+t_{n-j}}{-t_{n-k}+t_{n-j}} \Big) \cdot ds$$

$$= \int_{-1}^{1} \bar{p}(s) \cdot \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{s-t_{n-j}}{t_{n-k}-t_{n-j}} ds \overset{l=n-j}{=} \int_{-1}^{1} \bar{p}(s) \cdot \prod_{\substack{l=0 \\ l \neq n-k}}^{n} \frac{s-t_l}{t_{n-k}-t_l} ds$$

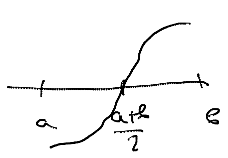
$$= A_{n-k} \quad \boxtimes$$

Lema 2 : Ako su x_k simetrično raspoređeni u odnosu na sredinu $[a,b]$ ako je f neparna f-ja i ako je p parna, onda je KF tačna.

Ⓓ $I(f) = S_n(f)$?

$$I = \int_a^b \underbrace{p(x) \cdot f(x)} dx$$

$f \cdot p \to$ neparna

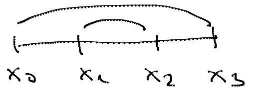


$I(f) = 0$

$S_n(f) \overset{?}{=} 0$

$$S_n(f) = \frac{b-a}{2} \left[A_0 \cdot f(x_0) + A_1 \cdot f(x_1) + \dots + A_{n-1} f(x_{n-1}) + A_n f(x_n) \right]$$

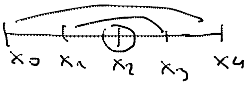
n neparno : br. čvorova paran



$f(x_k) = -f(x_{n-k})$ neparna

$$S_n(f) = \frac{b-a}{2} \Big[\underbrace{(A_0 - A_n)}_{\substack{(\text{Lema 1}) \\ =0}} \cdot f(x_0) + \underbrace{(A_1 - A_{n-1})}_{0} \cdot f(x_1) + \dots + \underbrace{(A_{n/2} - A_{n/2-1})}_{=0} \cdot f(x_{n/2}) \Big]$$

$= 0$



n parno : br. čvorova neparan

$$S_n(f) = \frac{b-a}{2} \Big[\underbrace{(A_0 - A_n)}_{=0} \cdot f(x_0) + \dots + \underbrace{(A_{n/2-1} - A_{n/2+1})}_{=0} \cdot f(x_{n/2-1}) + A_{n/2} \cdot \underbrace{f(x_{n/2})}_{= f\left(\frac{a+b}{2}\right) = 0} \Big]$$

$= 0$

$= 0$ $- + (2) -$

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Lema 3: Ako je n parno, $f \overset{= P_n(x)}{}$ polinom, p parno, simetrično rasp. x_k onda će KF da bude tačna i za polinome stepena $n+1$

Ⓓ $f = P_n(x) \rightsquigarrow L_n(x) = P_n(x)$

$I(f) = S_n(f)$ jeste tačna za poly st. n

$$\underset{\uparrow}{P_{n+1}(x)} = P_n(x) + \underbrace{c\left(x - \frac{a+b}{2}\right)^{n+1}}_{\heartsuit} \quad , \; C\text{-const}$$

$\heartsuit$ neparna fja (n je parno)

$$R_{n+1}(\overline{x}) = \underbrace{R_n}_{=0} + R(\heartsuit)$$

Lema 2 $\Rightarrow R(\heartsuit) = 0$

$$R_{n+1} = 0$$

n parno, simetrični čvorovi, p parna, f - proizvoljna

$P_n \to n+1$ tačka

$P_{n+1} \to n+2$ tačke

$$\overline{\omega}_{n+2}(t) = (t-t_0)\cdot(t-t_1)\cdot\ldots\cdot(t-t_{n-1})(t-t_n) \quad , \; t_i = -t_{n-i}$$

$$= (t^2-t_0^2)(t^2-t_1^2)\cdot\ldots\cdot(t^2-0)$$

↳ središnji na $[-1,1]$ $\overset{t_i \cdot 0}{\urcorner}$ dupli

$$= t^2 \prod_{i=0}^{\frac{n}{2}-1} (t^2 - t_i^2) \quad , \; \left(\prod_{i=0}^{\frac{n}{2}-1} (t^2-t_i^2)\right)_{n=0} \overset{\text{def}}{=} 1$$

$$|R_n(f)| \leq \frac{1}{(n+\cancel{1}^{\,2})!}\cdot\left(\frac{b-a}{2}\right)^{n+\cancel{2}^{\,3}}\cdot M_{n+\cancel{1}^{\,2}}\cdot\int_{-1}^{1} |\overline{p}(t)\, t\, \overline{\omega}_{n}(t)|\,dt$$

$$\leq \frac{1}{(n+2)!}\cdot\left(\frac{b-a}{2}\right)^{n+3}\cdot M_{n+2}\cdot\int_{-1}^{1}|\overline{p}(t)\cdot t\cdot \overline{\omega}_{n+1}(t)|\,dt$$

$\overline{\omega}_{\cancel{n+2}}\cancel{(t)}$

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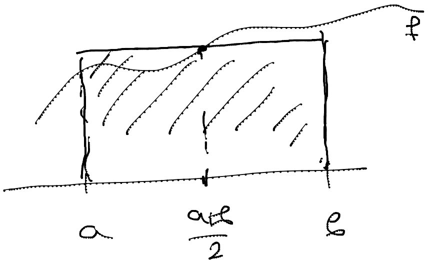
Osnovne KF Njutn-Kotesovog tipa ($p(x) \equiv 1$)

Osnovna f-la pravougaonika $f \rightsquigarrow L_0$



$n=0 \Rightarrow t_0 = 0$

$$S_0(f) = \frac{b-a}{2} \cdot A_0 \cdot \underbrace{f(t_0)}$$

$$\left(A_0 = \int_{-1}^{1} 1 \cdot dx = x\Big|_{-1}^{1} = 2 \right.$$

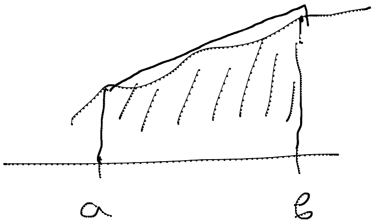
$$S_0(f) = \frac{b-a}{2} \cdot 2 \cdot \underbrace{f\left(\frac{a+b}{2}\right)}$$

$$\boxed{S_0(f) = (b-a) \cdot f\left(\frac{a+b}{2}\right)}$$

$$|R_0(f)| \le \frac{1}{2!} \cdot \left(\frac{b-a}{2}\right)^3 \cdot \max_{x \in [a,b]} |f''(x)| \cdot \underbrace{\int_{-1}^{1} t(t-t_0)dt}_{\int_{-1}^{1} t^2 dt = \frac{t^3}{3}\Big|_{-1}^{1} = \frac{2}{3}}$$

$$= \frac{(b-a)^3}{24} \cdot \max_{x \in [a,b]} |f''(x)|$$

Formula trapeza $f \rightsquigarrow L_1$



$n=1, \ t_0 = -1, \ t_1 = 1$

$$S_1(f) = \frac{b-a}{2} \left(A_0 \cdot f(a) + A_1 \cdot f(b) \right)$$

$$A_1 = \int_{-1}^{1} \frac{t-t_1}{t_0-t_1} dt = \int_{-1}^{1} \frac{t-1}{-2} dt = \frac{(t-1)^2}{-4}\Big|_{-1}^{1} = \underline{1}$$

$$A_2 \overset{\text{lema 1}}{=} A_1$$

$$\boxed{S_1(f) = \frac{b-a}{2}\left(f(a) + f(b)\right)}$$

$$|R_1(f)| = \frac{1}{2} \cdot \left(\frac{b-a}{2}\right)^3 \cdot \max_{x \in [a,b]} |f''(x)| \cdot \underbrace{\int_{-1}^{1} |(t-1)(t+1)| dt}_{\vdots}$$

$$= \frac{(b-a)^3}{12} \cdot \max_{x \in [a,b]} |f''(x)|$$