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(T) Neka C konveksna oblast u $\mathcal{X}$, $x_0 \in C$, $F: \mathcal{X} \to \mathcal{Y}$ neprekidni operator t.d. $\exists F'(x), \forall x \in C$ koji ima sl. osobine:

(a) $\| F'(x) - F'(y) \| \le \gamma \|x - y\| \quad , \forall x, y \in C$

(b) $\exists [F'(x)]^{-1} \quad ; \quad \| [F'(x)]^{-1} \| \le \beta \quad ; \forall x \in C$

(c) $\| [F'(x_0)]^{-1} \cdot F(x_0) \| \le \alpha$

gde su α, β, γ konstante t.d. $h = \frac{\alpha \beta \gamma}{2} < 1$.

Neka je $S(x_0, r) = \{ x \mid \|x - x_0\| < r \} \subseteq C$, pri čemu je $r = \frac{\alpha}{1-h}$. Tada:

1) Svi članovi niza određenog rekurentnom formulom

$$x_{n+1} = x_n - [F'(x_n)]^{-1} \cdot F(x_n) \quad , n = 0, 1, \dots$$

sa početnom tačkom x_0, pripadaju $S(x_0, r)$

2) $\exists x^* \in \overline{S(x_0, r)}$, t.d. $\lim\limits_{n \to \infty} x_n = x^*$ i $F(x^*) = 0$

3) $\forall n \ge 0 \quad : \quad \|x_n - x^*\| \le \alpha \cdot \dfrac{h^{2^n - 1}}{1 - h^{2^n}}$

(D) 1) $x_0 \in C$, $x_0 \in S$

$$x_1 = x_0 - [F'(x_0)]^{-1} \cdot F(x_0)$$

$$\|x_1 - x_0\| = \| \cancel{x_0} - [F'(x_0)]^{-1} \cdot F(x_0) - \cancel{x_0} \| \overset{(c)}{\le} \alpha < r$$

$r \cdot (1-h) < r$

$h < 1$

$$\Rightarrow x_1 \in S(x_0, r)$$

$$\|x_{n+1} - x_0\| = \| x_{n+1} - x_n + x_n - x_{n-1} + \dots + x_1 - x_0 \|$$

$$\le \underbrace{\|x_{n+1} - x_n\|} + \underbrace{\|x_n - x_{n-1}\|} + \dots + \underbrace{\|x_1 - x_0\|} \qquad (*)$$

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$$\|x_{n+1} - x_n\| = \| \cancel{x_n} - \underbrace{[F'(x_n)]^{-1}} \cdot F(x_n) \ \cancel{- x_n} \|$$

$$\overset{(b)}{\leq} \beta \cdot \|F(x_n)\|$$

$$\boxed{\begin{array}{l} \text{oblik Njut. met.} \\ F(x_{n-1}) + F'(x_{n-1})(x_n - x_{n-1}) = 0 \end{array}}$$

$$= \beta \cdot \| F(x_n) - \underbrace{F(x_{n-1}) - F'(x_{n-1})(x_n - x_{n-1})}_{=0} \|$$

$$\overset{\text{Lema}}{\leq} \beta \cdot \frac{\gamma}{2} \|x_n - x_{n-1}\|^2 \qquad (C \text{ konveksan}, (a))$$

$$= \frac{h}{\alpha} \|x_n - x_{n-1}\|^2 \qquad / \frac{h}{\alpha}$$

$$\frac{h}{\alpha}\|x_{n+1} - x_n\| \leq \left(\frac{h}{\alpha} \underbrace{\|x_n - x_{n-1}\|}\right)^2$$

$$\leq \left(\frac{h}{\alpha} \|x_{n-1} - x_{n-2}\|\right)^2$$

$$\leq \left(\frac{h}{\alpha} \|x_{n-1} - x_{n-2}\|\right)^4$$

$$\leq \dots$$

$$\leq \left(\frac{h}{\alpha} \underbrace{\|x_1 - x_0\|}_{\leq \alpha}\right)^{2^n}$$

$$\leq h^{2^n} \qquad / \cdot \frac{\alpha}{h}$$

$$\|x_{n+1} - x_n\| \leq \alpha \cdot h^{2^n - 1} \qquad (\heartsuit)$$

$$(*) : \ \|x_{n+1} - x_0\| \leq \alpha \cdot h^{2^n - 1} + \alpha \cdot h^{2^{n-1} - 1} + \dots + \alpha \cdot h^{2^1 - 1} + \alpha \cdot h^{2^0 - 1}$$

$$= \alpha \left(1 + h + h^3 + \dots + h^{2^n - 1}\right)$$

$$< \alpha \cdot \sum_{k=0}^{\infty} h^k = \frac{\alpha}{1 - h} = r$$

$$\Rightarrow x_{n+1} \in S(x_0, r)$$

2) $k > n$

$$\| x_{k+1} - x_n \| = \| x_{k+1} - x_k + x_k - x_{k-1} + \dots + x_{n+1} - x_n \|$$

$$\leq \| x_{k+1} - x_k \| + \| x_k - x_{k-1} \| + \dots + \| x_{n+1} - x_n \|$$

$$\overset{(\heartsuit)}{\leq} \alpha \cdot h^{2^k - 1} + \alpha \cdot h^{2^{k-1} - 1} + \dots + \alpha \cdot h^{2^n - 1}$$

$$\leq \alpha \cdot h^{2^n - 1} \left(1 + h^{2^n} + (h^{2^n})^2 + \dots \right)$$

$$= \underbrace{\alpha \cdot h^{2^n - 1}}_{(n \to \infty) \to 0,\ h<1} \cdot \frac{1}{1 - h^{2^n}} \xrightarrow[h<1]{n \to \infty} 0$$

$\Rightarrow \{x_k\}$ Коши ред

$\Rightarrow \exists$ ка некои $x^* \in \overline{S(x_0, r)}$

$\Rightarrow \lim\limits_{n \to \infty} x_n = x^*$

$F(x^*) = 0$???

(a) $\| F'(x_n) - F'(x_0) \| \leq \delta \underbrace{\| x_n - x_0 \|}_{\leq r} \leq \delta \cdot r$

$\| \cdot \| - \| \cdot \| \leq \| * + * \| \leq \| \cdot \| + \| \cdot \|$

$$\| F'(x_n) \| - \| F'(x_0) \| \leq \| F'(x_n) - F'(x_0) \| \leq \delta \cdot r$$

$$\| F'(x_n) \| \leq \delta \cdot r + \underbrace{\| F'(x_0) \|}_{\text{const}} = C$$

Њут. мет:

$$F(x_n) = -F'(x_n)(x_{n+1} - x_n) \quad / \| \cdot \|$$

$$\| F(x_n) \| = \| F'(x_n)(x_{n+1} - x_n) \|$$

$$\leq \underbrace{\| F'(x_n) \|} \cdot \| x_{n+1} - x_n \| .$$

$$\leq C \cdot \underbrace{\| x_{n+1} - x_n \|}_{\to 0} \to 0$$

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$$\Rightarrow \lim_{n\to\infty} \|F(x_n)\| = 0$$

$$\Rightarrow \lim_{n\to\infty} \|F(x_n)\| \overset{\text{nepr.}}{=} \|F(\lim_{n\to\infty} x_n)\| = \|F(x^*)\| = 0$$

$\|*\| = 0 \Leftrightarrow * = 0$

$$\Rightarrow \quad F(x^*) = 0$$

3) $\|x_{k+1} - x_n\| \le \alpha \cdot \dfrac{q^{2^n-1}}{1-q^{2^n}}$ (it 2))

$$\lim_{k\to\infty} \|x_{k+1} - x_n\| = \underbrace{\|x^* - x_n\| \le \alpha \cdot \frac{q^{2^n-1}}{1-q^{2^n}}}$$

$$\|x^* - x_n\| \le \alpha \cdot \frac{q^{2^n-1}}{1-q^{2^n}} \le \varepsilon$$

$$\alpha \cdot q^{2^n-1} \le \varepsilon \cdot (1 - q^{2^n})$$

$$\frac{\alpha}{q} \cdot q^{2^n} \le \varepsilon - \varepsilon \cdot q^{2^n}$$

$$q^{2^n}\left(\frac{\alpha}{q} + \varepsilon\right) \le \varepsilon$$

$$q^{2^n} \le \frac{\varepsilon \cdot q}{\alpha + \varepsilon \cdot q} \quad / \log_2$$

$$\log_2 q^{2^n} \le \log_2 \frac{\varepsilon q}{\alpha + \varepsilon q}$$

$$2^n \cdot \log_2 q \le \quad -\|- \qquad | : \log_2 q$$

$$2^n \overset{q<1}{\ge} \frac{\log_2 \frac{\varepsilon q}{\alpha + \varepsilon q}}{\log_2 q} \qquad | \log_2$$

$$\boxed{n \ge \log_2 \frac{\log_2 \frac{q \cdot \varepsilon}{\alpha + \varepsilon q}}{\log_2 q}}$$

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Dogovor: $\| x_{n+1} - x_n \| < \varepsilon$

(T) ne obezbeđuje jedinstvenost $F(x) = 0$

str. 171 jedinstvenost.

Modifikovana Njutnova met.

$$x_{n+1} = x_n - \underbrace{[F'(x_0)]}_{\text{fiksirano}}^{-1} \cdot F(x_n)$$

$n_0 = 0, n_1, n_2, \ldots$
$\quad\quad\quad\ \ 5 \quad 10 \quad \ldots$

$$x_{n+1} = x_n - [F'(x_{n_k})]^{-1} \cdot F(x_n)$$

iter = $\cancel{0}$, 1, 2, 3, 4, 5 $\to F'(x_0)$
iter = 6, 7, 8, 9, 10 $\to F'(x_5)$

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$$\varepsilon = 10^{-2}$$

$$f_1 \equiv x + 3\log x - y^2 = 0$$

$$f_2 \equiv 2x^2 - xy - 5x + 1 = 0$$

$$F = \begin{pmatrix} x + 3\log x - y^2 \\ 2x^2 - xy - 5x + 1 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$(3.4,\ 2.2) = (x_0, y_0)$$

$$F' = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 + \frac{3}{x \ln 10} & -2y \\ 4x - y - 5 & -x \end{pmatrix}$$

$$F'(x_0, y_0) = \begin{bmatrix} 1.3832 & -4.4 \\ 6.4 & -3.4 \end{bmatrix}$$

$$X^{(1)} = X^{(0)} - (F'(x_0, y_0))^{-1} \cdot F(X^{(0)})$$

$$= \begin{pmatrix} 3.4 \\ 2.2 \end{pmatrix} - \begin{bmatrix} \cdots \end{bmatrix}^{-1} \cdot F(3.4,\ 2.2)$$

$$= \begin{pmatrix} 3.4899 \\ 2.2634 \end{pmatrix}$$

$$\|X^{(1)} - X^{(0)}\|_\infty < 10^{-2}$$

$$X^{(2)} = X^{(1)} - [F'(x_1, y_1)]^{-1} \cdot F(x_1, y_1)$$

$\vdots$