KONAČNE RAZLIKE

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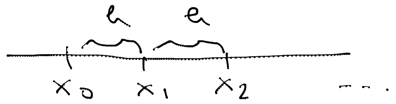
ekvidistantna tablica



$$x_i = x_0 + i\cdot h \quad , i = 0, 1, 2, \ldots \qquad f(x_i) = f_i$$

$\Delta f_i = f_{i+1} - f_i$ – KR unapred

$\nabla f_{i+1} =$ -||- – KR unazad

$\delta f_{i+1/2} =$ -||- – centralna KR

Reda 1

$$\Delta^k f_i = \Delta(\Delta^{k-1} f_i) = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i$$

(T) $$\boxed{\Delta^k f_i = \sum_{j=0}^{k} (-1)^j \cdot C_k^j \, f_{i+k-j}} \quad , \quad C_k^j = \binom{k}{j} = \frac{k!}{j!(k-j)!}$$

(D): $k=1$: $\Delta f_i = f_{i+1} - f_i$ (def.)

$$\Delta f_i = 1 \cdot C_1^0 \cdot f_{i+1} - C_1^1 f_i \qquad C_1^0 = \frac{1!}{0!\cdot 1!} = 1$$

$$= f_{i+1} - f_i \ \checkmark \qquad C_1^1 = \frac{1!}{1!\cdot 0!} = 1$$

$k \le n \overset{?}{\Rightarrow} k = n+1$: $$\Delta^{n+1} f_i \overset{\text{def.}}{=} \Delta^n f_{i+1} - \Delta^n f_i \overset{?}{=} \sum_{j=0}^{n+1} (-1)^j C_{n+1}^j f_{i+n+1-j}$$

$$\Delta^{n+1} f_i = \sum_{j=0}^{n} (-1)^j C_n^j f_{i+1+n-j} - \sum_{j=0}^{n} (-1)^j C_n^j f_{i+n-j}$$

$$= \underbrace{(-1)^0 \cdot C_n^0 \cdot f_{i+1+n}}_{j=0} + \sum_{j=1}^{n} (-1)^j C_n^j f_{i+1+n-j} - \underbrace{(-1)^n C_n^n f_i}_{j=n} - \sum_{j=0}^{n-1} (-1)^j C_n^j f_{i+n-j}$$

$$C_n^0 = \frac{n!}{0!(n-0)!} = 1 = C_{n+1}^0 \qquad C_n^n = \frac{n!}{n!(n-n)!} = 1 = C_{n+1}^{n+1}$$

$$= C_{n+1}^0 f_{i+1+n} + \sum_{j=0}^{n-1} (-1)^{j+1} C_n^{j+1} f_{i+n-j} - (-1)^n C_{n+1}^{n+1} f_i - \sum_{j=0}^{n-1} (-1)^j C_n^j f_{i+n-j}$$

$$= C_{n+1}^0 f_{i+1+n} - (-1)^n C_{n+1}^{n+1} f_i + \sum_{j=0}^{n-1} f_{i+n-j} \Big(\underbrace{(-1)^{j+1} C_n^{j+1} - (-1)^j C_n^j}_{(-1)^{j+1} \left(C_n^{j+1} + C_n^j \right)} \Big)$$

$$C_n^{j+1} + C_n^j = \frac{n!}{(j+1)!(n-j-1)!} + \frac{n!}{j!(n-j)!} = \frac{n!(n-j) + n!(j+1)}{(j+1)!\,(n-j)!}$$

$$= \frac{n!(n-j+j+1)}{(j+1)!(n-j)!} = \frac{(n+1)!}{(j+1)!(n-j)!} = C_{n+1}^{j+1}$$

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$$= C_{n+1}^{0} f_{i+1+n} - (-1)^{n} C_{n+1}^{n+1} f_i + \sum_{j=0}^{n-1} f_{i+n-j} (-1)^{j+1} C_{n+1}^{j+1}$$

$$= C_{n+1}^{0} f_{i+1+n} + (-1)^{n+1} C_{n+1}^{n+1} f_i + \sum_{j=1}^{n} f_{i+n-(j-1)} (-1)^{j} C_{n+1}^{j}$$

$$= \sum_{j=0}^{n+1} (-1)^{j} C_{n+1}^{j} f_{i+n+1-j}$$

(veza PR i KR)

(L) $x_i = x_0 + i \cdot h \qquad f[x_i, \dots, x_{i+k}] = \dfrac{\Delta^k f_i}{h^k \cdot k!}$

(D): $\underline{k=1}$: $f[x_i, x_{i+1}] \overset{\text{def.}}{=} \dfrac{f_{i+1} - f_i}{x_{i+1} - x_i} = \dfrac{\Delta f_i}{h} = \dfrac{\Delta f_i}{h^1 \cdot 1!}$ ✓

$k \leq n \overset{?}{\Rightarrow} n+1$:

$$f[x_i, \dots, x_{i+n+1}] \overset{\text{def.}}{=} \frac{f[x_{i+1}, \dots, x_{i+n+1}] - f[x_i, \dots, x_{i+n}]}{\underbrace{x_{i+n+1} - x_i}_{(n+1)\cdot h}}$$

$h \quad h$

$x_i \quad x_{i+1} \quad x_{i+2}$

$$= \frac{1}{h(n+1)} \left(\frac{\Delta^n f_{i+1}}{h^n \cdot n!} - \frac{\Delta^n f_i}{h^n \cdot n!} \right)$$

$$= \frac{\Delta^{n+1} f_i}{h^{n+1} (n+1)!}$$

$$f[x_i, \dots, x_{i+k}] = \frac{f^{(k)}(\xi)}{k!} \quad \text{(veza PR i izvoda)}$$

$$f[x_i, \dots, x_{i+k}] = \frac{\Delta^k f_i}{h^k \cdot k!}$$

$$\frac{\Delta^k f_i}{h^k \cdot \cancel{k!}} = \frac{f^{(k)}(\xi)}{\cancel{k!}}$$

$$\Rightarrow \boxed{\Delta^k f_i = f^{(k)}(\xi) \cdot h^k}$$

$$x_i \leq \xi \leq x_{i+k}$$

veza KR i izvoda

$|x_0 - x| < h \qquad L_n(x) = ?$

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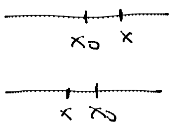
$x_i = x_0 + i \cdot h, \quad i = 0, \pm 1, \pm 2, \dots$

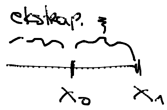
$x = x_0 + q \cdot h, \quad q \in (-1, 1)$



$q > 0$

$q < 0$

$q = \frac{x - x_0}{h}$



Ako se x nalazi na početku tablice

Njutn sa PR:

$$\begin{cases} L_n(x) = f_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots \\ \quad + f[x_0, \dots, x_n](x - x_0) \cdots (x - x_{n-1}) \\ f[x_0, \dots, x_k] = \frac{\Delta^k f_0}{h^k \cdot k!} \\ q = \frac{x - x_0}{h} \end{cases}$$

$$\Rightarrow L_n(x) = f_0 + \underbrace{\frac{\Delta f_0}{h} \cdot (x - x_0)}_{= q} + \frac{\Delta^2 f_0}{h^2 \cdot 2!} \cdot \underbrace{(x - x_0)}_{= q} (x - x_1) + \dots$$

$$+ \frac{\Delta^n f_0}{h^n \cdot n!} \cdot (x - x_0) \cdots (x - x_{n-1})$$

$$\frac{x - x_1}{h} = \frac{x - (x_0 + h)}{h} = q - 1$$

$$\boxed{L_n(x) = f_0 + q \cdot \Delta f_0 + \frac{1}{2!} \cdot \Delta^2 f_0 \cdot q(q-1) + \dots + \frac{q(q-1) \cdots (q - n + 1)}{n!} \cdot \Delta^n f_0}$$

Njutnov int. polinom sa konačnim razlikama za interpolaciju unapred

(I Njutnov)

Greška: (Lagranž): $f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot \omega_{n+1}(x)$

$$\omega_{n+1}(x) = \underbrace{(x - x_0)}_{q \cdot h} \underbrace{(x - x_1)}_{(q-1) \cdot h} \cdots \underbrace{(x - x_n)}_{(q-n) \cdot h} \qquad q = \frac{x - x_0}{h}$$

$$= h^{n+1} \cdot \prod_{j=0}^{n} (q - j)$$

$$\Rightarrow f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot h^{n+1} \prod_{j=0}^{n} (q - j) \qquad x_0 \le \xi \le x_n$$

$\Delta^k f_i = h^k \cdot f^{(k)}(\xi)$

$$\Rightarrow \boxed{f(x) - L_n(x) = \frac{\prod_{j=0}^{n} (q - j)}{(n+1)!} \cdot \Delta^{n+1} f}$$

* max po modulu n+1 kolone
* srednja vrednost

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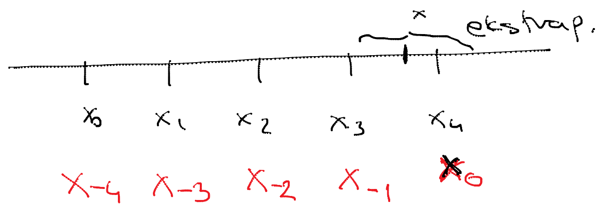
Ako se x nalazi na kraju tablice?



$$L_n(x) = f_0 + q\cdot\Delta f_{-1} + \frac{q(q+1)}{2!}\cdot\Delta^2 f_{-2} + \dots + \frac{q(q+1)\cdot\ldots\cdot(q+n-1)}{n!}\Delta^n f_{-n}$$

$$f_n + q\cdot\Delta f_{n-1} + \frac{q(q+1)}{2!}\cdot\Delta^2 f_{n-2} + \dots \quad —\|— \quad \Delta^n f_0$$

$$f(x) - L_n(x) = \frac{q(q+1)\cdot\ldots\cdot(q+n)}{(n+1)!}\cdot\Delta^{n+1} f$$

Njutnov int. poly. sa KR za interpolaciju unazad
(II NNF)

Kako se širi greška kroz tablicu?

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	
				ε	
x_{i-2}	f_{i-2}		ε		
		ε		-3ε	
x_{i-1}	f_{i-1}				6ε
x_i	$f_i+\varepsilon$		-2ε		
		$-\varepsilon$		3ε	
x_{i+1}	f_{i+1}		$+\varepsilon$		
x_{i+2}	f_{i+2}			$-\varepsilon$	

$\varepsilon = \frac{1}{2}\cdot 10^{-k}$ tačnost f_i

$$|\Delta^m f_i| < \frac{1}{2}\cdot 10^{-k}\cdot 2^m$$

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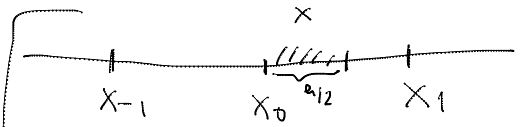
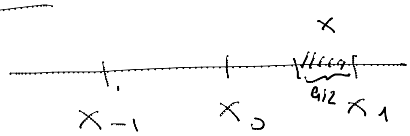


$x \in (x_0, x_0 + \frac{h}{2})$

$(*)$

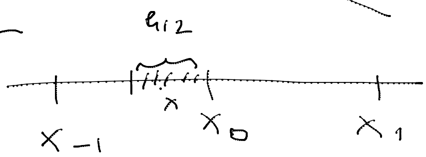
$x_0, x_1, x_{-1}, x_2, x_{-2}, \ldots$

Gaus unapred



$x_1, x_0, x_2, x_{-1}, x_3, x_{-2}, \ldots$

Gaus unazad $(**)$



$x_0, x_{-1}, x_1, x_{-2}, x_2 \ldots$

Gaus unazad $(***)$

$|q| \leq 0.25$

$(*)$ i $(***)$ aritmetička sredina $\Rightarrow$ Stirling

$(*)$ i $(**)$ —||— $\Rightarrow$ Besel

$0.75 \geq |q| \geq 0.25$