Inverzna interpolacija

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$f(x) = ? \rightsquigarrow L_n(x) = ?$

$y = f(?)$

I) neekvidistantna tablica $f(x_i) = y_i$, $i = 0,\dots,n$

$Y = f(x)$, $x = ?$

čvorovi

x	x_0	x_1	$\dots$	x_n
y	y_0	y_1	$\dots$	y_n

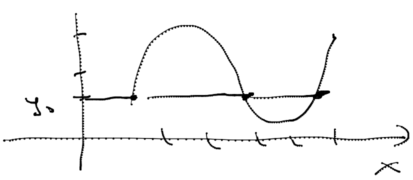
$\uparrow y$

$\rightsquigarrow$

$y \downarrow$

y	y_0	y_1	$\dots$	y_n	čvorovi
x	x_0	x_1	$\dots$	x_n	vrednosti f-je

1-1, ograničimo na podinterval gde jeste monotona



II ekvidistantna tablica, monotona

$y_0 < Y < y_1$

$$f(x) = \underbrace{Y}_{\text{dato}} \approx f_0 + \frac{q}{1!}\cdot\underbrace{\Delta f_0}_{\text{znamo}} + \frac{q(q-1)}{2!}\cdot\Delta^2 f_0 + \dots + \frac{q(q-1)\cdots(q-n+1)}{n!}\cdot\Delta^n f_0$$

$q = ?$, $q = \frac{x-x_0}{h} \Rightarrow x = q\cdot h + x_0$

$$q^{(n+1)} = \frac{Y-f_0}{\Delta f_0} - \frac{1}{\Delta f_0}\cdot\left[\frac{q^{(n)}(q^{(n)}-1)}{2}\cdot\Delta^2 f_0 + \dots + \frac{q^{(n)}(q^{(n)}-1)\cdots(q^{(n)}-n+1)}{n!}\cdot\Delta^n f_0\right]$$

$\Delta f_0 \neq 0$, $f_1 \neq f_0$ monotona

Metoda iteracije $q = F(q)$

$q_0 = 0$

$q_1 = F(q_0)$, $q_2 = F(q_1)$, ...

$q_0, q_1, q_2, \dots \xrightarrow{K} q$ pri određenim uslovima

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$y_{n-1} < Y < y_n \Rightarrow$ II Njutna

$L_n'(x) \approx y = f(x)$ $\qquad P_n(y) = x$

$f^{-1}(x) = P_n(x)$? Ne

NUM. DIFERENCIRANJE

$f(x)$, $f^{(k)}(x) = ?$

$f \rightsquigarrow L_n \quad /^{(k)}$

$f^{(k)} \rightsquigarrow L_n^{(k)}$

$R_n(x) = f(x) - L_n(x)$

$R_n^{(k)}(x) = (f(x) - L_n(x))^{(k)}$

$= f^{(k)}(x) - L_n^{(k)}(x)$

I Njutn:

$$L_n(x) = f_0 + q\cdot\Delta f_0 + \frac{\overbrace{q(q-1)}^{q^2-q}}{2}\Delta^2 f_0 + \frac{\overbrace{q(q-1)(q-2)}^{q^3-3q^2+2q+2}}{6}\cdot\Delta^3 f_0 + \dots$$

$$L_n'(x) = \frac{dL_n}{dx} = \frac{dL_n}{dq}\cdot\frac{dq}{dx} \underset{q=\frac{x-x_0}{h}}{=} \frac{1}{h}\cdot\frac{dL_n}{dq}$$

$$L_n'(x) = \frac{1}{h}\left[\Delta f_0 + \frac{2q-1}{2}\cdot\Delta^2 f_0 + \frac{3q^2-6q+2}{6}\cdot\Delta^3 f_0 + \frac{4q^3-18q^2+22q-6}{24}\cdot\Delta^4 f_0 + \dots\right]$$

$$L_n''(x) = \frac{1}{h^2}\left[\Delta^2 f_0 + (q-1)\Delta^3 f_0 + \underbrace{\frac{12q^2-36q+22}{12}}_{?}\Delta^4 f_0 + \dots\right]$$

$6q^2 - 18q + 11$

$$L_n'(x_0) = \frac{1}{h}\cdot\left[\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 + \dots\right]$$

$$R_n^{(k)}(x) = f^{(k)}(x) - L_n^{(k)}(x) = (f(x) - L_n(x))^{(k)}$$

$$= \left(f[x, x_0, \dots, x_n]\cdot\omega_{n+1}(x)\right)^{(k)}$$

$$= \sum_{j=0}^{k} C_k^j \left(f[x, x_0, \dots, x_n]\right)^{(j)}\cdot\omega_{n+1}^{(k-j)}(x), \quad C_k^j = \binom{k}{j}$$

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$$= \sum_{j=0}^{k} C_k^j \; \underbrace{\underbrace{\left(f[x, x_0, \ldots, x_n] \right)}_{g(x)}}^{(j)} \cdot \omega_{n+1}^{(k-j)}(x)$$

$$f[\underbrace{x_i, \ldots, x_i}_{p+1}] = \frac{f^{(p)}(x_i)}{p!}$$

$$g[\underbrace{x, x+\varepsilon, \ldots, x+j\cdot\varepsilon}_{j+1}] = \frac{g^{(j)}(\xi_\varepsilon)}{j!}, \quad \xi_\varepsilon \in \underbrace{[x, x+j\cdot\varepsilon]}_{\downarrow \; x}$$

x

$$\varepsilon \to 0 \; : \; g[\underbrace{x, x, x, \ldots, x}_{j+1}] = \frac{g^{(j)}(x)}{j!}$$

$$g^{(j)}(x) = j! \cdot g[\underbrace{x, \ldots, x}_{j+1}]$$

$$\left(f[x, x_0, \ldots, x_n] \right)^{(j)} = j! \; f[\underbrace{x, x, \ldots, x}_{j+1}, x_0, \ldots, x_n]$$

$$= \sum_{j=0}^{k} \frac{k!}{\cancel{j!}(k-j)!} \cdot \cancel{j!} \cdot f[\underbrace{x, \ldots, x}_{j+1}, \underbrace{x_0, \ldots, x_n}_{n+1}] \cdot \omega_{n+1}^{(k-j)}(x)$$

$$= \sum_{j=0}^{k} \frac{k!}{(k-j)!} \cdot \frac{f^{(n+j+1)}(\xi)}{(n+j+1)!} \cdot \omega_{n+1}^{(k-j)}(x)$$

$$\left| f^{(k)}(x) - L_n^{(k)}(x) \right| \le \sum_{j=0}^{k} \frac{k!}{(k-j)!\,(n+j+1)!} \cdot \max_{\xi \in [y_1, y_2]} \left| f^{(n+j+1)}(\xi) \right| \cdot \left| \omega_{n+1}^{(k-j)}(x) \right|$$

$$y_1 = \min(x, x_0, \ldots, x_n)$$
$$y_2 = \max(x, x_0, \ldots, x_n)$$

$$h = x_k - x_{k-1}$$

$j=0$

$k=1$:

$$f'(x_0) - L_n'(x_0) \le 1 \cdot \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot \omega_{n+1}'(x_0) + 1 \cdot \frac{f^{(n+2)}(\xi)}{(n+2)!} \cdot \underbrace{\omega_{n+1}^{(0)}(x_0)}_{=0}$$

$$\omega_{n+1}'(x_0) = \left((x-x_0) \cdots (x-x_n) \right)'_{x=x_0}$$

$$= \underbrace{(x_0 - x_1)}_{-x_0 - h} \cdot \underbrace{(x_0 - x_2)}_{-x_0 - 2h} \cdots \underbrace{(x_0 - x_n)}_{-x_0 - n\cdot h}$$

$$= -h \cdot (-2h) \cdots (-nh) = (-1)^n \cdot n! \cdot h^n$$

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$$f'(x_0) - L_n'(x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)\cancel{!}} \cdot (-1)^n \cdot \cancel{n!} \cdot h^n = \frac{f^{(n+1)}(\xi)}{n+1} \cdot (-1)^n h^n$$

h manje → R_M manja

→ R_R veća

$$\boxed{f'(x_0) = \frac{f(x_1) - f(x_0)}{h}}$$

, $x_1 = x_0 + h$

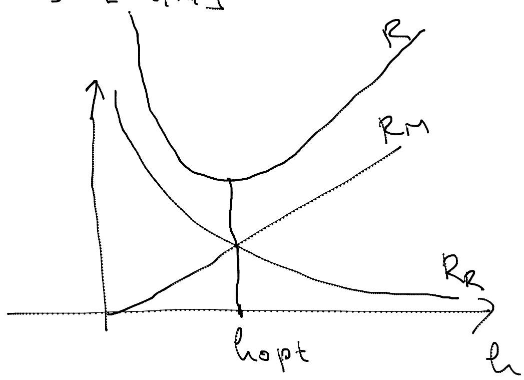
$$|R_M| \le \left| \frac{f''(\xi)}{2} \cdot h \right| \le \frac{M_2 h}{2}, \quad M_2 = \max_{\xi \in [x_0, x_1]} |f''(\xi)|$$

$$|R_R| \le \frac{\varepsilon + \varepsilon}{h} = \frac{2\varepsilon}{h}$$

$$R = R_M + R_R = \frac{M_2 h}{2} + \frac{2\varepsilon}{h} \to \min$$

$$R_h' = \frac{M_2}{2} - \frac{2\varepsilon}{h^2} = 0$$

$$h_{opt} = \sqrt{\frac{4\varepsilon}{M_2}}$$



$$f(x_1) = f(x_0 + h) = f(x_0) + h \cdot f'(x_0) + \frac{h^2}{2} \cdot f''(\xi)$$

$$f'(x_0) = \frac{1}{h}\left(f(x_1) - f(x_0) - \frac{h^2}{2} f''(\xi) \right)$$

$$\boxed{f'(x_0) = \frac{f(x_1) - f(x_0)}{h}} - \frac{h}{2} f''(\xi)$$ R_M

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$$f''(x_0) = \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2}$$

$$f(x_0-h) = \cancel{f(x_0)} - h\cdot \cancel{f'(x_0)} + \frac{h^2}{2}\cdot f''(x_0) - \cancel{\frac{h^3}{6}\cdot f'''(x_0)} + \frac{h^4}{24}\cdot f^{IV}(\xi_1)$$

$+$

$$f(x_0+h) = \cancel{f(x_0)} + h\cdot \cancel{f'(x_0)} + \frac{h^2}{2} f''(x_0) + \cancel{\frac{h^3}{6} f'''(x_0)} + \frac{h^4}{24}\cdot f^{IV}(\xi_2)$$

$$-2f(x_0) = -2\cancel{f(x_0)}$$

$$f(x_0-h) - 2f(x_0) + f(x_0+h) = h^2 f''(x_0) + \frac{h^4}{12}\cdot\left(f^{IV}(\xi_1) + f^{IV}(\xi_2)\right)$$

$$f''(x_0) = \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2} - \underbrace{\frac{h^2}{24}\left(f^{IV}(\xi_1) + f^{IV}(\xi_2)\right)}_{R_M}$$

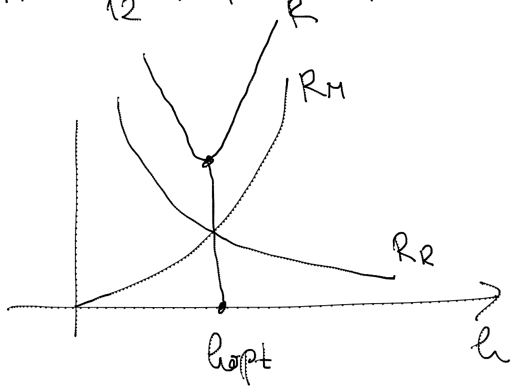
Lema: Ako je $f(x)$ neprekidna f-ja i $\alpha_i > 0$ tada postoji ξ takva da:

$$\alpha_1 f(x_1) + \dots + \alpha_n f(x_n) = (\alpha_1 + \dots + \alpha_n)\cdot f(\xi)$$

$$\xi \in [y_1, y_2] \qquad y_1 = \min(x_1 \dots x_n)$$
$$y_2 = \max(x_1, \dots, x_n)$$

$$f''(x_0) = \text{—}\|\text{—} \quad \frac{h^2}{24}\cdot \overset{\alpha_1+\alpha_2}{(1+1)} \underbrace{f^{IV}(\xi)}$$

$$R_M \le \frac{h^2}{12} M_4, \quad R_E \le \frac{4\varepsilon}{h^2}, \quad R = R_M + R_E = \frac{h^2}{12} M_4 + \frac{4\varepsilon}{h^2}$$



$$R'_h = \dots = 0$$

$$h_{opt} = \sqrt[4]{\frac{48\varepsilon}{M_4}}$$