$$Af^* = \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right| \cdot Ax_i^*$$

$\hookrightarrow \frac{\partial f}{\partial x_i}(x_1^*, \ldots, x_n^*)$

Tuesday, March 02, 2021
5:59 PM

1) zbir

$$f(x_1, \ldots, x_n) = x_1 + \ldots + x_n = S$$

$$\Delta S^* = |S - S^*| = \sum_{i=1}^{n} \underbrace{\left| \frac{\partial S}{\partial x_i} \right|}_{1} \cdot \Delta x_i^* = \sum_{i=1}^{n} \Delta x_i^*$$

$$AS^* = \sum_{i=1}^{n} Ax_i^*$$

2) razlika

$$f(x_1, x_2) = x_1 - x_2 = R$$

$$\Delta R^* = |R - R^*| = \sum_{i=1}^{2} \underbrace{\left| \frac{\partial R}{\partial x_i} \right|}_{|\pm 1| = 1} \cdot \Delta x_i^* = \sum_{i=1}^{2} \Delta x_i^*$$

$$AR^* = Ax_1^* + Ax_2^*$$

3) proizvod i količnik

$$f(x_1, \ldots, x_n) = x_1^{e_1} \cdot \ldots \cdot x_n^{e_n} = P, \quad e_i = \pm 1$$

$$AP^* = \sum_{i=1}^{n} \left| \frac{\partial P}{\partial x_i} \right| \cdot Ax_i^*$$

$$\frac{\partial P}{\partial x_k} = e_k \cdot x_1^{e_1} \cdots x_{k-1}^{e_{k-1}} \cdot \underline{x_k^{e_k - 1}} \cdot x_{k+1}^{e_{k+1}} \cdots x_n^{e_n}$$

$$= e_k \cdot \frac{P}{x_k}$$

$$AP^* = \sum_{i=1}^{n} |e_i| \cdot \left| \frac{P}{x_k} \right| \cdot Ax_i^* \qquad / : |P^*| \neq 0$$

$$RP^* = \frac{AP^*}{|P^*|} = \sum_{i=1}^{n} \underbrace{|e_i|}_{1} \cdot \left| \frac{\cancel{P}}{x_k} \right| \cdot \frac{1}{\cancel{|P|}} \cdot Ax_i^* = \sum_{i=1}^{n} Rx_i^*$$

4) stepen $\quad f(x) = x^k \quad \ldots \quad Rf^* = |k| \cdot Rx^*$

$f(x) = \sqrt[k]{x} \quad \ldots \quad Rf^* = \left|\frac{1}{k}\right| \cdot Rx^*$

Obrtan problem greške

Tuesday, March 02, 2021
6:36 PM

$$\begin{cases} \Delta x_1^*, \dots, \Delta x_n^* \quad ? \\ \Delta f^* \le \varepsilon \end{cases}$$

$$\Delta f^* = \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right| \cdot \Delta x_i^* \le \varepsilon$$

a) $f : D \to \mathbb{R},\ D \subseteq \mathbb{R}$

$f(x)$, $n = 1$

$$|f'(x^*)| \cdot \Delta x^* \le \varepsilon \Rightarrow \Delta x^* \le \frac{\varepsilon}{|f'(x^*)|}, \quad f'(x^*) \ne 0$$

$$\Delta x^* \le \frac{\Delta f^*}{|f'(x^*)|}$$

b) $f : D \to \mathbb{R},\ D \subseteq \mathbb{R}^n$

$f(x_1, \dots, x_n)$

1) Princip jednakih uticaja

$$\left|\frac{\partial f}{\partial x_1}\right| \cdot \Delta x_1^* = \left|\frac{\partial f}{\partial x_2}\right| \cdot \Delta x_2^* = \dots = \left|\frac{\partial f}{\partial x_n}\right| \cdot \Delta x_n^*$$

$$\Rightarrow \Delta f^* = n \cdot \left|\frac{\partial f}{\partial x_i}\right| \cdot \Delta x_i^*$$

$$\Rightarrow \boxed{\Delta x_i^* = \frac{\Delta f^*}{n \cdot \left|\frac{\partial f}{\partial x_i}\right|}} \qquad i = 1, \dots, n$$

2) Princip jednakih apsolutnih grešaka

$$\Delta x_1^* = \dots = \Delta x_n^*$$

$$\Delta f^* = \Delta x_i^* \sum_{j=1}^{n} \left|\frac{\partial f}{\partial x_j}\right|$$

$$\boxed{\Delta x_i^* = \frac{\Delta f^*}{\sum_{j=1}^{n} \left|\frac{\partial f}{\partial x_j}\right|}}$$

3) princip jednakih relativnih grešaka

$$R x_1^* = \dots = R x_n^*, \quad \frac{\Delta x_i}{|x_i^*|} = R x_i^*$$

$$\Delta f^* = \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right| \cdot \Delta x_i^* = \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right| \cdot \underbrace{\frac{\Delta x_i^*}{|x_i^*|}}_{R x_i^*} \cdot |x_i^*|$$

$$= \frac{\Delta x_k^*}{|x_k^*|} \cdot \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right| \cdot |x_i^*|$$

$$\Rightarrow \boxed{\Delta x_k^* = \frac{\Delta f^* \cdot |x_k^*|}{\sum_{i=1}^{n} \left| x_i^* \cdot \frac{\partial f}{\partial x_i} \right|}}$$

Aproksimacija

Tuesday, March 02, 2021
7:05 PM

$f(x) =$:(

x	x_0	$\dots$	x_n
f	f_0	$\dots$	f_n

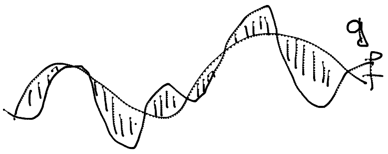
$$f(x) \approx g(x)$$

$$g(x) = \sum_{i=1}^{n} \underbrace{\phi_i(x)} \cdot c_i$$

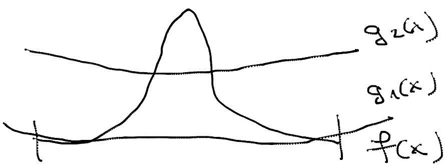
$1, x, x^2, \dots, x^n$

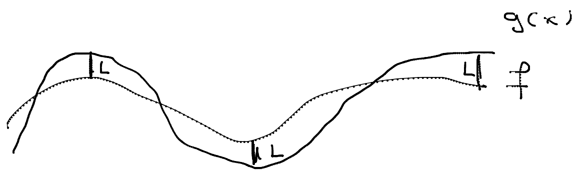
$1, \cos x, \sin x, \cos 2x, \dots$

$\vdots$



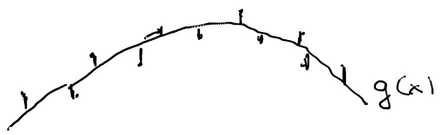
srednjekvadratna aproks.

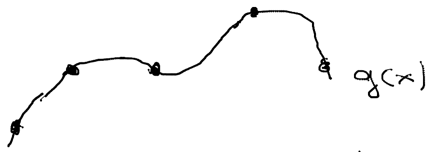




$$|f(x) - g(x)| \leq L$$

ravnomerna aproks.





$f(x_k) = g(x_k)$

Interpolacija

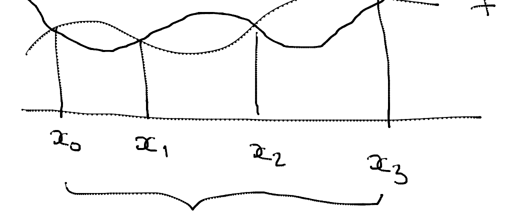
$$g(x) = \sum_{i=0}^{n} c_i \cdot x^i$$

n+1 nepoznata

n+1 uslova

sistem lin. jna (ima 1 rešenje)

loše uslovljen



$x \in [x_0, x_n]$ interpolacija

$x \notin [x_0, x_n]$ ekstrapolacija

Interpolacioni polinom Lagranža

Tuesday, March 02, 2021
7:25 PM

$$L_n(x) = \sum_{i=0}^{n} c_i \cdot x^i$$

(T) Postoji jedinstveno određen polinom $L_n(x)$ stepena n koji u $n+1$ različitoj tački x_k, $k=0,\dots,n$ zadovoljava uslove interpolacije $L_n(x_k) = f(x_k)$

Dokaz: pps. $\exists\, L_n^1(x) \neq L_n^2(x)$

$$L_n^1(x_k) = f(x_k) = L_n^2(x_k)$$

$$Q(x) = L_n^1(x) - L_n^2(x) \qquad st(Q) \leq n$$

$$Q(x_k) = L_n^1(x_k) - L_n^2(x_k) = f(x_k) - f(x_k) = 0 \quad , k=0,\dots,n$$

$\hookrightarrow$ $n+1$ tačka $\qquad Q(x) \equiv 0 \Rightarrow L_n^1(x) = L_n^2(x)$ $\lightning$

Egzistencija:

$$l_i(x) \text{ , stepena } n \text{ , } l_i(x_j) = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases} \quad i,j=0,\dots,n$$

x_k, $k=0,\dots,i-1, i+1,\dots,n$ su nule $l_i(x)$

$$\Rightarrow l_i(x) = C \cdot (x-x_0) \cdot \ldots \cdot (x-x_{i-1}) \cdot (x-x_{i+1}) \cdot \ldots \cdot (x-x_n)$$

$$= C \cdot \prod_{\substack{j=0 \\ j\neq i}}^{n} (x-x_j) \quad , C - \text{const}$$

$$l_i(x_i) = 1 = C \cdot \prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j) \Rightarrow C = \frac{1}{\prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)}$$

$$l_i(x) = \frac{\prod_{\substack{j=0 \\ j\neq i}}^{n} (x-x_j)}{\prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i-x_j)}$$

$$L_n(x) = \sum_{i=0}^{n} l_i(x) \cdot f(x_i)$$

$$\boxed{L_n(x) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \cdot f(x_i)}$$

$\boxtimes$

Tuesday, March 02, 2021
7:43 PM

$$\omega_{n+1}(x) = \prod_{j=0}^{n}(x-x_j)$$

$$\omega'_{n+1}(x) = \sum_{k=0}^{n}\Big(\prod_{\substack{j=0\\ j\neq k}}^{n}(x-x_j)\Big)$$

$$\omega'_{n+1}(x_i) = \prod_{\substack{j=0\\ j\neq i}}^{n}(x_i-x_j)$$

$$L_n(x)=\sum_{i=0}^{n} \frac{\omega_{n+1}(x)\cdot f(x_i)}{(x-x_i)\cdot \omega'_{n+1}(x_i)}$$

$f \xrightarrow{\text{greška met.}} L_n$ $\qquad f(x)-L_n(x)=R(x)$

(T) Ako je f diferencijabilna $n+1$ puta tada $\forall$ argument $\bar{x}$ postoji tačka ξ koja pripada minimalnom intervalu koji sadrži sve tačke $x_0,\dots,x_n,\bar{x}$ takva da

$$f(\bar{x})-L_n(\bar{x}) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot \omega_{n+1}(x)$$

a) $\bar{x}=x_j$, $j=0,\dots,n$

$f(x_j)=L_n(x_j) \Rightarrow R(\bar{x})=R(x_j)=0$

$\overset{R}{0}=\overset{\omega}{0}$

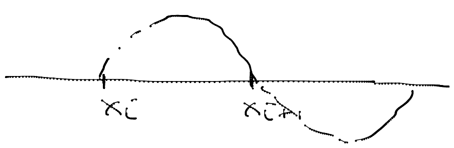
b) $\bar{x}\neq x_j$

$F(x) = \underbrace{f(x)-L_n(x)} - \underset{\equiv}{k}\cdot \omega_{n+1}(x)$, k - const

x_k, $k=0,\dots,n$: $F(x_k)=0$

$\bar{x}$: $F(\bar{x})=0 \Rightarrow k=\dfrac{f(\bar{x})-L_n(\bar{x})}{\omega_{n+1}(\bar{x})}$

$\left.\right\}$ $n+1$ $+$ 1 $\left.\right\}$ F $n+2$ nule $x_0,\dots,x_n,\bar{x}$



Rolova t.

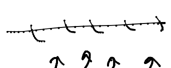
F neprekidna ✓

F diferenc. ✓

$F(x_i)=F(x_{i+1})=0$

$\Rightarrow \exists c\ F'(c)=0$

$n+2$ tačke



$n+1$ interval