SISTEMI NELINEARNIH J-NA

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$$\left.\begin{aligned} f_1(x_1,\dots,x_n)&=0\\ &\vdots\\ f_n(x_1,\dots,x_n)&=0 \end{aligned}\right\}\qquad \begin{aligned} x&=(x_1,\dots,x_n)^T\\ f&=(f_1,\dots,f_n)^T\\ 0&=(0,\dots,0) \end{aligned}$$

$$f(x)=0$$

funkcional:

$$F(x)=\|f(x)\|_2^2=f^T(x)\cdot f(x)=\sum_{i=1}^{n} f_i^2(x)$$

$F(x)\geq 0$ (norma)

$F(x)=0 \iff f_i(x)\equiv 0, \forall i=1,\dots,n$

Rešenje sistema $f(x^*)=0$ je tačka min. funkcionala

I obrnuto

$$f(x^*)=0 \iff F(x^*)=\min_x F(x)$$

Iterativne: $x_{n+1}=G(x_0,\dots,x_n)$

$\boxed{x_{n+1}=G(x_n)}$ dvoslojna metoda stacionarne (G ne zavisi od n)

$$\|x_{n+1}-x^*\|\leq c\cdot\|x_n-x^*\|^p, \quad c\text{ - const}$$

p - red metode

Iterativna reda $p=1$

Nepokretna tačka operatora G : $G(x)=x$

G kontrakcija : $\exists\, 0\leq q<1$

$$\forall x,y\in S(x_0,r)\subset\mathbb{R}^n$$
$$S(x_0,r)=\{x \mid \|x-x_0\|\leq r\}$$

$$\|G(x)-G(y)\|\leq q\cdot\|x-y\|$$

Jakobijeva met.

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$f(x)=0 \rightarrow x=G(x)$ nije jednoznačna

$$\left.\begin{aligned} x_1^{(n+1)} &= g_1(x_1^{(n)}, \dots, x_m^{(n)}) \\ &\vdots \\ x_m^{(n+1)} &= g_m(x_1^{(n)}, \dots, x_m^{(n)}) \end{aligned}\right\} \quad x_{n+1} = G(x_n)$$

(*) g dif. u $S(x_0, r)$

S: x_0-r, x_0, x_0+r

$m=1$: (unm)

$$\forall\, x,y \in S(x_0,r): \quad g(x)-g(y) = g'(\xi)\cdot(x-y), \quad \xi \in S(x_0,r)$$

$$|g(x)-g(y)| = |g'(\xi)|\cdot|x-y|$$

$$\leq \underbrace{\max_{\xi\in S}|g'(\xi)|}_{q}\cdot|x-y|$$

$$q = \max_{\xi \in S}|g'(\xi)| < 1 \Rightarrow K$$

$$> 1$$

$m>1$: $$\|g(x)-g(y)\|_\infty = \max_{1\leq i\leq m}|g_i(x)-g_i(y)|$$

$$g_i(x)-g_i(y) = \sum_{j=1}^{m}\frac{\partial g_i(\xi_i)}{\partial x_j}\cdot(x_j-y_j), \quad \xi_i \in S(x_0,r) \quad \forall i=1,\dots,m$$

$$|g_i(x)-g_i(y)| \leq \sum_{j=1}^{m}\left|\frac{\partial g_i(\xi_i)}{\partial x_j}\right|\cdot|x_j-y_j|$$

$$\|g(x)-g(y)\|_\infty \leq \max_{1\leq i\leq m}\sum_{j=1}^{m}\left|\frac{\partial g_i(\xi_i)}{\partial x_j}\right|\cdot|x_j-y_j|$$

$$\leq \underbrace{\max_{1\leq j\leq m}|x_j-y_j|}\cdot\max_{1\leq i\leq m}\sum_{j=1}^{m}\left|\frac{\partial g_i(\xi_i)}{\partial x_j}\right|$$

$$= \|x-y\|_\infty \cdot \underbrace{\max_{1\leq i\leq m}\sum_{j=1}^{m}\left|\frac{\partial g_i(\xi_i)}{\partial x_j}\right|}_{\|J(\xi_i)\|_\infty}$$

$$\leq \|x-y\|_\infty \cdot \max\|J(\xi)\|_\infty$$

$$= \|x-y\|_\infty \cdot \underbrace{\max_{1 \le i \le m} \sum_{j=1} \frac{\partial}{\partial x_j}}_{\|J(\xi_i)\|_\infty}$$

$$\le \|x-y\|_\infty \cdot \max_{z \in S} \|J(z)\|_\infty$$

$$\underline{q} = \max_{z \in S} \|J(z)\|_\infty$$

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$$J(x)=\begin{pmatrix} \frac{\partial g_1(x)}{\partial x_1} & \cdots & \frac{\partial g_1(x)}{\partial x_m} \\ \vdots & & \\ \frac{\partial g_m(x)}{\partial x_1} & \cdots & \frac{\partial g_m(x)}{\partial x_m} \end{pmatrix}$$

Apriorna: $\|x^* - x_n\| \le \underbrace{\frac{q^n}{1-q} \|x_1 - x_0\| < \varepsilon}_{n \geq \dots ?}$

Aposteriorna: $\|x^* - x_n\| \le \underbrace{\frac{q}{1-q} \|x_n - x_{n-1}\| < \varepsilon}_{k?}$

$\|\underbrace{x_{n+1}}_{G(x_n)} - \underbrace{x^*}_{G(x^*)}\| \le \underbrace{q}_{c} \|x_n - x^*\|^1$

(*) Sa tačnošću 10^{-2} reši sistem:

$$x + 3\log x - y^2 = 0$$
$$2x^2 - xy - 5x + 1 = 0$$

u okolini tačke $(3.4, 2.2)$.

- lokalizacija S ✗
- $x^* \in S$

Dogovor 1: dato $x_0 \Rightarrow r = 0{,}1$
dato S

$$S = \{(x,y) \mid |x-3{,}4| \le 0{,}1 \,,\, |y-2.2| \le 0.1\}$$

$$x = -3\log x + y^2 \equiv g_1(x,y)$$

$$y = \frac{\cancel{2x^2 - 5x - 1}}{x} = 2x - 5 - \frac{1}{x} \equiv g_2(x,y)$$

$$J(x,y) = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{3}{x \cdot \ln 10} & 2y \\ 2 - \frac{1}{x^2} & 0 \end{pmatrix}$$

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$$\max_{x \in S} \left\{ \left| \frac{-3}{x \ln 10} \right| + |2y| ,\ \left| 2 - \frac{1}{x^2} \right| \right\}$$

Dogovor 2 :

$$\leq \max \left\{ \frac{3}{3{,}3 \cdot \ln 10} + 2 \cdot 2{,}3 ,\ 2 + \frac{1}{3{,}5^2} \right\}$$

$$\max \{ 4{,}9948 ,\ \ldots \}$$

$$4.9948 < 1 \ ?$$ nije kontrakcija

$$x^{(n+1)} = \sqrt{\tfrac{1}{2}\left(x^{(n)} y^{(n)} + 5 x^{(n)} - 1\right)} \equiv g_1(x,y)$$

$$y^{(n+1)} = \sqrt{x^{(n)} + 3 \log\left(x^{(n)}\right)} \equiv g_2(x,y)$$

$$J(x,y) = \begin{pmatrix} \dfrac{\frac{1}{2}(y+5)}{2\sqrt{\frac{1}{2}(xy+5x-1)}} & \dfrac{\frac{1}{2}x}{2\sqrt{\frac{1}{2}(xy+5x-1)}} \\ \dfrac{1 + \frac{3}{x \ln 10}}{2\sqrt{x + 3\log x}} & 0 \end{pmatrix}$$

$$\max_{(x,y) \in S} \left\{ \left| \frac{\frac{1}{2}(y+5)}{2\sqrt{\frac{1}{2}(xy+5x-1)}} \right| + \left| \frac{\frac{1}{2}x}{2\sqrt{}} \right| ,\ \left| \frac{1 + \frac{3}{x \ln 10}}{2\sqrt{x + 3\log x}} \right| \right\}$$

$$\leq \max \left\{ \frac{\frac{1}{2}(2{,}3+5)}{2\sqrt{\frac{1}{2}(3{,}3 \cdot 2{,}1 + 5 \cdot 3{,}3 - 1)}} + \frac{\frac{1}{2} \cdot 3{,}5}{-\text{"}-} ,\ \frac{1 + \frac{3}{3{,}3 \cdot \ln 10}}{2\sqrt{3{,}3 \cdot 3 \log 3{,}3}} \right\}$$

$$= \max \{ 0.8 ,\ 0.32 \} = 0.8 < 1$$ ✓

0.24 ?

$\sim q$

$$\| x_n - x_{n-1} \|_\infty \leq \frac{1-q}{q} \cdot \varepsilon$$

$$\max \left| x^{(n)} - x^{(n-1)} ,\ y^{(n)} - y^{(n-1)} \right| \leq \frac{1-0.8}{0.8} \cdot 10^{-2} = 0{,}25 \cdot 10^{-2}$$

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$x_0 = 3.4,\quad y_0 = 2.2$

$x_1 = \dots\quad y_1 = \dots$

$\vdots$

$x_8 = 3.48,\quad y_8 = 2.26$ ✓

(*) Dat je sistem nelin. jna:

$$x^3y^2 + 17y = 8.5$$
$$9x - x^4 \cdot \sin y^2 = 4.5$$

a) Metodom iteracije sa tačnošću $5\cdot 10^{-5}$ odrediti rješ u oblasti $D = \{(x,y) \mid 0.4 \le x,y \le 0.6\}$

b) Procijeniti broj iteracija koje je potrebno uraditi da bi se postigla tačnost 10^{-7}

a)
$$x^{(n+1)} = \frac{1}{9}\left(4.5 + x^4 \sin y^2\right) = g_1$$
$$y^{(n+1)} = \frac{1}{17}\left(8.5 - x^3y^2\right) = g_2$$

$$J(x,y) = \begin{pmatrix} \frac{1}{9}(4x^3 \cdot \sin y^2) & \frac{1}{9}(x^4 \cdot \cos y^2 \cdot 2y) \\ \frac{1}{17}(-3x^2 \cdot y^2) & \frac{1}{17}(-x^3 \cdot 2y) \end{pmatrix}$$

$$\max_{(x,y)\in D} \left\{ \left|\frac{1}{9}(4x^3 \cdot \sin y^2)\right| + \left|\frac{1}{9}(x^4 \cdot \cos y^2 \cdot 2y)\right|,\ \left|\frac{1}{17}(-3x^2y^2)\right| + \left|\frac{1}{17}(-x^3 \cdot 2y)\right| \right\}$$

$$\le \max\left\{ \frac{1}{9}\cdot(4\cdot 0.6^3 \cdot \sin 0.6^2) + \frac{1}{9}(0.6^4 \cdot \cos 0.4^2 \cdot 2\cdot 0.6),\right.$$
$$\left.\frac{1}{17}\cdot(3\cdot 0.6^2 \cdot 0.6^2) + \frac{1}{17}(0.6^3 \cdot 2 \cdot 0.6)\right\}$$

$$= \max\{0.0509,\ 0.0461\} = \underset{q}{0.0509} < 1 \ ✓$$

$x_0 = 0.5,\quad y_0 = 0.5$

$$\max\left|x^{(n)} - x^{(n-1)},\ y^{(n)} - y^{(n-1)}\right| \le \frac{1-q}{q}\cdot \underbrace{\varepsilon}_{5\cdot 10^{-5}} = 0.00093\dots$$

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$x_0 = 0,5 \qquad y_0 = 0,5$

$x_1 = 0,50174 \qquad y_1 = 0,49816$

$\boxed{x_2 = 0,50173 \qquad y_2 = 0,49816}$ ✓

b) . . . . . (sami)

$\varepsilon = 10^{-7}$

$$n \geq \frac{\ln \frac{\varepsilon(1-q)}{\|x^{(1)} - x^{(0)}\|_\infty}}{\ln q} = 3,3144\ldots$$

$n = 4$

(*) Metodom iteracije odrediti SVA REŠENJA sistema

$5x - 6y + 20 \ln x = -16 \qquad (1)$

$2x + y - 10 \ln x = 4 \qquad (2)$

$(1) \quad y_1 = \frac{5x + 20\ln x + 16}{6} \qquad (2) \quad y_2 = -2x + 10 \ln x + 4$

$$\Upsilon = y_1 - y_2 = \frac{1}{6}(5x + 20 \ln x + 16 + 12x - 60 \ln x - 24)$$

$$= \frac{1}{6}(17x - 40 \ln x - 8)$$

$$= 0$$

$\Upsilon = 0$

$\Upsilon' = \frac{1}{6}\left(17 - \frac{40}{x}\right) = 0 \Rightarrow x = \frac{40}{17} \approx 2.35$

Υ def. ~~za~~ $x > 0$ $(\ln x)$

$x \in (0, 2.35), \ \Upsilon' < 0 \Rightarrow \Upsilon \searrow$

$x \in (2,35, \infty), \ \Upsilon' > 0 \Rightarrow \Upsilon \nearrow$

$\left.\begin{array}{l} \lim\limits_{x \to 0} \Upsilon(x) = \infty \\ \Upsilon(2.35) = -0.37 \end{array}\right\} \exists!$ nula

$\left.\begin{array}{l} \Upsilon(2,35) = -0,37 \\ \lim\limits_{x \to \infty} \Upsilon(x) = \infty \end{array}\right\} \exists!$ nula

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$x \in (1.6, 1.7)$: $\Upsilon(1.6) = 0{,}07$

$\Upsilon(1.7) = -0.05$

$x \in (3.2, 3.3)$: $\Upsilon(3.2) = -0.02$

$\Upsilon(3.3) = 0.06$

...

I tačka

$(x_0, y_0) = (1.6, 5.5) \Rightarrow (x_1, y_1) = (1.653, 5.718)$

II tačka $(x_0, y_0) = (3.2, 9.2) \Rightarrow (x_2, y_2) = (3.227, 9.262)$

Dogovor 3: (Matlab)

za x_0 : $\|J(x_0)\| < 1$?

Gaus-Zajdelova metoda

$$x_1^{(n+1)} = g_1(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})$$

$$x_2^{(n+1)} = g_2(x_1^{(n+1)}, x_2^{(n)}, \dots, x_m^{(n)})$$

$$\vdots$$

$$x_m^{(n+1)} = g_m(x_1^{(n+1)}, x_2^{(n+1)}, \dots, x_{m-1}^{(n+1)}, x_m^{(n)})$$

Njutnova metoda

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$f(x)=0$ $\qquad f=(f_1,\dots,f_m)^T$

$x=G(x)$

$G(x) = x - D(x)\cdot f(x)$, D regularna

$G(x^*)=x^* \rightarrow \underbrace{G(x^*)}_{=x^*} = x^* - D(x^*)\cdot f(x^*)$

$\underbrace{D(x^*)}_{\neq 0}\cdot \underbrace{f(x^*)}_{=0} = 0 \Rightarrow f(x^*)=0$

$f(x^*)=0 \Rightarrow G(x^*) = x^* - D(x^*)\cdot \underbrace{f(x^*)}_{0} = x^*$

$$x^{(n+1)} = G(x^{(n)}) = x^{(n)} - \underbrace{D(x^{(n)})}\cdot f(x^{(n)})$$

$$D(x) = (J(x))^{-1} = \left(\frac{\partial f_i(x)}{\partial x_j}\right)^{-1} \rightarrow \text{Njutn - Raphson-ova metoda}$$

nije isto J iz iterativne

spec. slučaj: $Ax=b$

$f(x) = Ax - b$

$f'(x) = A$

$x^{(n+1)} = x^{(n)} - \underbrace{A^{-1}}\cdot(Ax^{(n)}-b) = A^{-1}b$

nije def. iterativni postupak

$$\left.\begin{array}{l} f_1(x_1,\dots,x_m)=0 \\ \vdots \\ f_m(x_1,\dots,x_m)=0 \end{array}\right\} \qquad F: X \to Y$$

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Def: Linearni operator $P: X \to Y$ naziva se izvod Frechet-a operatora F u tački x ako

$$\| F(x+h) - F(x) - P \cdot h \| = \sigma(\|h\|), \quad \|h\| \to 0$$

$$P \overset{\text{oznaka}}{=} F'(x)$$

f_i - nep. dif.

$$F(x+h) = F(x) + F'(x) \cdot h + \sigma(\|h\|)$$

$$\begin{pmatrix} f_1(x_1+h_1, \dots, x_m+h_m) \\ \vdots \\ f_m(x_1+h_1, \dots, x_m+h_m) \end{pmatrix} = \begin{pmatrix} f_1(x_1, \dots, x_m) \\ \vdots \\ f_m(x_1, \dots, x_m) \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_m} \end{pmatrix}}_{\text{izvod Frecheta (= Jakobijeva mat.)}} \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} + \sigma\left(\sqrt{\sum h_i^2}\right)$$

$$F(x^*) = 0, \quad h = x^* - x_n$$

$$\| F(x_n + h) - F(x_n) - P \cdot h \| = \sigma(\|h\|)$$

$$\| F(\cancel{x_n} + x^* - \cancel{x_n}) - F(x_n) - F'(x_n) \cdot (x^* - x_n) \| = \sigma(\|\underbrace{x^* - x_n}_{\xrightarrow[n\to\infty]{} 0}\|)$$

$$\| \cdot \| \approx 0$$

$$F(x^*) - F(x_n) - F'(x_n) \cdot (x^* - x_n) \approx 0$$

$$F(x_n) + F'(x_n)(x^* - x_n) \approx F(x^*) = 0$$

$$F(x_n) + F'(x_n)(\cancel{x_{n+1}} - x_n) \approx 0$$

$$\boxed{x_{n+1} = x_n - [F'(x_n)]^{-1} \cdot F(x_n)}, \quad n = 0, 1, \dots$$

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Lema: Ako $\exists$ izvod $F'(x)$ neprekidnog operatora $F: \mathcal{X} \to \mathcal{Y}$ $\forall x \in C$ gde je C konveksan skup u $\mathcal{X}$ i ako $\exists$ konstanta γ t.d. $\|F'(x) - F'(y)\| \leq \gamma \|x-y\|$, $\forall x,y \in C$

tada $\forall x,y \in C$ važi:

$$\|F(x) - F(y) - F'(y)(x-y)\| \leq \frac{\gamma}{2} \|x-y\|^2$$

D) C konveksan ako $\forall x,y \in C$, $0 \leq t \leq 1$ važi:

$$tx + (1-t)y \in C$$

$$tx + y - ty = y + t(x-y)$$

$$\phi(t) = F(y + t(x-y))$$

def. $\forall x,y \in C$, $t \in [0,1]$

$\phi : [0,1] \to \mathcal{Y}$

ϕ dif. $\forall t \in [0,1]$:

$$\phi'(t) = F'(y + t(x-y)) \cdot (x-y)$$

$$\phi(0) = F(y), \quad \phi(1) = F(y + x - y) = F(x), \quad \phi'(0) = F'(y)(x-y)$$

$$F(x) - F(y) - F'(y)(x-y) = \underbrace{\phi(1) - \phi(0)} - \phi'(0)$$

$$= \int_0^1 \phi'(t)\,dt - \int_0^1 \phi'(0)\,dt$$

$$= \int_0^1 (\phi'(t) - \phi'(0))\,dt$$

$$\|F(x) - F(y) - F'(y)(x-y)\| \leq \int_0^1 \underbrace{\|\phi'(t) - \phi'(0)\|}\,dt \qquad (*)$$

$$\|\phi'(t) - \phi'(0)\| = \|F'(y + t(x-y))(x-y) - F'(y)\cdot(x-y)\|$$

$$= \|(F'(y + t(x-y)) - F'(y)) \cdot (x-y)\|$$

$$\leq \|\cancel{F'(y)} + t \cdot F'(x) - t \cdot F'(y) - \cancel{F'(y)}\| \cdot \|x-y\|$$

$$= t \cdot \underbrace{\|F'(x) - F'(y)\|} \cdot \|x-y\|$$

$$\leq t \cdot \gamma \cdot \|x-y\| \cdot \|x-y\| = \gamma \cdot t \cdot \|x-y\|^2$$

$$(*): \quad \| \; - - - - \; \| \leq \int_0^1 \gamma t \, \|x-y\|^2 \, dt$$

$$= \gamma \, \|x-y\|^2 \cdot \int_0^1 t \, dt$$

$$= \gamma \, \|x-y\|^2 \qquad \boxtimes$$