Metoda Bairstow

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(*) $P_n(x) = \underset{=}{1} \cdot x^n + a_1 x^{n-1} + \dots + a_{n-1}x + a_n$

$P_n(x) = 0$

$P_n(x) = \underbrace{(x^2 + px + q)}_{x_{1,2}} \underbrace{(x^{n-2} + b_1 x^{n-3} + \dots + b_{n-3}x + b_{n-2})}$

$p, q = ?$

(*) $P_n(x) = (x^2+px+q)(x^{n-2} + b_1 x^{n-3} + \dots + b_{n-3}x + b_{n-2}) + \underbrace{r \cdot x + s}_{\text{ostatak}}$

$r(p,q) = 0, \quad s(p,q) = 0$

(*) koeficijenti:

$x^{n-1}: \quad a_1 = b_1 + p$

$x^{n-2}: \quad a_2 = b_2 + p \cdot b_1 + q$

$x^{n-3}: \quad a_3 = b_3 + p \cdot b_2 + q \cdot b_1$

$\vdots$

$x^{n-k}: \quad a_k = b_k + p \cdot b_{k-1} + q \cdot b_{k-2}$

$\vdots$

$x^2: \quad a_{n-2} = b_{n-2} + p \cdot b_{n-3} + q \cdot b_{n-4}$

$x^1: \quad a_{n-1} = p \cdot b_{n-2} + q \cdot b_{n-3} + r$

$x^0: \quad a_n = q \cdot b_{n-2} + s$

$\boxed{b_k = a_k - p \cdot b_{k-1} - q \cdot b_{k-2}}$ $\quad k = \overset{1,2}{3}, 4, \dots, n-2, \overset{n-1,n}{\dots}, \quad b_0 = 1, b_{-1} = 0$

$b_2 = a_2 - p \cdot b_1 - q \cdot b_0 \Rightarrow b_0 = 1$

$b_1 = a_1 - p \cdot b_0 - q \cdot b_{-1} \Rightarrow b_{-1} = 0$

$r = a_{n-1} - p \cdot b_{n-2} - q \cdot b_{n-3}$

$= b_{n-1}$

$s = \underbrace{a_n - q \cdot b_{n-2} - p \cdot b_{n-1}} + p \cdot b_{n-1}$

$= b_n + p \cdot b_{n-1}$

$$\left.\begin{aligned} r(p,q) &= b_{n-1}(p,q) = 0 \\ s(p,q) &= b_n(p,q) + p \cdot b_{n-1}(p,q) = 0 \end{aligned}\right\} \text{ sistem nelin. jna } (\square)$$

NjtM: $x_{n+1} = x_n - [F'(x_n)]^{-1} \cdot F(x_n)$

$F'(x_n) \,/\quad \underbrace{x_{n+1} - x_n}_{\Delta x_n} = -[F'(x_n)]^{-1} \cdot F(x_n)$

$$F'(x_n) \cdot \Delta x_n = -F(x_n)$$

$$\Rightarrow \quad F(x_n) + F'(x_n) \cdot \Delta x_n = 0$$

$$F = \begin{pmatrix} r \\ s \end{pmatrix}, \quad x_n = \begin{pmatrix} p_n \\ q_n \end{pmatrix}, \quad \Delta x_n = x_{n+1} - x_n = \begin{pmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{pmatrix} = \begin{pmatrix} \Delta p_n \\ \Delta q_n \end{pmatrix}$$

$$F' = \begin{pmatrix} \frac{\partial r}{\partial p} & \frac{\partial r}{\partial q} \\ \frac{\partial s}{\partial p} & \frac{\partial s}{\partial q} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} r(p_n, q_n) \\ s(p_n, q_n) \end{pmatrix} + \begin{pmatrix} \frac{\partial r}{\partial p} & \frac{\partial r}{\partial q} \\ \frac{\partial s}{\partial p} & \frac{\partial s}{\partial q} \end{pmatrix} \cdot \begin{pmatrix} \Delta p_n \\ \Delta q_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \frac{\partial r}{\partial p} \cdot \Delta p_n + \frac{\partial r}{\partial q} \cdot \Delta q_n + r(p_n, q_n) &= 0 \\ \frac{\partial s}{\partial p} \cdot \Delta p_n + \frac{\partial s}{\partial q} \cdot \Delta q_n + s(p_n, q_n) &= 0 \end{aligned}\right\}$$

$\Delta p_n, \Delta q_n = ?$

$\left.\begin{aligned} r &= b_{n-1} \\ s &= b_n + p b_{n-1} \end{aligned}\right\} (\square)$

$$\Rightarrow \begin{cases} \frac{\partial b_{n-1}}{\partial p} \cdot \Delta p_n + \frac{\partial b_{n-1}}{\partial q} \cdot \Delta q_n + b_{n-1} = 0 \\ \Big(\underbrace{\frac{\partial b_n}{\partial p} + 1 \cdot b_{n-1} + p \cdot \frac{\partial b_{n-1}}{\partial p}}_{\frac{\partial s}{\partial p}}\Big) \cdot \Delta p_n + \Big(\underbrace{\frac{\partial b_n}{\partial q} + p \cdot \frac{\partial b_{n-1}}{\partial q}}_{\frac{\partial s}{\partial q}}\Big) \cdot \Delta q_n + \underbrace{b_n + p \cdot b_{n-1}}_{s} = 0 \end{cases}$$

$$p \cdot \Big(\underbrace{\frac{\partial b_{n-1}}{\partial p} \cdot \Delta p_n + \frac{\partial b_{n-1}}{\partial q} \cdot \Delta q_n + b_{n-1}}_{= \text{ prva jednadžba } = 0}\Big) = p \cdot 0 = 0$$

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$$\Rightarrow \begin{cases} \dfrac{\partial b_{m-1}}{\partial p}\cdot \Delta p_n + \dfrac{\partial b_{m-1}}{\partial q}\cdot \Delta q_n + b_{m-1} = 0 \\ \left(\dfrac{\partial b_m}{\partial p} + b_{m-1}\right)\cdot \Delta p_n + \dfrac{\partial b_m}{\partial q}\cdot \Delta q_n + b_m = 0 \end{cases} \quad (\heartsuit)$$

$$\frac{\partial b_k}{\partial p}, \frac{\partial b_k}{\partial q} = ?$$

$$b_0 = 1,\ b_{-1} = 0 \Rightarrow \frac{\partial b_0}{\partial p} = \frac{\partial b_0}{\partial q} = \frac{\partial b_{-1}}{\partial p} = \frac{\partial b_{-1}}{\partial q} = 0$$

$$b_k = a_k - p\cdot b_{k-1} - q\cdot b_{k-2} \quad , k = 1, 2, \dots, m-2$$

$$\frac{\partial b_k}{\partial p} = -1\cdot b_{k-1} - p\cdot \frac{\partial b_{k-1}}{\partial p} - q\cdot \frac{\partial b_{k-2}}{\partial p} \quad (1)$$

$$\frac{\partial b_k}{\partial q} = -p\cdot \frac{\partial b_{k-1}}{\partial q} - 1\cdot b_{k-2} - q\cdot \frac{\partial b_{k-2}}{\partial q} \quad (2)$$

(1): $\quad c_{k-1} = -\dfrac{\partial b_k}{\partial p}$ (smena)

$$c_{k-1} = b_{k-1} - p\cdot c_{\substack{k-1\\k-2}} - q\cdot c_{\substack{k-2\\k-3}}$$

$$\boxed{c_{k-1} = b_{k-1} - p\cdot c_{k-2} - q\cdot c_{k-3}}$$

$k=2$: $\quad c_1 = -\dfrac{\partial b_2}{\partial p} = -\dfrac{\partial(a_2 - pb_1 - q)}{\partial p} = b_1 + p\cdot \underbrace{\dfrac{\partial b_1}{\partial p}}_{-c_0} = b_1 - p\cdot c_0$

$k=1$: $\quad c_0 = -\dfrac{\partial b_1}{\partial p} = -\dfrac{\partial(a_1 - p)}{\partial p} = 1 \Rightarrow c_0 = 1$

$$c_1 = b_1 - p\cdot c_0 - q\cdot c_{-1} \Rightarrow c_{-1} = 0$$

$$c_0 = 1 = b_0 = \underbrace{b_0}_{1} - p\cdot \underbrace{c_{-1}}_{0} - q\cdot \underbrace{c_{-2}} \Rightarrow c_{-2} = 0$$

$$\Rightarrow \boxed{c_k = b_k - p\cdot c_{k-1} - q\cdot c_{k-2},} \quad k = 1, \dots, m-2$$
$$c_0 = 1,\ c_{-1} = 0$$

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(2) : $\frac{\partial b_k}{\partial q} = -b_{k-2} - q \cdot \frac{\partial b_{k-2}}{\partial q} - p \cdot \frac{\partial b_{k-1}}{\partial q}$

$d_{k-2} = \frac{\partial b_k}{\partial q}$ (smena)

$\vdots$

$$\begin{cases} d_k = b_k - p \cdot d_{k-1} - q \cdot d_{k-2}, & k = 1, \dots, n-2 \\ d_0 = 1, \; d_{-1} = 0 \end{cases}$$

$\Rightarrow$ (isto) $\boxed{\begin{array}{l} c_k = b_k - p \cdot c_{k-1} - q \cdot c_{k-2}, \quad k = 1, \dots, n-2 \\ c_{-1} = 0, \; c_0 = 1 \end{array}}$

$$\left.\begin{array}{l} \frac{\partial b_{n-1}}{\partial p} \cdot \Delta p + \frac{\partial b_{n-1}}{\partial q} \cdot \Delta q + b_{n-1} = 0 \\ \left(\frac{\partial b_n}{\partial p} + b_{n-1}\right) \cdot \Delta p + \frac{\partial b_n}{\partial q} \cdot \Delta q + b_n = 0 \end{array}\right\} \text{ sistem } (\heartsuit)$$

$\frac{\partial b_{n-1}}{\partial p} = -c_{n-2}$, $\frac{\partial b_{n-1}}{\partial q} = -c_{n-3}$ $\left(c_{k-1} = -\frac{\partial b_k}{\partial p}\right)$

$c_{k-2} = -\frac{\partial b_k}{\partial q}$

$\frac{\partial b_n}{\partial p} = -c_{n-1}$ $\frac{\partial b_n}{\partial q} = -c_{n-2}$

$$\Rightarrow \left.\begin{array}{l} c_{n-2} \cdot \Delta p + c_{n-3} \cdot \Delta q = b_{n-1} \\ (c_{n-1} - b_{n-1}) \cdot \Delta p + c_{n-2} \cdot \Delta q = b_n \end{array}\right\} \Delta p, \Delta q = \dots \quad (\star)$$

$a_1, \dots, a_n \longrightarrow b_1, \dots, b_n$ $(b_k = a_k - p \cdot b_{k-1} - q \cdot b_{k-2})$

$b_i \longrightarrow c_i$ $(c_k = b_k - p \cdot c_{k-1} - q \cdot c_{k-2})$

$D = \begin{vmatrix} c_{n-2} & c_{n-3} \\ c_{n-1} - b_{n-1} & c_{n-2} \end{vmatrix} = c_{n-2}^2 - c_{n-3}(c_{n-1} - b_{n-1})$ Kramerovo pravilo

$D_p = \begin{vmatrix} b_{n-1} & c_{n-3} \\ b_n & c_{n-2} \end{vmatrix} = \dots$

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$(\star)$: $\underbrace{c_{n-1}}$ $- b_{n-1}$

$\underbrace{\cancel{b_{n-1}} - p \cdot c_{n-2} - q \cdot c_{n-3}} - \cancel{b_{n-1}}$

$\Rightarrow \; -(p\, c_{n-2} + q\, c_{n-3}) \cdot \Delta p + c_{n-2} \cdot \Delta q = b_n$ (2 jednačine u $\star$)

Rešavanjem sistema $(\star) \Rightarrow \Delta p, \Delta q$

$\Delta p = p_{n+1} - p_n \Rightarrow p_{n+1} = p_n + \Delta p$

$\Delta q = q_{n+1} - q_n \Rightarrow q_{n+1} = q_n + \Delta q$

$(p_0, q_0) \to (p_1, q_1) \to \dots \to (p^*, q^*)$

Norma $\|\Delta x_n\| <$ tol

Krit. zaust: $\Delta p_n <$ tol i $\Delta q_n <$ tol

$P_n(x) : (x^2 + p \cdot x + q) = \underbrace{\dots}_{\text{za narednu iteraciju}}$

$(*)$ $p_0 = q_0 = 0$

$x_1 \approx d_1, \quad x_2 \approx d_2 \Rightarrow p_0 \approx -(d_1 + d_2)$

$q_0 \approx d_1 \cdot d_2$

(*) $P_4(x) = 2x^4 - 20x^3 + 68x^2 - 100x + 50$, $\varepsilon = 10^{-4}$

$n = 4$

$p_0 = -5.8$, $q_0 = 5$

$$P_4(x) = 0 = \overset{a_0}{1} \cdot x^4 - \overset{a_1}{10}x^3 + \overset{a_2}{34}x^2 - \overset{a_3}{50}x + \overset{a_4}{25}$$

		$n=0$		$n=1$		$n=2$		$n=3$	
p		-5.8		-6.022		-5.9996		-6	
q		5		4.9324		4.9941		+5	
k	a	b	c	b	c	b	c	b	c
0	1	1	1	1	1	1	1	1	1
1	-10	-4.2	1.6	-3.978	2.0439	-4.0004	1.9992	-4	2
2	34	4.64	8.92	5.1120	12.4879	5.0050	12.0051	5	12
3	-50	-2.088		0.4055		0.0068		0	
4	25	-10.3104		2.2275		0.0448		0	
Δp		-0.222		0.0224		$4 \cdot 10^{-4}$		$2 \cdot 10^{-6}$	$< \varepsilon$
Δq		-0.0676		0.0617		0.0058		$1 \cdot 10^{-5}$	$< \varepsilon$

$b_0 = 1$, $c_0 = 1$, $b_{-1} = 0$, $c_{-1} = 0$

$b_k = a_k - p \cdot b_{k-1} - q \cdot b_{k-2}$

$b_1 = a_1 - p_0 \cdot b_0 - q_0 \cdot b_{-1} = -10 + 5.8 \cdot 1 - 5 \cdot 0 = -4.2$

$b_2 = a_2 - p_0 \cdot b_1 - q_0 \cdot b_0 = 34 + 5.8 \cdot -4.2 - 5 \cdot 1 = 4.64$

$b_3 = a_3 - p_0 \cdot b_2 - q_0 \cdot b_1 = -50 + 5.8 \cdot 4.64 + 5 \cdot 4.2 = -2.088$

$b_4 = a_4 - p_0 \cdot b_3 - q_0 \cdot b_2 = 25 - 5.8 \cdot 2.088 - 5 \cdot 4.64 = -10.3104$

$c_k = b_k - p \cdot c_{k-1} - q \cdot c_{k-2}$

$c_1 = b_1 - p_0 \cdot c_0 - q_0 \cdot c_{-1} = -4.2 + 5.8 \cdot 1 - 5 \cdot 0 = 1.6$

$c_2 = b_2 - p_0 \cdot c_1 - q_0 \cdot c_0 = 4.64 + 5.8 \cdot 1.6 - 5 \cdot 1 = 8.92$

$$\left.\begin{aligned} &c_2 \cdot \Delta p + c_1 \cdot \Delta q = b_3 \\ &-(c_2 \cdot p_0 + c_1 \cdot q_0) \cdot \Delta p + c_2 \cdot \Delta q = b_4 \end{aligned}\right\} \quad \begin{aligned} \Delta p &= -0.222 \\ \Delta q &= -0.0676 \end{aligned}$$

$p_1 = p_0 + \Delta p = -6.022$

$q_1 = q_0 + \Delta q = 4.9324$

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$p^* = -6, \; q^* = 5$

$P_4(x) = \underbrace{(x^2 - 6x + 5)}_{x_{1,2} = 1,\ 5} \underbrace{(\ldots)}$

a) $(x^2 + b_1 \cdot x + b_2) = (x^2 - 4x + 5)$

b) $P_4(x) : (x^2 - 6x + 5) = (\ldots)$

$x_{3,4} = 2+i,\ 2-i$

$$\underset{a_1}{x^4} - \underset{a_2}{10x^3} + \underset{a_3}{34x^2} - \underset{a_4}{50x} + 25 : x^2 \underset{p}{- 6x} \underset{q}{+ 5} = \underbrace{x^2 \underset{a_1}{- 4x} \underset{a_2}{+ 5}}_{\text{novi}}$$

$$x^4 - 6x^3 + 5x^2 \quad \ominus \qquad (=q)$$

$$-4x^3 + 29x^2 - 50x + 25$$

$$-4x^3 + 24x^2 - 20x \quad \ominus \qquad (= a_1(\text{novo}) \cdot p)$$

$$\Rightarrow \boxed{5}x^2 - 30x + 25$$

$$5x^2 - 30x + 25$$

$$0$$

$a_1(\text{novo}) = a_1 - p$

$a_2(\text{novo}) = a_2 - a_1(\text{novo}) \cdot p - q \cdot 1$

$a = [-\cancel{10} \quad 34 \quad -50 \quad 25]$
-4